Active factor completion strategies*

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Abstract

Embracing the concept of factor investing, we design a flexible framework for building out different factor completion strategies for traditional multi-asset allocations. Our notion of factor completion comprises a maximally diversified reference portfolio anchored in a multi-asset multi-factor risk model that acknowledges market factors such as equity, duration, and commodity, as well as style factors such as carry, value, momentum, and quality. The specific nature of a given factor completion strategy varies with investor preferences and constraints. We tailor a select set of factor completion strategies that include factor-based tail hedging, constrained factor completion, and a fully diversified multi-asset multi-factor proposition. Our framework is able to organically exploit tactical asset allocation signals while not sacrificing the notion of maximum diversification. To illustrate, we additionally embed the common trend style that permeates many asset classes.

Keywords: Diversification, Risk Parity, Factor Completion, Multi-asset Multi-factor Investing

JEL Classification: G11; D81

The global financial crisis severely challenged traditional asset allocation frameworks, because the correlation breakdown across most asset classes led to an erosion of portfolio diversification. Given the corresponding desire for more stable portfolio building blocks, there has since been considerable interest and research in structuring portfolios by using relevant risk and return factors rather than traditional asset classes alone (see Ilmanen and Kizer (2012); Idzorek and Kowara (2013)). Style factors such as quality, value, and momentum have emerged as natural candidates to meaningfully expand the investment opportunity set. Not only do these concepts help explain the cross-section of equity returns, but they also carry over to other asset classes, such as commodities, foreign exchange, and fixed income.

Many extant studies have focused on generating optimal style factor portfolios (e.g., Clarke, de Silva, and Murdock (2005); Bender, Briand, Nielsen, and Stefek (2010); Idzorek and Kowara (2013)). However, there has been less focus on deriving an integrated framework that blends traditional market and style factors. Similarly, a majority of asset owners continue to follow a more traditional asset allocation, so there is a need for a holistic approach to integrate the concept of style factors into the asset allocation toolkit (e.g., Koedijk, Slager, and Stork (2016)).

We contribute to satisfying this need in five primary ways. First, we put forward a global factor model that combines three market factors with four style factors. Equipped with such a risk model, a given traditional asset allocation will typically have little (if any) style factor exposure. Second, to complete a portfolio's lack of style factor exposure, we must choose a reasonable reference portfolio. We suggest a maximally diversified risk parity portfolio that captures the salient multi-asset multi-factor drivers. Third, the nature of a factor completion strategy should accommodate investors' preferences and constraints, so we demonstrate examples such as factor-based tail hedging, constrained factor completion, and a fully diversified multi-asset multi-factor proposition. Fourth, although the framework rests on the notion of maximum diversification, we illustrate how to include tactical asset allocation signals as well. Incorporating technical trading signals for the traditional asset classes, we adequately operationalize the well-known trend style that permeates many asset classes. Fifth, a natural extension of this process is to investigate the possibility of return predictability in style factors. In doing so, we provide evidence of the potential relevance of time series momentum in style factors.

The key ingredient of the factor completion strategies presented here is the maximally diversified reference portfolio. As Markowitz's (1952) seminal work notes, diversifying is crucial for deriving an optimal risk-return trade-off. Thus, diversification prompts utilization of assets' dependence structure in order to reduce any subsequent portfolio volatility. This notion of diversification motivates the search for uncorrelated assets that can help diversify portfolios. Meucci (2009) suggests using principal component analysis (PCA), which can be formed from any set of portfolio assets, to synthetically construct uncorrelated portfolios. We can think of the uncorrelated principal portfolios as embedded risk factors that span the investable asset universe. Meucci (2009) perceives a portfolio as well-diversified when uncorrelated risk factors contribute evenly to overall portfolio volatility.

Lohre, Opfer, and Ország (2014) build on this idea, and investigate a corresponding maximum diversification strategy in a classic multi-asset framework. Maximum diversification is obtained by following a risk parity strategy that budgets equal risk to the principal portfolios rather than to the underlying assets. This approach blends well with the work of Bruder and Roncalli (2012) and Roncalli and Weisang (2016), who advocate the use of risk parity strategies with respect to risk factors rather than single assets.

Meucci's (2009) framework uses a purely statistical PCA risk model. While the PCA implies orthogonality of risk factors by construction, it is also prone to certain drawbacks. Most notably, principal portfolios derived from a PCA often lack sound economic intuition, and their weights may prove to be unstable over time. Moreover, because the risk factors are omni-directional, it is not immediately apparent whether a given principal portfolio should be bought or sold. Therefore, Meucci, Santangelo, and Deguest (2015) consider an alternative orthogonal decomposition of the assets' variance-covariance matrix, instead of the eigenvector decomposition underlying the PCA. They suggest using minimum torsion factors extracted from an optimization procedure, with the aim of minimizing the tracking error between the original risk factors and the de-correlated risk factors. Their approach largely rectifies the PCA's drawbacks: Choosing the original risk factors as an anchor for factor orthogonalization ensures they are closely aligned with the uncorrelated minimum torsion factors, which enhances interpretation and stability. Moreover, the trade direction of a given minimum torsion factor is then in line with the corresponding original risk factor. This approach intuitively seeks to derive the best orthogonal decomposition of a given factor model, and it has recently been applied in the context of commodity factor investing (Bernardi, Leippold, and Lohre (2018)). Similarly, Martellini and Milhau (2017) demonstrate the benefits of minimum torsion portfolios for classic multi-asset allocations. Our work contributes to this strand of the literature.

In the world of multi-asset multi-factor investing, we advocate a maximum diversification strategy that pursues a diversified risk parity allocation along orthogonalized risk factors that are closely aligned with the most salient market and style factors. Utilizing this anchor we build out completion strategies for any given asset allocation based on investors' preferences. Our framework allows to precisely gauge the associated diversification trade-offs implicit in the various factor completion choices.

The remainder of this article is thus organized as follows. Section 1 formulates our rationale of multi-asset multi-factor investing, and describes our choice of market and style factors. Section 2 reviews how Meucci (2009) and Meucci, Santangelo, and Deguest (2015) manage diversification, and describes the mechanics of pure diversified risk parity strategies within the multi-asset multi-factor domain. Section 3 introduces our factor completion framework and its various applications, including the use of Black and Litterman (1990, 1992) to capture the notions of trend style investing and style factor momentum. Section 4 concludes.

1 The case for multi-asset multi-factor investing

The concept of factor investing has received a great deal of attention over the past decade. At its heart, factor investing suggests a major shift in the investment process, away from allocating across asset classes and picking single securities, and toward obtaining exposure to factors that have emerged as relevant in describing the cross-section of asset returns. Efficiently capturing factor exposure requires investing in single securities. However, these investments are typically broadly diversified across many securities so as to maximize the likelihood of capitalizing on a given factor premium.

1.1 Equity origins of style factor investing

We note that factor strategies are not overly engineered, but tend to relate to specific investment styles. They are thus commonly referred to as "style factors". Style factor investing has a long history in theory and practice. For example, the notion of value investing can be traced back to Graham and Dodd's (1934) seminal book, and it has since found many disciples who are seeking to buy securities that are cheap relative to their fundamental values. In the academic literature, the benefits of value investing were documented in the cross-section of U.S. stocks as early as the 1970s (e.g., Basu (1977)). The natural route to diversify value investments is to pursue a price momentum strategy. Price momentum builds on the notion of recent winners outperforming recent losers; the effect can be verified on relatively short-term lookback periods, and on investment horizons of up to twelve months (Jegadeesh and Titman, 1993). Therefore, a momentum investor must actively monitor and rebalance portfolio positions on a short-term basis, while a value investor, in contrast, needs to wait longer for securities to revert to their fundamental values. In addition to momentum and value, quality has recently emerged as a further style factor in equities. Quality style investors are not necessarily looking to buy companies cheaply, but rather to buy high-quality companies with strong balance sheets. Novy-Marx (2013) and Ball, Gerakos, Linnainmaa, and Nikolaev (2016) establish that quality style investing is profitable across a broad set of financial statement indicators, such as accruals or cash-based profitability. A related investment style is defensive investing. Defensive assets are characterized in terms of their relatively low-risk characteristics, as measured by volatility or market beta. Moreover, defensive, or low-risk, assets have been shown to outperform high-risk ones on a risk-adjusted basis as far back as the 1970s (Haugen and Heins, 1975; Black, 1972) ; more recent evidence includes Ang, Hodrick, Xing, and Zhang (2006) and Frazzini and Pedersen (2014).

1.2 Extending style factor investing to other asset classes

The notion of style factors has long been established in the literature. However, the major promise of factor investing is not to follow a single style, but to create a portfolio that allows for simultaneous harvesting of multiple style factor premia. Given the distinct nature of the described style factors, a multi-factor portfolio is expected to capitalize on style factor diversification. While traditional fund managers are often more prone to "high-conviction" investing, often following one particular investment style, diversified multi-factor investing has been the traditional domain of quantitative investment managers.

We expect similar notions of style factor investing to also permeate other asset classes. Obviously, for a given style factor to be a meaningful portfolio building block, at least one of the following factor rationale must apply. First, the candidate factor is tied to a risk-based argument, suggesting that the factor premium compensates for systematic risk. Second, the factor is rooted in irrational yet persistent investor behavior, allowing rational investors to constantly exploit the factor premium. Third, a given factor premium is associated with a certain market or industry structure that promotes specific return patterns. Likewise, the investment constraints of certain market participants can fuel abnormal return effects that persist in the continued presence of these constraints.[1]

1.3 The multi-asset multi-factor universe

We build on academic research that documents how style factors not only explain the cross-section of equity returns but also extend to other asset classes such as commodity, foreign exchange (FX), and fixed income. Notable contributions are Asness, Moskowitz, and Pedersen (2013) for value and momentum and Koijen, Moskowitz, Pedersen, and Vrugt (2018) for carry. Carry investing relates to the observation that, all else being equal, high-yielding assets outperform low-yielding ones. The most prominent example originates from FX investing, where the carry trade exploits the return differential of high- versus low-yield currencies, provided that FX rate movements do not nullify the yield advantage.

We categorize the carry, value, momentum, and quality factor strategies within the four asset classes equities, commodities, FX, and government bond rates. Appendix A.1 provides a detailed overview of the style factors used along with their construction principles. Rather than using single-factor strategies in a "kitchen" sink fashion, we adopt a pure style factor investing view, and aggregate single-factor strategies according to their underlying rationale. That is, we cluster all momentum factors into an aggregate momentum factor by applying a risk parity weighting scheme to FX, commodity, rates, and equity momentum strategies. In the same way, we synthesize aggregate carry, value, and quality factors. We then consider three traditional market risk factors that each derive from the risk parity weightings of investable market indices.

The first factor is global equity risk, as represented by the S&P 500, EuroSTOXX 50, FTSE 100, Nikkei 225, and MSCI Emerging Market indices. Given their considerable positive correlation to equities, we amend the first factor by a duration-hedged investment-grade and high-yield credit component. The second factor is duration risk, represented by ten-year U.S. Treasury notes, German bunds, ten-year Japanese government bonds, and U.K. gilts. The third factor is commodity risk, as represented by gold, oil, copper, and agriculture

[1] There has been a proliferation of factor research in recent asset pricing literature. Obviously, this development calls for safeguarding against data-snooping biases when testing for the relevance of given factors. Our goal here is not to add to the "factor zoo", but to collect and allocate style factor strategies that permeate many asset classes.

indices. To obtain exposure to traditional market factors, we consider efficient investment vehicles such as equity index futures, bond futures, and exchange-traded commodities (ETC).

Panel A of Table 1 gives the descriptive statistics for the above market factors. For all indices, we report monthly local excess returns. In particular, we measure equity and bond indices in terms of futures returns; for commodity indices, we use their total returns versus three-month U.S. Treasury bills. Over the entire January 2001 – October 2018 sample period, we observe that the MSCI Emerging Markets index was the best performing equity index, with a 7.59% annualized excess return, albeit also with the highest volatility, at 21.55%. This translates into a Sharpe ratio of 0.35.

Conversely, the EuroSTOXX50 experienced the lowest excess return (2.47%) and the lowest Sharpe ratio (0.13) across equity indices over the sample period. The bond indices' excess returns range from 1.96% (Japanese government bonds) to 4.32% (bunds), with standard deviations between 2.51% (Japanese government bonds) and 6.01% (gilts). The highest dispersion in excess returns is observed for the commodity indices. The best performing commodity was copper, with a 10.57% annualized excess return, while agriculture and oil were the weakest (1.08% and 1.25%, respectively). Oil suffered from a severe drawdown of -92.40% over our sample period.

[Table 1 about here.]

Panel B of Table 1 summarizes the performance of the style factors. All style factors earned a positive premium in excess of the risk-free rate, ranging from 0.89% for Rates Value, to 5.77% for Commodity Carry. In general, the Rates factors exhibit the weakest performance, which translate to the lowest Sharpe ratios across style factors as well. Commodity Quality obtains the highest Sharpe ratio, at 1.61. Compared to the market factors, the style factors exhibit lower volatility figures, in the single digits. In turn, the worst drawdown is -24.08% (Equity Momentum), which is much lower than that of any equity index.

Panel C of Table 1 gives the descriptive statistics of the aggregate factors, which are risk parity combinations of the respective market and style factors. On an aggregate level, Value exhibits the lowest return with 2.56%, while Commodity is the strongest market factor, with a 5.82% annualized excess return. All aggregated style factors have Sharpe ratios above 1 (1.98 for Quality, 1.28 for Carry, 1.08 for Momentum, and 1.02 for Value). Duration has a Sharpe ratio of 0.86, Commodity 0.37, and Equity 0.33.

The upper chart in Figure 1 depicts the average correlation of single asset classes and style factors over the entire sample period. We observe high correlations within the equity bucket (reaching a maximum of 0.84 for EuroSTOXX50 and FTSE100) and the bond bucket (a maximum of 0.85 between bund and gilt). The cross-correlation between the equity and bond buckets is slightly negative. The commodity bucket is also discernible from the correlation analysis, but we note that commodities are generally more heterogeneous than equities or bonds.

[Figure 1 about here.]

Style factor correlations do not naturally lend themselves to clustering certain buckets, as they typically range between ± 0.3. We observe the highest correlation for the Rates Momentum factor and for U.S. ten-year Treasury notes (0.69). In addition, we find FX Carry to be positively correlated with equity and credit markets (around 0.5) due to their severe downside comovements throughout the global financial crisis. Interestingly, we find that equity style factors are not positively correlated with broader market indices, but are actually negatively correlated. As expected, aggregating single asset classes and style factors renders building blocks that are fairly uncorrelated; see the lower chart of Figure 1. The highest correlation is obtained for the aggregates of Carry and Equity-Credit (0.43) and the lowest for Duration and Equity-Credit (-0.39).

2 Diversified risk parity for maximum diversification

To benefit from diversifying a multi-asset portfolio with style factors, we could simply turn to Markowitz's (1952) classical mean-variance optimization. But we would need to determine which inputs will best guide the resulting portfolio toward complementing the existing allocation by a factor completion allocation. Indeed, while the mean-variance paradigm is the model of choice to balance expected risks and return, it actually leads to portfolios that are fairly concentrated and anything but efficient ex post (Best and Grauer (1991)).

Given that mean-variance inputs come with a certain amount of estimation error, academics and practitioners have turned to more robust allocation paradigms that do not require estimation of expected returns. The simplest approach, $1/N$, diversifies portfolio weights by allocating capital equally to each asset or factor. However, depending on the heterogeneity of the investment universe, this approach may cause an imbalance in risk allocation. Conversely, one could adopt a risk parity strategy that allocates so that each asset or factor contributes equally to portfolio risk.

While it is intuitive to aim for a balanced risk allocation, there is an ad hoc nature to a naïve risk parity approach. Therefore, in this section, we begin with a definition of diversification, and then describe how to motivate and execute meaningful diversified risk parity strategies within a multi-asset multi-factor universe. This diversified risk parity paradigm then serves as an anchor from which to guide subsequent factor completion optimizations based on the mean-variance framework.

2.1 Search for low-correlated factors

According to standard portfolio theory, diversification is geared at eliminating unsystematic risk. In this vein, a common notion is to avoid exposure to single shocks or risk factors. Naturally, diversification is especially effective when one combines low-correlated assets. Taking this idea to extremes, Meucci (2009) constructs uncorrelated risk sources by applying a PCA to the variance-covariance matrix of the portfolio assets. In particular, he considers a portfolio consisting of N assets with return vector $\mathbf{R}$. Given weights $\mathbf{w}$, the resulting portfolio return is $R_w = \mathbf{w}'\mathbf{R}$. According to the spectral decomposition theorem, the covariance matrix $\mathbf{\Sigma}$

can be expressed as a product of

$$\mathbf{\Sigma} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' \tag{1}$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, ..., \lambda_N)$ is a diagonal matrix consisting of $\mathbf{\Sigma}$'s eigenvalues, assembled in descending order, $\lambda_1 \geq ... \geq \lambda_N$. The columns of matrix $\mathbf{E}$ represent the eigenvectors of $\mathbf{\Sigma}$. These eigenvectors define a set of N *principal portfolios*[2] whose returns, given by $\tilde{\mathbf{R}} = \mathbf{E}'\mathbf{R}$, are uncorrelated and whose variances equal $\lambda_1, ..., \lambda_N$. As a consequence, a given portfolio can be expressed either in terms of its weights $\mathbf{w}$ in the original assets, or in terms of its weights $\tilde{\mathbf{w}} = \mathbf{E}'\mathbf{w}$ in the principal portfolios.

More importantly, we obtain maximum diversification when following a risk parity strategy along the extracted principal portfolios (Lohre, Opfer, and Ország, 2014). While the underlying rationale is intuitive, the ensuing PCA-driven portfolio allocation will suffer from the classic PCA critique: Because principal components are statistical in nature, they often lack sound economic interpretation, which complicates the buy-or-sell decision. Given orthogonalization, we expect investors to be indifferent, from a diversification perspective, to buying or selling a principal portfolio. Moreover, principal components are known to be rather unstable, which tends to translate into excessive strategy turnover (Bernardi, Leippold, and Lohre, 2018).

Recalling that PCA is just one possible decomposition of covariance matrix $\mathbf{\Sigma}$, Meucci, Santangelo, and Deguest (2015) conceive a new technique that uncovers alternative sets of uncorrelated yet economically meaningful factors. They suggest using a parsimonious factor model that can be orthogonalized so that the ensuing de-correlated factors have minimum tracking error to the original factors. In particular, the starting point is a K-factor model with factors $\mathbf{F}$. Portfolio returns R_w for a portfolio with weights $\mathbf{w}$ can then be represented in terms of factor returns as follows:

$$R_w = \mathbf{w}'\mathbf{R} = \mathbf{b}'\mathbf{F} \tag{2}$$

where $\mathbf{b}$ denotes the portfolio returns' factor loadings with respect to factor model $\mathbf{F}$. Meucci, Santangelo, and Deguest (2015) advocate changing the representation in (2) to instead reflect uncorrelated factors $\mathbf{F}_T$:

$$R_w = \mathbf{b}'\mathbf{F} = \mathbf{b}_T'\mathbf{F}_T \tag{3}$$

where $\mathbf{b}_T$ denotes the portfolio returns' factor loadings with respect to factor model $\mathbf{F}_T$. Using $\mathbf{R} = \mathbf{B}'\mathbf{F}$ with $\mathbf{B} \in \mathbb{R}^{K \times N}$, the covariance matrix $\mathbf{\Sigma}$ can be decomposed as follows:

$$\mathbf{\Sigma} = \mathbf{B}'\mathbf{\Sigma}_F\mathbf{B} + \mathbf{u} \tag{4}$$

where $\mathbf{u}$ is idiosyncratic risk that is not captured by the chosen factor structure. The critical step is to construct an orthogonal decomposition of $\mathbf{F}$ using a linear transformation $\mathbf{F}_T = \mathbf{t}\mathbf{F}$, where $\mathbf{t}$ is a $K \times K$ torsion matrix. Meucci, Santangelo, and Deguest (2015) devise an algorithm to back out an uncorrelated factor representation that closely mimics the original

[2]Partovi and Caputo (2004) coined the term "principal portfolios" in their recasting of the efficient frontier.

factor model $\mathbf{F}$. Among all de-correlating linear transformations $\mathbf{t}$, their algorithm selects the minimum torsion $\mathbf{t}_{MT}$ that minimizes the distance to the original factors:

$$\mathbf{t}_{MT} = \underset{Corr(\mathbf{tF})=\mathbf{I}_{K\times K}}{\arg\min} \sqrt{\frac{1}{K}\sum_{k=1}^{K} Var\left(\frac{\mathbf{t}'\mathbf{F}_k - \mathbf{F}_k}{\sigma_k^F}\right)} \tag{5}$$

where σ_k^F denotes the volatility of factor $\mathbf{F}_k$. Equipped with this minimum torsion, $\mathbf{t}_{MT}$ we can decompose the systematic risk of a given portfolio as follows:

$$\mathbf{B}'\mathbf{\Sigma}_F\mathbf{B} = \mathbf{B}'\mathbf{t}_{MT}^{-1}\mathbf{\Sigma}_{MT}\mathbf{t}'^{-1}_{MT}\mathbf{B} \tag{6}$$

where $\mathbf{\Sigma}_{MT} = \text{diag}(\sigma^2_{MT,1}, ..., \sigma^2_{MT,K})$ is a diagonal matrix of minimum torsion factors' variances. As with the principal portfolios, we can similarly rewrite a portfolio with weights $\mathbf{w}$ in the original assets to reflect weights $\mathbf{w}_{MT}$ in the minimum torsion factors $\mathbf{F}_{MT} = \mathbf{t}_{MT}\mathbf{F}$. Specifically, it holds that $\mathbf{w}_{MT} = \mathbf{t}'^{-1}_{MT}\mathbf{Bw}$.

2.2 Diversification distribution, entropy, and diversified risk parity

Regardless of the nature of uncorrelated factors—principal portfolios or minimum torsions—the beauty of this approach is that total portfolio variance emerges from simply computing a weighted average over the uncorrelated factor variances. Focusing on minimum torsion weights $w_{MT,k}$ onward, we have:

$$Var(R_w) = \sum_{k=1}^{K} w^2_{MT,k}\sigma^2_{MT,k}. \tag{7}$$

Normalizing the minimum torsion factors' contributions by portfolio variance yields what Meucci (2009) calls the *diversification distribution*:

$$p_{MT,k} = \frac{w^2_{MT,k}\sigma^2_{MT,k}}{Var(R_w)}, \qquad k = 1, ..., K. \tag{8}$$

By design, the diversification distribution is always positive, and the $p_{MT,k}$s sum to 1. Building on this concept, Meucci (2009) would conceive a portfolio as well-diversified when the $p_{MT,k}$s are "approximately equal and the diversification distribution is close to uniform". This definition of a well-diversified portfolio coincides with allocating equal risk budgets to the uncorrelated factors, which led Lohre, Opfer, and Ország (2014) to coin the approach *diversified risk parity*. Conversely, portfolios loading on a specific factor display a concentrated diversification distribution. It is thus straightforward to apply a dispersion metric to the diversification distribution to gauge overall portfolio diversification. Meucci (2009) suggests using the exponential of its entropy[3] to measure the *effective number of uncorrelated bets*,

[3]Entropy has been used before in portfolio construction (see, e.g., Woerheide and Persson (1993) and Bera and Park (2008)). Unlike Meucci (2009), these studies do not consider the entropy of the risk allocation, but of the weight allocation, thus disregarding the dependence structure of the portfolio assets.

$\mathcal{N}_{Ent}$:

$$\mathcal{N}_{Ent} = \exp\left(-\sum_{k=1}^{K} p_{MT,k} \ln p_{MT,k}\right) \tag{9}$$

To rationalize this choice, consider two extreme cases. For a completely concentrated portfolio, we have $p_{MT,k} = 1$ for one risk factor k, and $p_{MT,l} = 0$ for $l \neq k$, resulting in an entropy of 0, which in turn implies $\mathcal{N}_{Ent} = 1$ uncorrelated bet. This is a portfolio whose risk is driven solely by one risk factor. In contrast, $\mathcal{N}_{Ent} = K$ holds for a portfolio that is completely homogeneous in terms of uncorrelated risk sources. In this case, $p_{MT,k} = p_{MT,l} = 1/K$ holds for all k, l. As a result, the entropy is equal to $\ln(K)$, and therefore the maximum effective number of uncorrelated bets ($\mathcal{N}_{Ent} = K$) is obtained. All of the uncorrelated risk sources contribute equally to the risk of the portfolio. To achieve maximum diversification, the optimal diversified risk parity strategy is ultimately an inverse volatility strategy in the minimum torsion factors. Its weights can be computed analytically, as detailed in Appendix A.2.

3 Factor completion strategies

From a pure factor investing perspective, a maximum diversification strategy in the multi-asset multi-factor domain is the method of choice for optimally harvesting the available style and market factor premia. Therefore, this section investigates the mechanics of diversified risk parity strategies that have a natural appeal as anchors for factor completion. Still, the majority of investors tend to cling to traditional multi-asset allocations and may be more inclined to consider less radical steps toward integrating style factors into their overall portfolio solutions. The presented factor completion framework therefore caters to various investment objectives and constraints, while expanding the investment opportunity set in terms of style factors.

3.1 Diversified risk parity in the multi-asset multi-factor domain

3.1.1 Rationalizing minimum torsion factors

To further rationalize the uncorrelated risk sources synthesized from the underlying multi-asset multi-factor data, we investigate the minimum torsion factors (MTF) over our entire sample period of January 2001 to October 2018. The economic nature of the MTFs is best assessed in terms of torsion matrix $\mathbf{t}_{MT}$, which represents their weights with respect to the original factors. Given that the correlation of the original style factors is generally low, the interpretation of the corresponding de-correlated minimum torsion factors is straightforward (see Table 2). We observe a similarly uncontroversial pattern for Duration, which has slightly negative loadings on Carry and Momentum. Notably, MTF1 and MTF3 reasonably track equity and commodity market risk, respectively. Nevertheless, these two factors load on other factors as well. For example, MTF1 is mainly long on Equity and Momentum and short on Carry and Value. MTF3 is long on Commodity, Carry, and Momentum, and short on Quality. Overall, the MTFs are closely aligned with the underlying original factors.

[Table 2 about here.]

Our subsequent analysis requires estimation of MTFs over time. To capture time variation and adapt to potential structural breaks, we rely on a sixty-month rolling window estimation.[4]

3.1.2 Diversified risk parity over time

We next investigate the consequences of allocating according to the optimal diversified risk parity strategy. That is, we follow an inverse volatility strategy along the MTFs that can be computed analytically.[5] Rebalancing occurs monthly. Given that the first estimation consumes sixty months' of data, we assess the strategy performance from January 2006 to October 2018.

For benchmarking the diversified risk parity (DRP) strategy, we consider four alternative, purely risk-based asset allocation strategies: $1/N$, minimum-variance, risk parity, and the most-diversified portfolio (MDP) of Choueifaty and Coignard (2008). The construction of these alternatives is detailed in Appendix A.3. Note that all allocation paradigms are applied on an aggregate factor level, rather than in an overall haphazard fashion at the single asset and factor level. As a result, all allocation strategies generally may benefit from the imposed factor structure.

Table 3 provides the performance and risk statistics of the DRP strategy and the alternative risk-based asset allocation strategies. All returns are net of transaction costs.[6] The optimal DRP strategy earns 3.28% at 1.45% volatility, which is equivalent to a Sharpe ratio of 1.24. Among the alternative strategies, we observe the highest annualized return for the $1/N$ strategy (3.53%), but it comes with the highest volatility (3.49%) and the worst drawdown (-9.35%). Conversely, the minimum-variance strategy provides the lowest return (2.94%). Given that minimum-variance indeed exhibits the lowest volatility (1.31%), its 1.15 Sharpe ratio is nevertheless comparable to that of the DRP. Its drawdown statistics are also the least severe, and amount to a maximum loss of -1.04% during the entire sample period. Nevertheless, the DRP strategy's maximum drawdown is only slightly more severe (-1.92%).

[Table 3 about here.]

Note that the MDP exhibits a similar performance to that of the DRP, with a return of 3.23% and a volatility of 1.57%. This translates into a 1.14 Sharpe ratio. In addition to the DRP strategy, we also consider a standard risk parity strategy applied to the original (non-orthogonalized) factors. We observe that standard risk parity yields the same return as DRP, but at slightly higher risk. Therefore, we posit that the benefit of orthogonalizing

[4]In unreported results, we investigate the stability of these dynamic MTF weights. We find they are quite stable over time, and are well aligned with the static weights in Table 2.

[5]See Equation (20) in Appendix A.2.

[6]Transaction costs are accounted for as follows: First, we consider transaction costs of 70 bps for a turnover of 100% when constructing each portfolio representing a given style factor. This consideration does not apply for the market factor indices, which are traded via futures. Second, we assume a fee of 96 bps p.a. for holding those style factors in a swap. Again, this does not apply to the market factors. Third, we assume 30 bps transaction costs for a turnover of 100% in any market factor. For the style factors, we assume 35 bps for a turnover of 100%.

the original factors over our sample period is marginal (at least performance-wise). This observation is expected, given that the original factors are relatively uncorrelated.

Comparing the monthly two-way turnover of strategies, the DRP strategy falls in the middle, with an average of 3.66%. This compares to 1.53% for $1/N$, 2.97% for risk parity, 5.98% for minimum-variance, and 6.94% for MDP.[7]

As Lee (2011) argues, evaluating risk-based portfolio strategies by means of Sharpe ratios is hard to reconcile with the fact that returns fail to enter the respective objective functions. Instead, we should rather compare the different risk characteristics of the portfolios. While all such strategies appear somewhat similar in terms of their overall risk and return statistics, they differ in their effective number of bets ($\mathcal{N}_{Ent}$). Hence, their risk profiles also differ. By construction, the DRP strategy has seven bets throughout time, and the risk parity strategy comes closest, with 6.94 bets. While the MDP displays a sufficiently balanced risk profile (6.20 bets), the other alternatives are more concentrated, as reflected by 4.07 bets for minimum-variance, and 3.11 for $1/N$.

To gain a better understanding of the risk-based strategies' diversification, we next discuss their weights and risk allocations in depth, as seen in Figure 2, where risk is decomposed by MTFs. Specifically, Figure 2 gives the decomposition of the risk-based allocation strategies in terms of single asset and factors weights (left column), aggregate factor allocation weights (middle column), and risk (right column).

We first examine the DRP strategy. Exploring the risk decomposition in terms of the MTFs, the DRP shows equal risk by construction for all seven factors (right column of Figure 2). Portfolio weights are likewise diversified, with market factors having 25%-35% weights in total (left and middle column). The risk parity strategy is fairly close in terms of portfolio weights and risk allocation. Presumably, there was no correlation breakdown across the chosen factor structure, rendering orthogonalization seemingly obsolete.

[Figure 2 about here.]

Second, regarding the benchmark strategies, we note that up to 95% of the $1/N$-strategy's risk budget is consumed by commodities and equities. The risk contribution from equities is roughly 25%, so commodities makes up the bulk, and renders other asset classes and style factors irrelevant. This assessment is confirmed by the volatility decomposition along single assets and style factors. Third, we capture the archetypical weight distribution of minimum-variance, which is heavily concentrated in the two style factors Quality and Value. Both exhibit the least volatile returns during the sample period. While equities scarcely enter the minimum-variance portfolio, we observe a diversifying commodities position of 5%-10% at times. And, although there is a relatively large position in bonds, they do not significantly impact the risk budget. Fourth, we note that the MDP is more balanced from a risk perspective, with one-half of the risk budget consumed by market factors and the other

[7] We measure turnover based on the weight differences of successive models, which could understate the actual strategy turnover. However, we observe a non-zero turnover in the $1/N$ strategy because of embedded turnover to rebalance the underlying risk parity factor schemes.

half by style factors. Its weight decomposition over time is between that of risk parity and minimum-variance.

In order to directly compare the degree to which the risk-based asset allocation strategies accomplish the goal of diversifying across uncorrelated risk sources, we plot the effective number of uncorrelated bets over time in Figure 3. Confirming our interpretation of the risk contributions over time, we find the $1/N$-strategy to be largely surpassed by the other strategies. By construction, the DRP strategy constantly maintains the maximum number of seven bets over time. Given the original factor correlation structure, it is not surprising that standard risk parity finishes as a runner-up in this statistic. Between $1/N$ and DRP, we find that minimum-variance and MDP have 4.07 and 6.20 bets on average, respectively. While the MDP is stable throughout the sample period, minimum-variance begins with approximately five bets in 2006, and slowly loses ground, with just four bets left at the end of the sample period.

[Figure 3 about here.]

3.2 Traditional asset allocation through a factor lens

Now that we are equipped with MTFs, we are ready to examine the degree of diversification of traditional multi-asset allocations along market and style factors. Such an analysis illustrates the scope style factors command to enhance portfolio diversification and serve as a means of benchmarking any suggested factor completion solution.

To illustrate, we choose a common traditional asset allocation benchmark that invests 25% in global equities, 25% in corporate bonds, 10% in commodities, and the remainder in global government bonds.[8] To examine the relevance of factors, we explore this traditional multi-asset allocation in terms of its global asset and style factor exposures in Figure 4. Specifically, we use the three market and four style factors introduced in the preceding sections. The risk allocation is rather concentrated in traditional market factors, with equity and commodities making up three-quarters of the risk budget. However, there are several minor implicit style factor exposures associated with the traditional asset class allocation, such as Carry and Quality.

Invoking Meucci's (2009) concept of the effective number of bets, we compute the benchmark to contain 3.07 bets through time (first column in Table 4). We thus conclude that the chosen traditional asset allocation exploits less than half the available spectrum of factor bets.

[Figure 4 about here.]

[Table 4 about here.]

[8] Note that we apply equal portfolio weights within any given asset class using the assets in panel A of Table 1.

3.3 Factor-based tail hedging

A prudent first step in entering multi-asset multi-factor investing would be to consider style factors as a means to protect a given asset allocation against downside risk. Taking this risk-based perspective, we can determine a factor completion portfolio by running a minimum-variance portfolio optimization that fixes the original asset allocation of the investor as described in section 3.2, and only allows dynamic allocation to the style factors to help reduce portfolio risk. In essence, we optimize

$$\mathbf{w} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \tag{10}$$

subject to non-trading constraints in the benchmark assets. Figure 4 depicts the subsequent factor completion portfolio over time.

Note that we find that long-short equity factors constitute a significant part of the tail hedge factor completion portfolio (left column). Similarly, there are allocations to defensive factors such as Equity Quality and FX Value. As for the primary portfolio objective, the tail-hedged portfolio has a volatility of 5.85% ex post, which compares to the benchmark volatility of 5.60%, shown in Table 4. Note that the risk reduction is more significant in terms of tail risk, because it leads to lowering the maximum drawdown from -17.18% to -15.15%.

Having included style factors in the portfolio mix, we would expect an increase in portfolio diversification; however, the increase is rather modest, and only raises portfolio diversification by one extra bet to 4.12. Judging from the right column in Figure 4, this increase is not attributable to a larger style factor exposure, but rather to an implicit increase in duration risk (as represented by the tail hedge factor completion portfolio). Therefore, the dominant benchmark exposure to equity and commodities is somewhat reduced. In terms of performance, the addition of style factors leads to an increase in the Sharpe ratio to 1.03, compared to 0.72 for the benchmark portfolio (see Table 4).

3.4 Diversified risk parity for factor completion

3.4.1 Constrained diversified risk parity

Next, we consider a factor completion portfolio that seeks to complete the underlying benchmark asset allocation with respect to the overall risk allocation in terms of style and market factors. As outlined above, the optimal solution would follow the DRP allocation in the absence of investment constraints, thereby maximizing portfolio diversification. However, the latter framework is less suited for dealing with investment constraints. We therefore propose couching the optimization in a mean-variance setting, augmented with implied views from the optimal DRP allocation. Without constraints, such an optimization would naturally capture the maximally diversified portfolio with perfectly balanced risk allocation. Given a fixed asset allocation, the optimizer will allocate style factors in a manner to best trade off diversification-centric views vis-à-vis benchmark constraints.

Weights are therefore derived by using the following utility maximization:

$$\mathbf{w} = \underset{\mathbf{w}}{\operatorname{argmax}}\, \mathbf{w}'\mathbf{r} - \frac{\lambda}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \tag{11}$$

subject to non-trading constraints in the benchmark assets, and using the implied views from the optimal DRP allocation for expected return inputs $\mathbf{r}$. The resulting factor completion portfolio ultimately consists of a diversified mix of style factors (see third row of Figure 4).

The associated risk allocation is considerably less concentrated in equity and commodities, and more exposed to the four style factors. The average number of effective bets is 6.00, indicating that the benchmark constraints come at the cost of one bet. This small decrease in diversification efficiency is explained by the fact that the benchmark asset allocation is already fairly close to the optimal DRP allocation in the three market factors. However, we anticipate that completing the factor profile may be less straightforward for alternative asset allocation constraints. In our case, the overall portfolio has a maximum drawdown similar to that of the tail-hedged portfolio (-15.60%) but with higher returns, thereby raising the Sharpe ratio to 1.14.

3.4.2 Pure diversified risk parity

Ideally, investors are not restricted, and can consider lifting investment constraints to fully capitalize on the maximum diversification allocation. In practice, this does not imply offloading the given physical investments. Rather, it suggest potentially complementing the existing portfolio positions in terms of a futures overlay to manage market beta exposure according to the diversified risk parity allocation.

The latter needs to be levered further to attain the desired level of portfolio risk, because the original DRP portfolio volatility is only 1.45%. In the specific example, we apply a leverage factor of 3.25 to obtain a similar risk level as the original asset allocation, raising the DRP volatility to 5.63% annually. We refer to this factor completion solution as "pure diversified risk parity," as we simply lever the implied DRP views in an unconstrained mean-variance portfolio optimization. Given the moderate benchmark allocation, we choose a similarly modest risk aversion coefficient $\gamma = 5$ in this optimization. Obviously, this levered pure DRP portfolio, with an average of 6.95 bets, conserves the maximum diversification properties of the original DRP allocation. Turnover is increased, but so are net total return (7.74%) and risk-adjusted returns (1.11 in terms of the Sharpe ratio, 0.77 in terms of the Calmar ratio) compared to the benchmark portfolio. Increasing portfolio diversification again helps mitigate tail risk; the maximum drawdown of the pure DRP portfolio is further reduced to -10.00%.

3.5 Diversified risk parity with a trend

The framework we present here naturally lends itself to exploiting tactical asset allocation signals while embracing the merits of DRP. Indeed, there is considerable evidence that asset classes tend to exhibit time series return predictability. There is especially strong evidence of the efficacy of time series momentum signals that are often used by investors pursuing a trend-following investment style (Moskowitz, Ooi and Pedersen, 2012; Hurst, Ooi and Pedersen, 2017).

A naïve way to include a trend style factor into a given portfolio would provide the optimizer with the rather smooth return time series of the corresponding trend-following strategy backtest. However, unlike the style factor “momentum”, which is based on cross-sectional information, a trend-following strategy seeks to exploit time series information by implementing directional long or short positions in the underlying asset classes. Thus, a smooth strategy backtest would most likely lead the optimizer to underestimate portfolio risk with respect to traditional market risk factors. Instead, one should build on the underlying traditional asset class time series together with the corresponding trend signal. Along this line of reasoning, we enhance the pure DRP strategy by providing the portfolio optimizer with explicit return forecasts for the three asset classes: equity, government bonds, and commodities (but not for the style factors). All are based on a simple trend-following signal: twelve-month price momentum (Moskowitz, Ooi, and Pedersen (2012)).

In this context, it is important to not drive out all of the diversification benefits associated with DRP. To incorporate trend signals in the pursuit of maximum diversification, we investigate signal blending based on Black-Litterman (1990, 1992) and He and Litterman (2002). In particular, we choose the pure DRP allocation as the reference portfolio in a Black-Litterman optimization, calibrated so that the resulting DRP Trend portfolio exhibits a tracking error of 2% relative to the pure DRP portfolio. We simply use the standard Black-Litterman master formulae for refining return and variance-covariance estimates. For example, the refined return input results from:

$$E(R) = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q\right] \tag{12}$$

where Σ refers to the variance-covariance matrix that combines all single asset classes and factors. Specifying the objective view Π is straightforward as well: We back out the resulting expected returns from the DRP base allocation. The projection matrix P is a diagonal matrix with 1s along the diagonal. We also shrink the respective variance-covariance matrix according to the traditional Black-Litterman formula as follows:

$$\Sigma_{BL} = \Sigma + \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1} \tag{13}$$

In the empirical application, we use a cautious specification of Ω and τ. In particular, we set $\Omega = \text{diag}(\Sigma)$ and $\tau = 0.015$, which prevent us from deviating too far from the DRP base allocation. The time series trend signals populating vector Q are rescaled to conform to the amplitude of monthly returns, following the spirit of Grinold and Kahn (2000):

$$Q_{abs} = 0.5 \cdot Volatility \cdot Scores \tag{14}$$

To allow for consistent exploitation of the trend signal relative to the DRP base allocation (across various leverage-gamma specifications), we construct a relative forecast centered around the implied views:

$$Q_{rel} = \Pi \cdot Q_{abs} \tag{15}$$

We explicitly set Q equal to Π for the assets that have no trend signal in order to minimize trading on correlation patterns alone. Because the trend signal is quite volatile, we introduce a transaction cost penalty in the Black-Litterman framework that creates sluggishness in the rebalancing process. Moreover, because the optimization problem is quadratic in nature, it is straightforward to expand the mean-variance objective function by a quadratic transaction cost (TC) penalty[9]:

$$\max_{w} w'\mu - \frac{\gamma}{2} w'\Sigma w - \lambda_{TC}\Gamma|\Delta w|^2 \tag{16}$$

Using $\Delta w = w - w_0$ and rearranging terms, we have:

$$\max_{w} w'(\mu + 2\lambda_{TC}\Gamma w_0') - w'\left(\frac{\gamma}{2}\Sigma + \lambda_{TC}\Gamma\right) w \tag{17}$$

Adding the TC penalty to the objective function has two implications for the optimization: First, the penalty leads to an increase in expected return for the current allocation. Second, perceived volatility is increased, thereby reducing the attractiveness of the assets. The transaction cost matrix Γ is typically populated on the basis of estimates derived from TC models (Gârleanu and Pedersen, 2013). While it is possible to impose a constant TC matrix, our empirical analysis uses $\lambda_{TC} = 0.3$, and assumes Γ is linear in the diagonal of the variance-covariance matrix.

As expected, the DRP Trend allocation is more active than the pure DRP allocation (see the last row of Figure 4 and Table 4); its mean monthly turnover is 10 percentage points higher. This increased turnover results in an active return of (10.35% − 7.74% =) 2.61% p.a., and a Sharpe ratio of 1.60 (see Table 4). We note that the presence of a trend style allocation leads to very little loss in portfolio diversification, with the number of bets still averaging to 6.76.

Figure 5 sheds more light on the number of bets contained in all the active factor completion strategies over time. First, the pure DRP strategy consistently maintains close to seven bets. Second, the DRP Trend portfolio is not far from this anchor allocation, except for during the first three years of the sample period (which include the global financial crisis). Third, the constrained DRP can compete during these three years in terms of bets, but it falls behind by one bet in the subsequent period. Fourth, the tail hedge and benchmark portfolios have fewer bets over time.

[Figure 5 about here.]

[9] See Dichtl, Drobetz, Lohre, Rother, and Vosskamp (2019) for an application of quadratic TC penalties in the context of dynamic equity factor allocations.

The concept of the effective number of uncorrelated bets is particularly insightful in the context of calibrating the degree of activity of the tactical asset allocation (TAA). The lower panel of Figure 5 illustrates the interplay between the aggressiveness of the TAA (as determined by τ) and the degree of diversification (as measured by $\mathcal{N}_{Ent}$). Obviously, diversification is reduced for more aggressive allocations; however, a confident choice of TAA activity ($\tau = 0.15$) still enables maintenance of at least five bets at all times.

3.6 Adding style factor momentum to the mix

In a similar way, we can also use style factor momentum signals to further tilt the overall allocation. While the subject of factor timing is a matter of contention among practitioners and academics, there is mounting evidence of predictability with regard to factor momentum, in the cross-section (Moskowitz and Grinblatt (1999); Arnott, Clements, Kalesnik, and Linnainmaa (2019)) and in the time series dimension (Avramov, Cheng, Schreiber, and Shemer (2017); Moskowitz, Ooi, and Pedersen (2012); Gupta and Kelly (2018)).

We extend the above Black-Litterman optimization framework by including a twelve-month trend signal for all style factors. Table 5 shows the effect on risk and return when we add the trend signals sequentially. Following this course, the Sharpe ratio of the corresponding active allocations increase from 1.11 in the pure DRP case, to 1.86 when all trend signals are included.

Unsurprisingly, trend style investing in market factors accounts for roughly two-thirds of the increase in risk-adjusted performance. Equity has the greatest impact, adding 130 bps to returns, while lowering volatility (from 5.63% to 5.34%). Nevertheless, given the average number of 6.94 bets, diversification benefits do not appear to suffer.

Duration and Commodity have a similar impact on risk-adjusted returns, while avoiding impacting the number of bets or volatility. Notably, the reduction is likewise modest when further considering style factors. For example, accounting for style factor momentum in equity factors further enhances the Sharpe ratio (1.74) for most. Rates factor momentum only marginally increases the Sharpe ratio, while reducing the number of bets from 6.55 to 6.38.

[Table 5 about here.]

4 Conclusion

Traditional asset allocations are only minimally balanced across the salient drivers of risk and return. Clearly, the corresponding portfolio allocation would benefit from explicit joint management of asset and style factor exposures. This paper provides a viable framework with which to complete a given portfolio allocation in terms of a factor completion portfolio. Naturally, factor completion strategies require the specification of a meaningful reference portfolio toward which to complete. We focus on diversified risk parity strategies in a multi-asset multi-factor domain as a factor completion anchor. However, the factor completion framework presented is not dependent on that choice. We demonstrate relevant cases of factor completion that cater to specific investor needs. In particular, our framework allows to precisely gauge the associated diversification trade-offs implicit in the various factor completion choices.

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A Appendices

A.1 Style factor definitions

A.1.1 Foreign exchange (FX) style factors

Carry	The FX Carry strategy has historically benefited from the tendency of FX forwards of high-yielding currencies to overestimate the actual depreciation of future FX spot. On a monthly basis, the strategy evaluates the implied carry rate (FX forwards versus FX spot) of a number of currencies (G10 and EM) against the USD, and ranks them based on that measure. The strategy goes long on single-currency indices (which roll FX forwards) for the currencies with the highest carry, and short on single-currency indices for the currencies with the lowest carry.
Value	The FX Valuation strategy relies on exchange rates reverting back to their fair value over medium- to long-term horizons. On a monthly basis, the strategy evaluates the valuation measure (based on GS DEER, Dynamic Equilibrium Exchange Rate model) of a number of currencies (G10 and EM) against the USD, and ranks them based on that measure. The strategy goes long on single-currency indices (which roll FX forwards) for the highest ranking currencies (i.e., most undervalued), and short on single-currency indices for the lowest ranking currencies (i.e., most overvalued).
Momentum	The Momentum factor capitalizes on the persistence of trends in forward exchange rate movements that are driven by both carry and spot movements. On a daily basis, the strategy evaluates the recent performance of twenty-seven currencies against the USD. It then takes either a long or short position on each against the USD, depending on whether actual performance has been positive or negative.

A.1.2 Commodity style factors

Carry	Carry captures the tendency of commodities with tighter timespreads to outperform due to low inventories that drive both backwardated futures curves and price appreciation, and to buy demand from consumer hedgers for protection against price spikes in undersupplied commodities The strategy goes long on the top third and short on the bottom third of the twenty-four commodities from the S&P GSCI universe, ranked by annualized strength of front month time spreads. The strategy is rebalanced daily based on signals over the last ten days. The strategy is net of cost.
Value	Value uses the weekly Commodity Futures Trading Commission (CFTC) positioning data to determine long and short positions in commodities. It will take long positions in commodities where the speculative positions are the most short, and short positions in commodities where the speculative positions are the most long.
Momentum	Momentum in commodity returns reflects initial underreactions, or subsequent overreactions, to changes in demand. Increases or decreases in supply can take many years to implement, and may subsequently overshoot the required changes to match demand. The strategy goes long on the top third and short on the bottom third of the twenty-four commodities from the S&P GSCI universe, ranked by rolling one-year excess returns of each commodity. The strategy is rebalanced daily based on signals over the last ten days. The strategy is net of cost.

Quality	Quality captures the tendency for deferred futures contracts to outperform nearer-dated contracts, due to producers hedging further out than consumers, and to passive investors investing near the front of the curve. The strategy goes long on selected points on the curve of each commodity, equally weighted among commodities. The strategy goes short on an equally weighted basket of the nearest commodity contracts, which are beta-adjusted at the basket level.

A.1.3 Rates style factors

Momentum	The Momentum factor capitalizes on the persistence of trends in short- and long-term interest rate movements. On a daily basis, the strategy evaluates the recent performance of a number of futures contracts for the U.S., Germany, Japan, and the U.K. It then takes either a long or short position on each, depending on whether actual performance has been positive or negative.
Quality	The Quality factor capitalizes on the observation that risk-adjusted returns at the short end of the curve tend to be higher than those at the long end. A leveraged long position on the former versus the latter tends to capture positive excess returns as compensation for the risk premium that stems from investors having leverage constraints and favoring long-term rates. The interest rates curve strategy enters a long position on five-year U.S. bond futures, and a short position on thirty-year bond futures, as well as a long position on five-year German bond futures and a short position on ten-year German bond futures, rolling every quarter. The exposure to each future is adjusted to approximate a duration-neutral position.

A.1.4 Equity style factors

Value	The Value factor refers to the finding that value stocks characterized by attractive valuation ratios offer higher long-run average returns than growth stocks characterized by high valuation ratios. To optimally combine factors in a multi-factor asset allocation exercise, we construct the factor so it has zero exposure to other factors, and does not change the exposure to other factors when added to the portfolio.
Momentum	The Momentum factor captures a medium-term continuation effect in returns by buying recent winners and selling recent losers. The factor combines price as well as earnings momentum information. To optimally combine factors in a multi-factor asset allocation exercise, we construct the factor so it has zero exposure to other factors, and does not change the exposure to other factors when added to the portfolio.
Quality	The Quality factor combines different measures of determining financial health and operating profitability. To optimally combine factors in a multi-factor asset allocation exercise, we construct the factor so it has zero exposure to other factors, and does not change the exposure to other factors when added to the portfolio.
Defensive	The Defensive factor refers to the finding that low-volatility stocks tend to outperform high-volatility stocks on a risk-adjusted basis. To capture this behavior, the factor is constructed to go long on a minimum-variance portfolio, and short on the beta-portion of the market.

All style factors are constructed in a long-short fashion and all non-equity style factors are sourced from Goldman Sachs (GS); see the subsequent table the style factor indices used. For the equity style factors equity value, momentum and quality, each follow a multi-factor approach that combines several metrics proxying for the respective style dimension. For equity defensive, we build on a long-short approach that is long a minimum-volatility portfolio while shorting a beta-adjusted market portfolio.

Overview of style factor series. This table shows the style factor series used throughout the paper.

Style factor	Equity	Fixed Income	Commodity	FX
Carry	–	GS Interest Rate Carry 05	GS Macro Carry Index RP14	GS FX Carry C0115
Value	Value	GS Interest Rate Value 05	GS Commodity COT Strategy COT3	GS FX Value C0114
Momentum	Momentum	GS Interest Rates Trend	GS Macro Momentum Index RP15	GS FX Trend C0038
Quality	Quality & Low Volatility	GS Interest Rates Curve C0210	GS Commodity Curve RP09	–

A.2 Derivation of optimal DRP weights

The kth minimum torsion factor has weight:

$$w_{MT,k}^{DRP} = \frac{1/\sigma_{MT,k}}{\sum_{k=1}^{K} 1/\sigma_{MT,k}} \tag{18}$$

Expressing the above in terms of single assets and single style factors (and not minimum torsion factors) requires accounting for the factors' torsion $\mathbf{t}_{MT}$, and the associated mapping $\mathbf{B}$ of single assets and single style factors on the original aggregate factors. In particular, we have:

$$\mathbf{w}^{DRP_B} = \frac{\mathbf{B}^{-1}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}}{\sum_{i=1}^{N}(\mathbf{B}^{-1}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP})_i} \tag{19}$$

where $\mathbf{B}^{-1}$ is the Moore-Penrose inverse of the factor model sensitivities. Intuitively, to back out the physical weights, we re-rotate the orthogonalized factors and reverse the mapping as given by the linear risk model coefficients $\mathbf{B}$.

The stability of weights (19) crucially depends on the statistical fit of the chosen parsimonious risk model with respect to each underlying single asset or style factor. Given that the seven aggregate factors are constructed from the underlying single assets and style factors, one can consider a natural alternative to determining a diversified risk parity allocation. Specifically, we simply invest into the minimum torsion factors based on the mapping implicit in the risk parity scheme used to form the aggregate factors. Collecting the risk parity weights for the seven factors in $\mathbf{w}_{ERC}^{\mathbf{F}}$, we can multiply these by the re-rotated inverse-volatility weights in the seven aggregate factors, as follows:

$$\mathbf{w}^{DRP_F} = \frac{\mathbf{w}_{ERC}^{\mathbf{F}}\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP}}{\sum_{k=1}^{K}(\mathbf{t}'_{MT}\mathbf{w}_{MT}^{DRP})_k} \tag{20}$$

A.3 Risk-based allocation methodologies

A.3.1 1/N

The $1/N$ strategy rebalances monthly to an equally weighted allocation scheme. Hence, for K factors, the portfolio weights $\mathbf{w}_{1/N}$ are:

$$\mathbf{w}_{1/N} = \frac{\mathbf{1}}{K} \tag{21}$$

A.3.2 Minimum-Variance

The minimum-variance (MV) portfolio weights $\mathbf{w}_{MV}$ derive from:

$$\mathbf{w}_{MV} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathbf{w}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{w} \tag{22}$$

subject to the full investment and positivity constraints, $\mathbf{w}'\mathbf{1} = 1$ and $\mathbf{w} \geq \mathbf{0}$.

A.3.3 Risk Parity

We construct the original risk parity (RP) strategy by allocating capital so that the seven aggregate factors' risk budgets contribute equally to overall portfolio risk. Since there are no closed-form solutions available, we follow Maillard, Roncalli, and Teiletche (2010) to obtain $\mathbf{w}_{RP}$ numerically via:

$$\mathbf{w}_{RP} = \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i=1}^{K}\sum_{j=1}^{K} \left(w_i(\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{w})_i - w_j(\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{w})_j\right)^2 \tag{23}$$

which essentially minimizes the variance of the factors' risk contributions. Again, the full investment and positivity constraints apply.

A.3.4 Most-Diversified Portfolio of Choueifaty and Coignard (2008)

To build maximum diversification portfolios, Choueifaty and Coignard (2008) define a portfolio diversification ratio $D(\mathbf{w})$ as follows:

$$D(\mathbf{w}) = \frac{\mathbf{w}' \cdot \boldsymbol{\sigma}_F}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{w}}} \tag{24}$$

where $\boldsymbol{\sigma}_F$ is the vector of aggregate factor return volatilities. Therefore, the most-diversified portfolio (MDP) simply maximizes the ratio between two distinct definitions of volatility, i.e., the ratio between the average portfolio factors' volatility and total portfolio volatility. We obtain the MDPs' weight vector $\mathbf{w}_{MDP}$ by numerically computing:

$$\mathbf{w}_{MDP} = \underset{\mathbf{w}}{\operatorname{argmax}}\, D(\mathbf{w}) \tag{25}$$

As before, we enforce the full investment and positivity constraints.

Table 1: Descriptive statistics of market, style, and aggregate factors. This table shows performance statistics for the market, style, and aggregate factors. Returns are annualized excess returns, and are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. Min and Max denote the lowest and highest monthly excess return during the sample period. MaxDD is the maximum drawdown. Return, Volatility, Min, Max and MaxDD are in percentage terms. The time period is from January 2001 – October 2018.

		Return	Vola	SR	t-stat	Min	Max	MaxDD
Panel A: Market factors								
SP500	*SP500*	5.04	14.43	0.35	1.47	-17.25	10.95	-53.41
EuroSTOXX50	*E50*	2.47	18.49	0.13	0.56	-18.52	16.81	-58.65
FTSE100	*FTSE*	3.31	13.86	0.24	1.01	-13.38	8.80	-47.00
Nikkei225	*Nikkei*	5.70	19.29	0.30	1.25	-25.35	12.86	-57.96
MSCI EM	*EM*	7.59	21.55	0.35	1.48	-27.50	16.66	-62.67
Credit.IG	*IG*	0.86	4.25	0.20	0.85	-7.27	4.72	-23.68
Credit.HY	*HY*	4.23	10.47	0.40	1.70	-15.81	13.49	-43.43
US10Y	*US10*	3.75	5.88	0.64	2.68	-5.81	8.68	-7.70
Bund	*Bund*	4.32	5.15	0.84	3.54	-2.90	4.78	-9.11
JGB10Y	*JGB10*	1.96	2.51	0.78	3.29	-3.14	2.26	-4.79
Gilt	*Gilt*	3.23	6.01	0.54	2.26	-4.89	5.69	-9.88
Gold	*Gold*	9.39	17.17	0.55	2.30	-18.40	13.77	-43.39
Oil	*Oil*	1.25	31.13	0.04	0.17	-32.43	27.47	-92.40
Copper	*Copper*	10.57	27.13	0.39	1.64	-36.42	31.81	-63.64
Agriculture	*Agri*	1.08	20.44	0.05	0.22	-18.97	16.21	-56.38
Panel B: Style factors								
Equity.Quality	*EQ.Q*	3.04	3.49	0.87	3.66	-1.93	4.20	-8.78
Equity.Defensive	*EQ.D*	4.00	5.12	0.78	3.30	-4.39	3.67	-11.64
Equity.Value	*EQ.V*	3.16	4.00	0.79	3.33	-2.83	4.20	-9.61
Equity.Momentum	*EQ.M*	2.77	5.60	0.49	2.08	-6.45	5.14	-24.08
FX.Carry	*FX.C*	4.94	6.44	0.77	3.23	-8.11	7.20	-15.22
FX.Value	*FX.V*	2.22	4.51	0.49	2.07	-6.25	5.06	-9.29
FX.Momentum	*FX.M*	4.89	4.95	0.99	4.16	-3.24	5.60	-11.85
Cmdty.Carry	*CM.C*	5.77	8.05	0.72	3.02	-5.79	7.41	-14.72
Cmdty.Quality	*CM.Q*	4.53	2.80	1.61	6.80	-1.55	2.93	-4.44
Cmdty.Momentum	*CM.M*	2.33	9.45	0.25	1.04	-7.80	9.53	-22.86
Cmdty.Value	*CM.V*	5.56	7.46	0.75	3.14	-5.55	6.94	-10.36
Rates.Value	*FI.V*	0.89	3.39	0.26	1.10	-3.93	3.28	-7.05
Rates.Momentum	*FI.M*	4.19	4.94	0.85	3.57	-4.26	5.29	-7.20
Rates.Quality	*FI.Q*	1.32	3.05	0.43	1.82	-2.22	3.85	-9.87
Rates.Carry	*FI.C*	2.66	3.29	0.81	3.40	-3.31	3.61	-6.33
Panel C: Aggregate factors								
Equity-Credit		3.13	9.36	0.33	1.41	-13.25	9.15	-38.37
Duration		2.90	3.36	0.86	3.63	-2.70	3.72	-5.23
Commodity		5.82	15.60	0.37	1.57	-24.00	13.05	-48.49
Carry		3.98	3.11	1.28	5.38	-3.20	3.52	-6.01
Value		2.56	2.53	1.02	4.28	-2.47	2.70	-4.86
Momentum		3.74	3.47	1.08	4.55	-2.33	3.77	-5.93
Quality		3.23	1.64	1.98	8.33	-0.85	1.67	-1.96

Table 2: Minimum torsion factor loadings to original factors. This table shows the loadings of uncorrelated risk sources w.r.t. to original factors as represented by the minimum torsion matrix t_{MT}. The time period is January 2001 – October 2018.

Original Factors	MTF1 *Equity*	MTF2 *Duration*	MTF3 *Commodity*	MTF4 *Carry*	MTF5 *Value*	MTF6 *Momentum*	MTF7 *Quality*
Equity	**1.11**	0.09	-0.13	-0.10	-0.02	0.08	-0.00
Duration	*0.40*	**1.21**	0.16	-0.28	0.24	-0.17	-0.12
Commodity	-0.03	0.01	**1.10**	0.02	0.00	0.05	-0.02
Carry	*-0.34*	-0.20	0.26	**1.11**	-0.17	-0.04	0.01
Value	-0.13	0.28	0.04	-0.27	**1.11**	-0.13	-0.07
Momentum	*0.44*	-0.20	**1.09**	-0.06	-0.13	**1.15**	-0.04
Quality	0.03	*0.58*	**-1.98**	0.09	*-0.32*	-0.15	**1.11**

Table 3: Descriptive statistics of risk-based allocation techniques. This table shows performance statistics for the risk-based allocation techniques. DRP shows diversified risk parity performance, RP shows the risk-parity strategy, MVP mean-variance and MDP apply the maximum diversification approach of Choueifaty and Coignard (2008). Annualized excess returns are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. MaxDD is maximum drawdown. Number of bets denotes the effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). Turnover is calculated as two-way turnover. Return, Volatility, MaxDD, CVaR, and Turnover are in percentage terms. The time period is January 2006 – October 2018.

	DRP	RP	$1/N$	MVP	MDP
Net Return	3.28	3.27	3.53	2.94	3.23
Volatility	1.45	1.48	3.49	1.31	1.57
Sharpe Ratio	1.24	1.23	0.61	1.15	1.14
MaxDD	-1.92	-1.79	-9.35	-1.04	-2.81
Calmar Ratio	1.71	1.83	0.38	2.82	1.15
CVaR	2.12	2.12	6.15	1.86	2.30
Number of Bets	7.00	6.94	3.11	4.07	6.20
Turnover	3.66	2.97	1.53	5.98	6.94

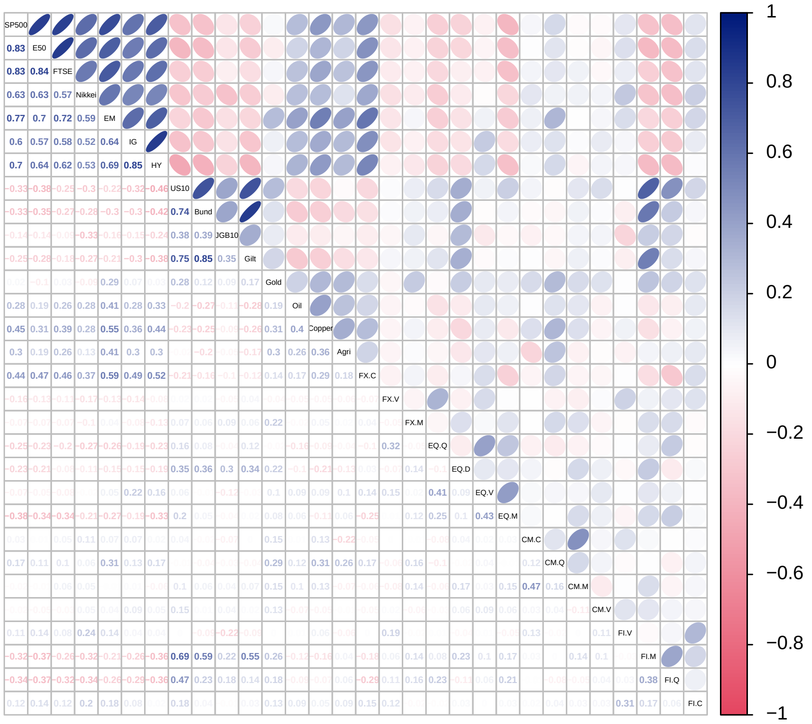
Table 4: Descriptive statistics of factor completion strategies. This table shows performance statistics for the factor completion strategies. Annualized excess returns are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. MaxDD is maximum drawdown. Number of bets denotes the effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). Turnover is calculated as two-way turnover. Return, Volatility, MaxDD, CVaR, and Turnover are in percentage terms. The time period is January 2006 – October 2018.

	BM	Tail hedge	Constrained	Pure	Trend
Net Return	5.41	7.49	8.87	7.74	10.35
Volatility	5.60	5.85	6.44	5.63	5.40
Sharpe Ratio	0.72	1.03	1.14	1.11	1.60
MaxDD	-17.18	-15.15	-15.60	-10.00	-7.60
Calmar Ratio	0.31	0.49	0.57	0.77	1.36
CVaR	9.95	9.88	10.72	9.37	8.21
Number of Bets	3.07	4.12	6.00	6.95	6.76
Turnover	0.00	11.31	18.94	26.22	36.46

Table 5: Descriptive statistics of trend style and style factor momentum. This table shows performance statistics for trend style and style factor momentum. Annualized excess returns are calculated using the arithmetic average of simple returns. The standard deviation and Sharpe ratio are annualized through multiplication by $\sqrt{12}$. MaxDD is maximum drawdown. Number of bets denotes the effective number of uncorrelated bets ($\mathcal{N}_{Ent}$). Turnover is calculated as two-way turnover. Return, Volatility, MaxDD, CVaR, and Turnover are in percentage terms. The time period is January 2006 – October 2018.

		Market factor trend				Style factor momentum			
	Pure	+Equity	+Credit	+Duration	+Cmdty	+FX	+Equity	+Cmdty	+Rates
Net Return	7.74	9.04	9.41	9.86	10.35	10.86	11.36	12.24	13.13
Volatility	5.63	5.34	5.45	5.44	5.40	5.54	5.48	5.69	6.02
Sharpe Ratio	1.11	1.39	1.43	1.50	1.60	1.64	1.74	1.82	1.86
MDD	-10.00	-7.77	-8.83	-8.71	-7.60	-8.36	-5.52	-5.41	-7.14
Calmar Ratio	0.77	1.17	1.07	1.13	1.36	1.65	2.06	2.26	1.84
CVaR	9.37	8.44	8.55	8.41	8.21	8.36	8.11	8.31	8.74
Number Of Bets	6.95	6.94	6.89	6.83	6.76	6.71	6.60	6.55	6.38
Turnover	26.22	25.71	28.09	33.40	36.46	40.99	41.74	42.43	43.07

Figure 1: Correlation Matrix. The upper chart depicts the correlation structure of the thirty market and style factors, while the lower chart shows the correlation of the seven aggregate factors. The time period is January 2001 – October 2018. Colors range from dark red (correlation of -1) to dark blue (correlation of 1).



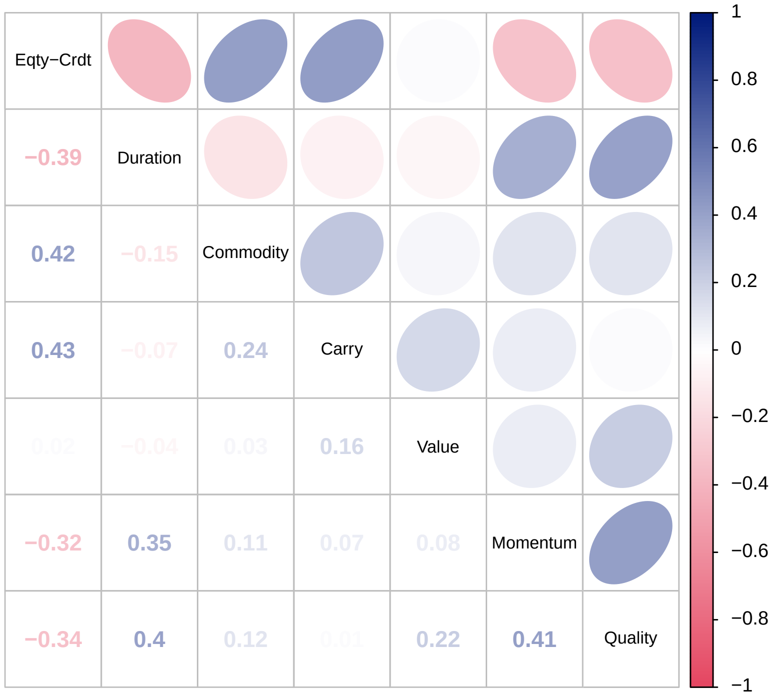


Figure 2: Weights and Risk Decompositions: Risk-Based Strategies. This figure depicts the decomposition of the risk-based allocation strategies in terms of single asset and factors weights (left column), aggregate factor allocation weights (middle column), and risk (right column), which is decomposed by minimum torsion factors. The results build on rolling window estimations over sixty months. The sample period is January 2006 – October 2018.

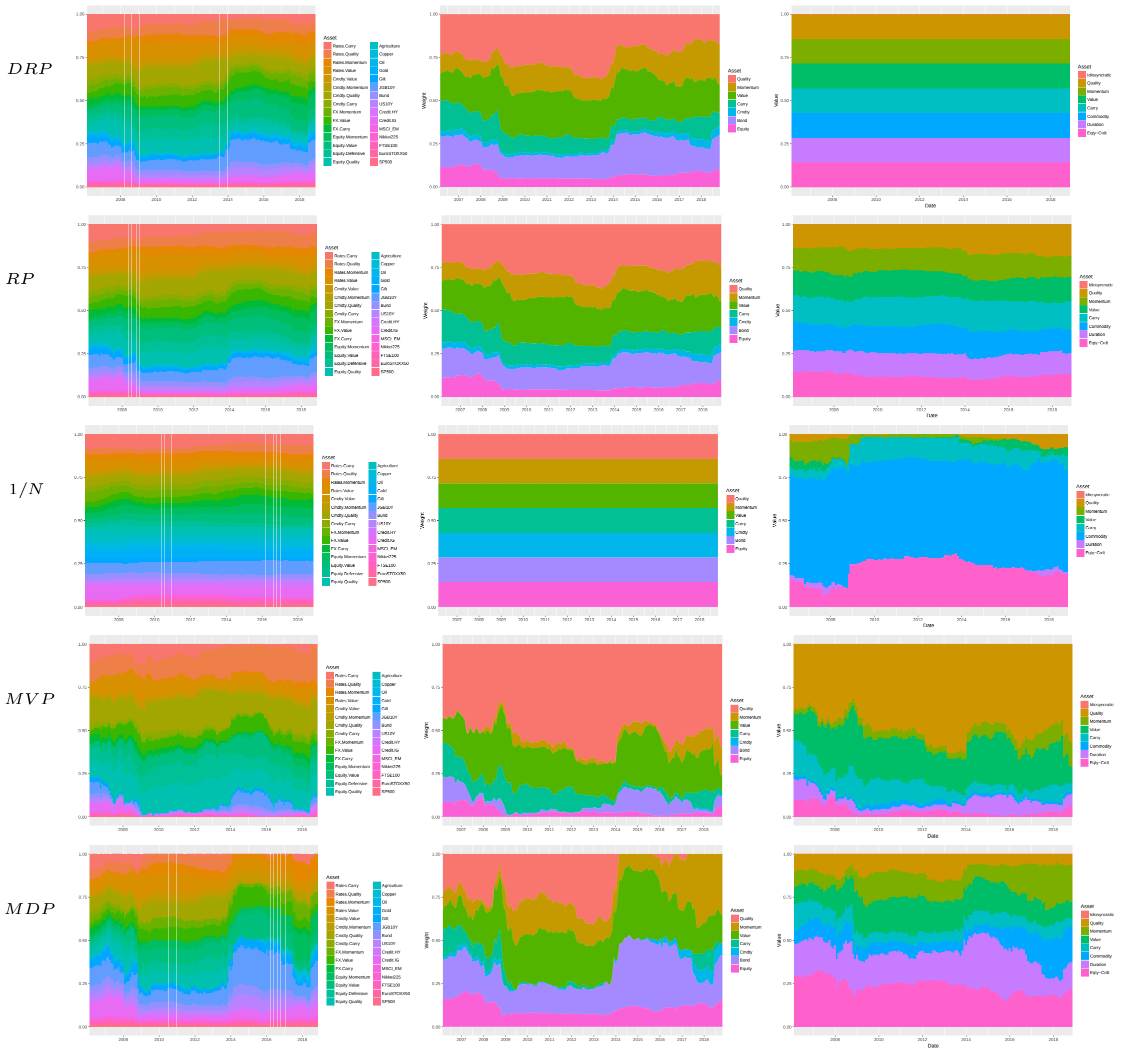


Figure 3: Effective Number of Uncorrelated Bets. We plot the number of uncorrelated bets for the risk-based asset allocation strategies when using rolling window estimation of sixty months. The sample period is January 2006 – October 2018.

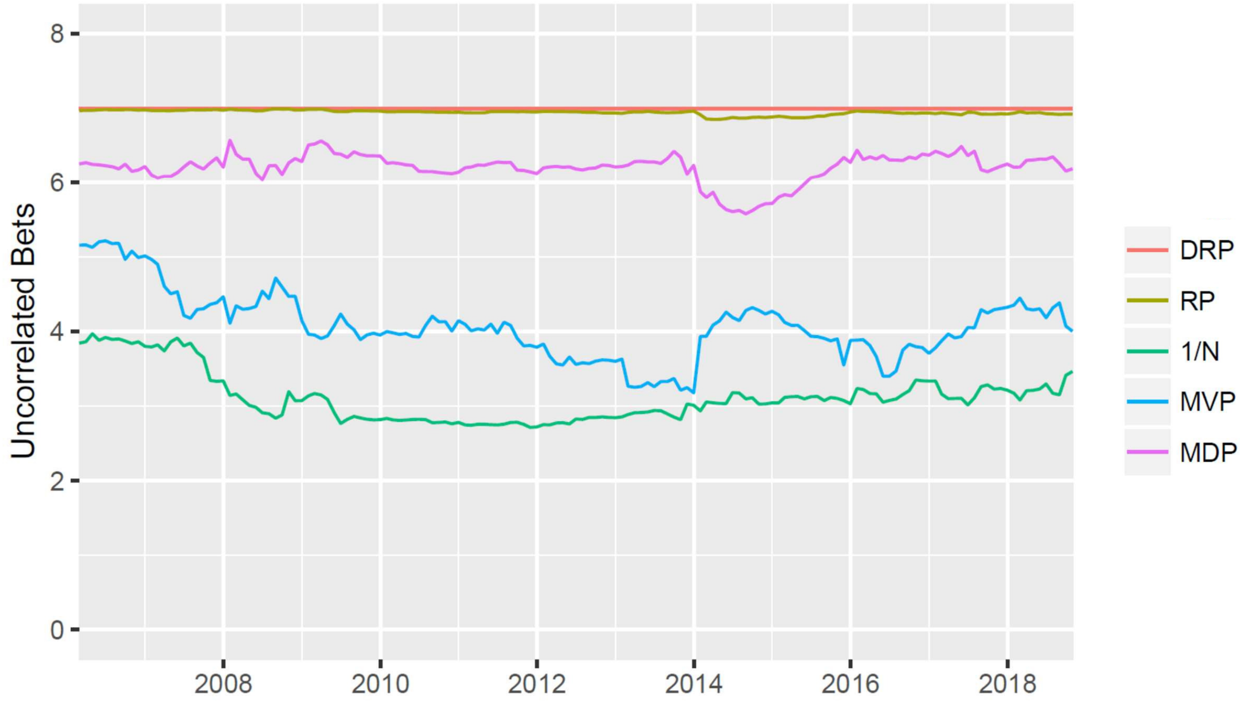


Figure 4: Weights and Risk Decompositions: Active factor completion strategies. This figure depicts the decomposition of the active factor completion strategies in terms of single asset and factor weights (left column), aggregate factor allocation weights (middle column), and risk (right column). which is being decomposed by minimum torsion factors. The results build on rolling window estimation over sixty months. The sample period is January 2006 – October 2018.

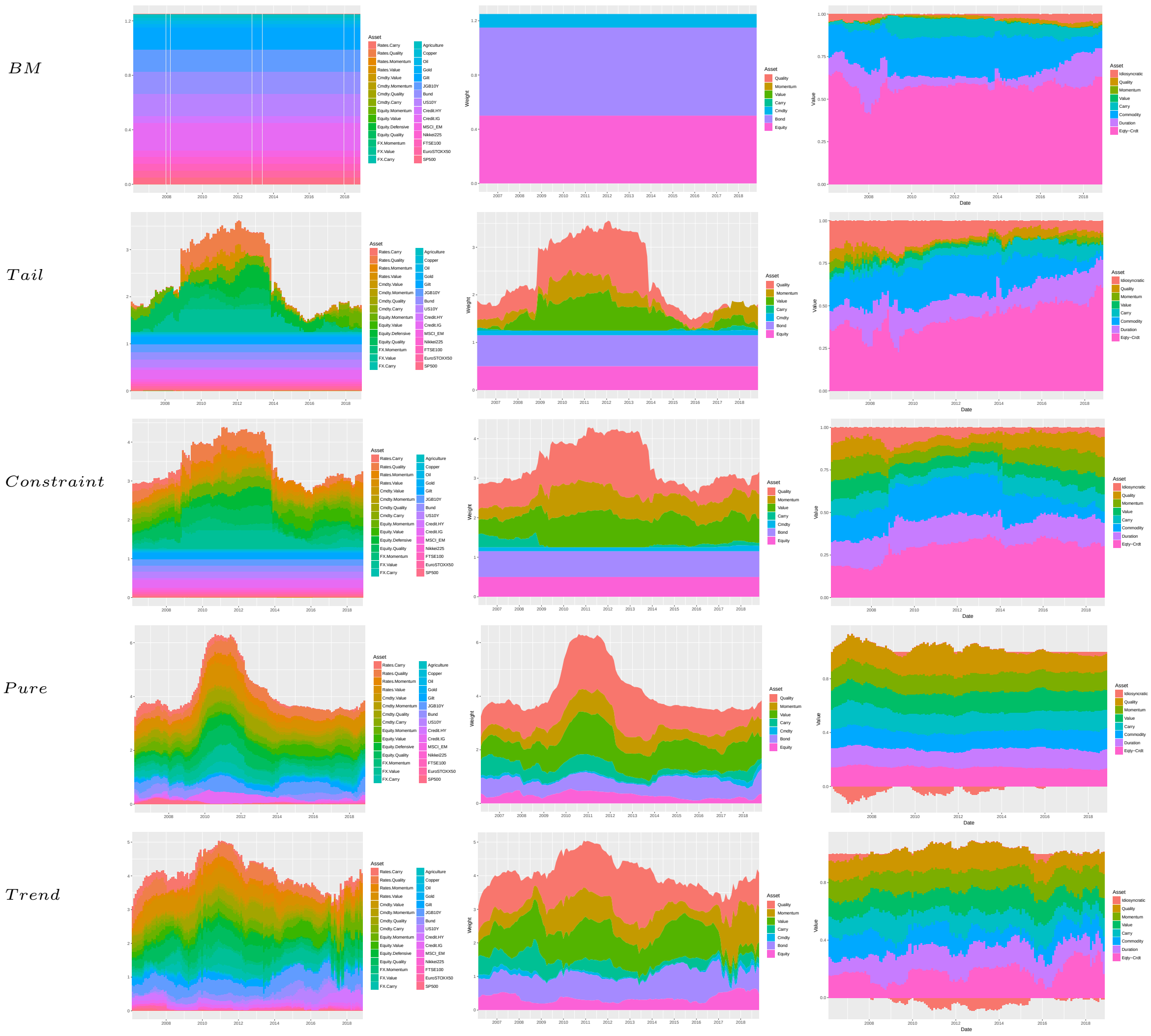


Figure 5: Effective Number of Uncorrelated Bets. The upper chart depicts the number of uncorrelated bets for the active factor completion strategies when using rolling window estimations of sixty months. The lower chart compares the effective number of bets when choosing different τ, where τ ranges from zero (Pure) to 15% (tau15). The sample period is January 2006 – October 2018.