THE INTERNATIONAL ACTIVE FUND MANAGEMENT INDUSTRY: CONCENTRATION CROSS EFFECTS

David Feldman,[1] Konark Saxena[2] and Jingrui Xu[3]

Latest Revision January 31, 2019

Abstract. We introduce a theoretical model of international active fund management industries, where performance and size depend on competitiveness (concentration). After Pastor and Stambaugh's (2012) and Feldman, Saxena and Xu's (2019) single-market studies, we study concentration cross effects in domestic/foreign active fund management industries (DAFMI/FAFMI), where competing managers select efforts and fees. In equilibrium, performance, size and "direct benefits" increase/decrease together. Higher FAFMI concentration improves DAFMI, if and only if DAFMI gains from higher alpha production exceed higher effort costs of producing it. Empirically, we find that 30 global DAFMIs' net alphas and sizes, on average, decrease with U.S. FAFMI concentration.

JEL Codes: G10, G15, G20, L10

Keywords: Active management, Mutual funds, Global Fund Markets, Effort, Performance, Market concentration, Competition, Herfindahl-Hirschman index, Industry size, Alpha

---

[1] Banking and Finance, UNSW Business School, UNSW Sydney, NSW 2052, Australia. Email: d.feldman@unsw.edu.au

[2] Banking and Finance, UNSW Business School, UNSW Sydney, NSW 2052, Australia. Email: k.saxena@unsw.edu.au

[3] Wang Yanan Institute for Studies in Economics (WISE) and Department of Finance, School of Economics, Xiamen University, China. Email: jingrui.xu@xmu.edu.cn

1 Introduction

Recent studies have shown that active funds markets' competition (concentration) affects performance, size, fees, and other characteristics [see, for example, Pastor and Stambaugh (2012) (PS), and Feldman, Saxena and Xu (2019) (FSX)]. Here, we study these questions in an international context: How, and by which mechanisms, do a foreign active fund management industry (FAFMI) concentration levels[4] affect a domestic active fund management industry (DAFMI)? We introduce a model of DAFMI/FAFMI equilibrium with endogenous performance, size, fees, and managerial effort under a continuum of DAFMI and FAFMI concentrations.

For simplicity, we consider a two-country international model. Each country has an active fund management industry (AFMI) with competing fund managers who invest their portfolios in both domestic and foreign stocks and with infinitely many mean-variance risk-averse investors who allocate their wealth across a passive international benchmark portfolio (which includes both domestic and foreign stocks) and domestic active funds. This framework corresponds to the real-world transaction and information costs advantage that funds have over individuals when investing in foreign stock markets.[5] We assume decreasing returns to scale in producing gross alphas at the industry level, as in PS and FSX, and at fund levels, as in Berk and Green (2004) and FSX.

Fund managers, competing in net (of management fees) alpha productions, spend two types of efforts exploring investment opportunities, one targeting the domestic stock market, the other the foreign market. Gross alpha production and managerial effort costs depend on concentrations. In particular, higher FAFMI concentration implies more unexplored investment opportunities in the foreign stock market, making effort spent in FAFMI more productive. Moreover, higher FAFMI concentration diverts managerial effort to FAFMI, as it has more unexplored investment opportunities, thus leaving more unexplored opportunities in DAFMI and making effort spent in DAFMI more productive. By symmetry, higher DAFMI concentration induces similar effects.

As in FSX, we set the numbers of funds, both in DAFMI and FAFMI, and define DAFMI and FAFMI sizes as the ratio of their assets under active fund management to total wealth. We use the term *direct benefits* for either DAFMI or FAFMI, also as in FSX, the sum,

[4] For brevity we use the term "concentration" to stand for "concentration levels."
[5] As we argue in Section 2, a case in which both countries' investors and active funds invest in both countries falls under the model that FSX solves.

over domestic and foreign stock markets, of gross alphas minus the sum of effort costs to produce it. We show that, in equilibrium, if and only if higher FAFMI concentration induces higher (lower) DAFMI direct benefits, then it induces higher (lower) DAFMI fund expected net alphas and size. By symmetry, a similar necessary and sufficient condition holds for higher DAFMI concentration.

We also provide second-order relations. In equilibrium, concave DAFMI expected net alphas in FAFMI concentration imply concave DAFMI direct benefits in FAFMI concentration. In turn, concave DAFMI direct benefits in FAFMI concentration imply concave DAFMI size in FAFMI concentration. On the other hand, equilibrium convex DAFMI size in FAFMI concentration implies convex direct benefits in FAFMI concentration and, consequently, convex DAFMI fund expected net alphas in FAFMI concentration.

We specialize our model to allow endogenous concentrations. We show that in equilibrium, although the relation between DAFMI concentration and DAFMI expected net alphas and the relation between DAFMI concentration and DAFMI size, are, in this case, more complex, we can still conclude that DAFMI fund expected net alphas and DAFMI size move in the same direction as FAFMI concentration. We believe that this endogenous concentration framework befits empirical concentration measures, which measure the relative fund size distribution in industries with a given number of funds.

Using the Normalized-Herfindahl-Hirschman Index (NHHI) and other indices as concentration measures, we perform empirical tests. We study 30 active global equity AFMIs, which we consider here as DAFMIs, and analyze their fund net alphas and size associations with the domestic and U.S. equity AFMI concentrations, which we consider here as FAFMI. We find that, pooling all the markets' data together, DAFMI fund net alphas and sizes are, on average, both significantly negatively associated with the U.S. NHHI (FAFMI NHHI). Consistent with our theoretical results, DAFMI net alphas and sizes move, on average, in the same direction in response to changes in FAFMI concentration.

We also empirically study all 30 individual pairs of the 30 DAFMIs with the FAMFI (U.S.). We find that six (one) DAFMI markets' fund net alphas and size, on average, are both significantly negatively (positively) associated with the FAFMI NHHI, whereas seven DAFMI markets' fund net alphas and size are both insignificantly associated with the FAFMI NHHI. On the other hand, on average, only one (one) DAFMI market's fund net alphas is significantly positively (negatively) associated with the FAMFI NHHI, but its size is significantly negatively (positively) associated with the FAMFI NHHI. These results show that, in general, DAFMI markets' fund net alphas and size are more likely to move in the same direction as FAMFI's NHHI than to move in other directions. This finding is consistent with the prediction of our theoretical model under both the exogenous concentration framework and the endogenous concentration framework.

We use Pastor, Stambaugh, and Taylor's (2015) (PST) recursive-demeaning estimator to address endogeneity and omitted-variable-related issues when studying the FAFMI concentration–DAFMI net alpha relations. In studying the FAFMI concentration–DAFMI size relation, we use vector auto-regression (VAR) techniques to account for simultaneity in determination of DAFMI size and FAFMI concentration in our robustness checks and find consistent results with those that do not use the VAR techniques. We control for survival bias by using Morningstar Direct's global database, which contains both surviving and terminated funds. Our empirical results are robust to the use of alternative methods and concentration measures.

Our findings provide implications for fund managers, investors, and regulators. The current low and decreasing concentration in the U.S. AFMI, under current U.S. AFMI parameters, benefit (harm) global DAFMIs whose fund net alphas and size are, on average, negatively (positively) associated with the U.S. AFMI concentration. Our results show that a large proportion of the global DAFMIs in our sample benefit from the decreasing U.S. AFMI concentration.

Current international studies report how a fund market's size, managerial fees, fund performance, flow-performance relationship, and portfolio choice differ with fund markets' fundamental characteristics, such as regulation, transaction costs, stock market developments, and sophistication of investors [see, for example, Khorana, Servaes and Tufano (2005), Khorana, Servaes and Tufano (2008), Ferreira, Keswani, Miguel and Ramos (2012a) and (2012b), and Chan, Covrig, and Ng (2005)]. This study complements the literature by demonstrating how foreign fund markets' characteristics affect domestic fund markets.

Some international studies analyze how investment activities in one country facilitate transmission of shocks to other countries, consequently influencing portfolio returns [see, for example, Jotikasthira, Lundblad and Ramadorai (2012) and Goldstein and Pauzner (2004)]. Other international studies analyze how regulation in one country affects funds' investments in other countries [see, for example, Defond, Hu, Hung and Li (2011) and Yu and Wahid (2014)]. Similar to these studies, which look at the effects of a country's information shocks and regulation changes on other countries, we study here the cross effects of changes in AFMI concentrations in a country on other countries' AFMIs.

Section 2 develops the theoretical model, Section 3 presents the empirical methods and results, and Section 4 concludes.

2 Theoretical Framework

We develop a theoretical framework for modeling the effects of DAFMI and FAFMI concentrations on DAFMI managerial efforts, fees, performance, size, and direct benefits. For simplicity, we consider a two-country international model, where each country has an AFMI with competing fund managers who invest in stocks and infinitely many mean-variance risk-averse investors who allocate their wealth to a passive international benchmark portfolio and active funds.

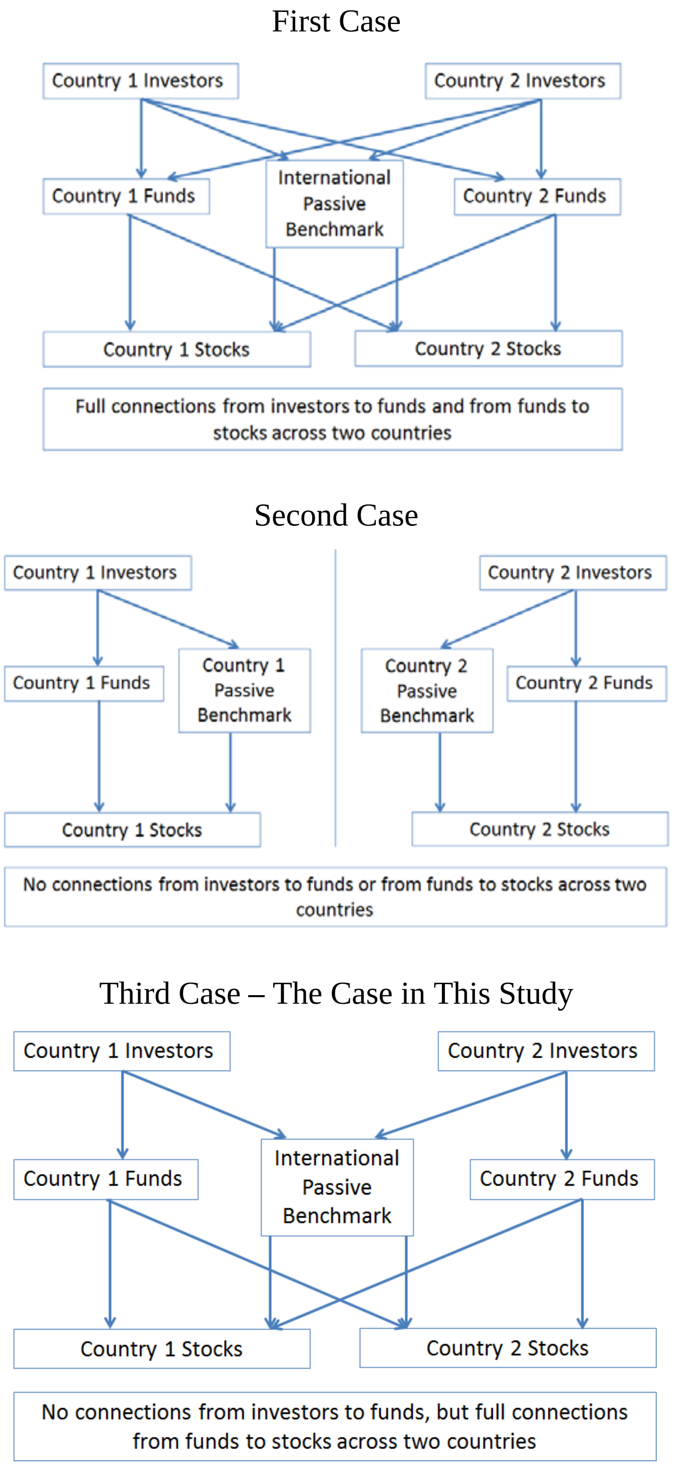
Consider three possibilities of investment across the two countries. If each country's investors and AFMI managers invest in both countries' AFMIs and stocks, respectively, we may consider the two countries as one "global village," with one type of investor and one AFMI. FSX studies this case.

If each country's investors and AFMI managers can invest only in their own country, we have two separate AFMIs. Each country's AFMI is, again, modeled in FSX.

If, however, due to transaction and information costs, each country's investors invest only in DAFMI, whereas fund managers, facing lower transaction and information costs, invest in both countries' stocks, a new DAFMI/FAFMI model is required. We introduce this model here. In this case, fund managers compete domestically for wealth to manage, but both DAFMI and FAFMI concentrations affect gross alpha production and effort costs. Figure 1 illustrates these three cross-country investment possibilities.

Figure 1. Three Cases of a Two-Country Model

This figure shows the three cases of two-country models. In the first case, investors invest in both countries' AFMIs and managers invest in both countries' stocks. The two countries can be regarded as one AFMI "global village." In the second case, investors can invest only in DAFMI and fund managers can only invest in domestic stocks. The two countries' AFMIs are separate. In the third case, investors invest only in DAFMI, whereas fund managers invest in stocks of both countries. Each country's fund managers compete for domestic investments, but both DAFMI and FAFMI concentration-levels affect gross alpha production and effort costs in each country's AFMI.



2.1 Setting

The economy consists of two countries, Country 1 and Country 2, and one period. We denote each Country k's, $k = 1{,}2$, parameters by superscripts. For simplicity and without loss of generality, we assume the countries' currency exchange rate is one. Country k has two types of agents: M^k, $M^k > 1$, active fund managers, and N^k, $N^k \to \infty$, infinitely many investors. Fund managers in both countries are risk-neutral, invest in both countries' stocks, and maximize fund profits by optimally choosing proportional management fees and effort levels. Mean-variance risk-averse investors in both countries allocate their wealth between a passive international (including both domestic and foreign stocks) benchmark portfolio and domestic active funds (DAFMI), maximizing their portfolios' Sharpe ratios. All investor are small; thus, individual investors' do not affect fund sizes.

Due to the economy's internal symmetry with respect to Country 1 and 2, it is sufficient to focus on one country only. We denote Country 1 (2) as domestic (foreign), and its AFMI as DAFMI (FAFMI).

Fund Managers' Problem

Manager i in Country 1 maximizes her economic profits by allocating effort levels in each country and fee rates, subject to nonnegative allocations and nonnegative profit rate. Mathematically,

$$\max_{e_i^{11}, e_i^{12}, f_i^1} s_i^1 [f_i^1 - C_i^1(e_i^{11}, e_i^{12};\ s_i^1, H^1, H^2)], \tag{1}$$

subject to

$$f_i^1 - C_i^1(e_i^{11}, e_i^{12}; s_i^1, H^1, H^2) \geq 0, \tag{2}$$

$$e_i^{11} \geq 0, \tag{3}$$

$$e_i^{12} \geq 0, \tag{4}$$

$$f_i^1 \geq 0, \tag{5}$$

where s_i^1, f_i^1, e_i^{11}, e_i^{12}, and $C_i^1(e_i^{11}, e_i^{12};\ s_i^1, H^1, H^2)$ represent manager i's fund size, (nonnegative) proportional management fee, (nonnegative) effort level spent in Country 1's stock market, (nonnegative) effort level spent in Country 2's stock market,[6] and average (per

[6] We remind the reader that the first superscript designates the manager's country, and the second superscript designates the country where stock effort was directed.

dollar) cost function, where H^1 and H^2 represent Country 1 and Country 2 AFMI (DAFMI and FAFMI) concentrations, respectively. We define the domain of H^1 and H^2 as $[0,1)$, where $\{0\}$ represents a fully competitive market and $\{1\}$ represents a monopolistic market.[7] Also, inequality (2) shows that manager i's profit rate should be nonnegative to survive.

Following FSX, we assume that the marginal diversification benefits of investing in an additional fund are trivial, such that managers compete for investments over net alphas. Manager i has to maximize her fund expected net alpha given fund size and AFMI concentrations. Thus, as in FSX, we can transform manager i's profit maximizing problem (1) to an equivalent problem of maximizing expected net alpha:

$$\max_{e_i^{11}, e_i^{12}, f_i^1} \mathrm{E}(\alpha_i^1 \middle| D), \tag{6}$$

subject to constraints (2), (3), (4), and (5).

Proof. See the Appendix.

The proof is similar to the one in FSX. Its intuition is as follows. Under competition, funds that offer higher expected net alphas draw (all) investments. The possibility (threat) that other managers increase fund profits by improving expected net alphas, and their fund sizes, forces managers to maximize expected net alphas to "survive." Thus, funds must offer similar expected net alphas, in a unique Nash equilibrium. We note that this aspect of the equilibrium is similar to that in PS, but in addition to their result, we show that it holds also in the case of finite number of managers.

Manager i's average cost function has the following form.[8]

$$\begin{aligned} C_i^1(e_i^{11}, e_i^{12};\ s_i^1, H^1, H^2) = c_0^1 + c_{1,i}^1 s_i^1 + c_2^{11}(e_i^{11};\ H^1, H^2) \\ + c_2^{12}(e_i^{12};\ H^1, H^2), \end{aligned} \tag{7}$$

where c_0^1 and $c_{1,i}^1$ are constants and $c_2^{11}(e_i^{11};\ H^1, H^2)$ and $c_2^{12}(e_i^{12};\ H^1, H^2)$ are costs, conditional on countries' concentrations, due to e_i^{11} and e_i^{12}, efforts directed at Country 1's stock market and Country 2's stock market, respectively. Each fund's operation cost is positive, so $c_0^1 > 0$. Also, we assume decreasing returns to scale at the fund level, so fund average cost

---

[7] The open right boundary of the concentrations' domain implies that managers are competing.

[8] To simplify our model, we assume there is no interaction between effort and size in the average cost function because it is unlikely that fund size affects managers' per dollar efforts. We also assume that there is no interaction between concentration and size in the average cost function because it is unlikely that concentration affects managers' average cost sensitivities to fund size. Nevertheless, even if these interacting effects exist, they tend to be small in comparison to the effects of other terms in the average cost function.

increases with fund size, i.e., $c_{1,i}^{1} > 0$. $c_{2}^{11}(e_i^{11};\ H^1, H^2)$ and $c_{2}^{12}(e_i^{12};\ H^1, H^2)$ have the following functional characteristics:

- nonnegative, i.e., $c_2^{11}(0;\ H^1, H^2) = 0$, $c_2^{12}(0;\ H^1, H^2) = 0$, $\forall\, H^1, H^2$ and $c_2^{11}(e_i^{11};\ H^1, H^2) > 0$, $\forall\, e_i^{11} > 0, H^1, H^2$, and $c_2^{12}(e_i^{12};\ H^1, H^2) > 0$, $\forall\, e_i^{12} > 0, H^1, H^2$;
- increasing convex in effort, as we assume increasing marginal cost for each unit of effort, i.e., $c_{2\ e_i^{11}}^{11}(e_i^{11};\ H^1, H^2) > 0$, $c_{2\ e_i^{11}, e_i^{11}}^{11}(e_i^{11};\ H^1, H^2) > 0$, $\forall\, e_i^{11}, H^1, H^2$, and $c_{2\ e_i^{12}}^{12}(e_i^{12};\ H^1, H^2) > 0$, $c_{2\ e_i^{12}, e_i^{12}}^{12}(e_i^{12};\ H^1, H^2) > 0$, $\forall\, e_i^{12}, H^1, H^2$;
- no cross partial effects of two countries' concentrations on costs of efforts, i.e., $c_{2\ H^1,H^2}^{11}(e_i^{11};\ H^1, H^2) = 0$, $\forall\, e_i^{11}, H^1, H^2$, $c_{2\ H^1,H^2}^{12}(e_i^{12};\ H^1, H^2) = 0$, $\forall\, e_i^{12}, H^1, H^2$. Alternatively, we can argue that these cross partial effects are negligible.

The different $c_{1,i}^{1}$s across DAFMI funds imply differences in DAFMI fund-level decreasing returns to scale parameters, as $c_{1,i}^{1}$'s measure differences in the rate at which managers' costs in generating gross alpha increase with size. We now introduce two terms, an DAFMI individual manager skill and DAFMI aggregate skill.

DAFMI fund manager skill. In our model, ${c_{1,i}^{1}}^{-1}$ is the source of heterogenous manager ability/skill. A more skilled manager is one who has lower total variable costs of active management for the same AUM and gross alpha.

Aggregate DAFMI skill. DAFMI aggregate skill is the sum of individual managers' skills, $\sum_{i=1}^{M^1}\left({c_{1,i}^{1}}^{-1}\right)$. In our model, DAFMI is more skilled when the sum of its mangers' skills is higher.

We show below that higher DAFMI aggregate skill corresponds to higher DAFMI size and that higher individual DAFMI fund manager skill, relative to other managers, corresponds to a higher relative size of their fund. (See Proposition 1 and the discussion following Lemma 1.)

By spending efforts, manager i improves her fund net alpha. Manager i's net alpha has the following form.

$$\alpha_i^1 = a^1 - b^1 \frac{S^1}{W^1} + A^{11}(e_i^{11};\ H^1, H^2) + A^{12}(e_i^{12};\ H^1, H^2) - f_i^1, \tag{8}$$

where a^1 and b^1 are positive constants and b^1 is the industry-level decreasing returns to scale rate, with conditional mean and variance

$$\mathrm{E}\begin{pmatrix} a^1 \\ b^1 \end{pmatrix} D \triangleq \begin{pmatrix} \widehat{a^1} \\ \widehat{b^1} \end{pmatrix}, \quad \mathrm{Var}\begin{pmatrix} a^1 \\ b^1 \end{pmatrix} D \triangleq \begin{pmatrix} \sigma_{a^1}^2 & \sigma_{a^1 b^1} \\ \sigma_{a^1 b^1} & \sigma_{b^1}^2 \end{pmatrix}, \tag{9}$$

where D is investors' information set. Equation (8) is based on the alpha production structure in PS and FSX. The information structure in Definitions (9) follows PS. For simplicity, we assume $\sigma_{a^1 b^1} = 0$. Parameter W^1 is the country's total wealth, and S^1 is DAFMI size (controlled by investors). The gross alpha production functions $A^{11}(e_i^{11};\ H^1, H^2)$ and $A^{12}(e_i^{12};\ H^1, H^2)$ quantify the impact of e_i^{11} and e_i^{12}, respectively. $A^{11}(e_i^{11};\ H^1, H^2)$ and $A^{12}(e_i^{12};\ H^1, H^2)$ have the following functional characteristics:

- nonnegative, i.e., $A^{11}(0;\ H^1, H^2) = 0$, $A^{12}(0;\ H^1, H^2) = 0$, $\forall\, H^1, H^2$ and $A^{11}(e_i^{11};\ H^1, H^2) > 0, \forall\, e_i^{11} > 0, H^1, H^2$, $A^{12}(e_i^{12};\ H^1, H^2) > 0, \forall\, e_i^{12} > 0, H^1, H^2$;
- increasing concave in effort, as we assume marginal productivity of efforts is decreasing, i.e., $A_{e_i^{11}}^{11}(e_i^{11};\ H^1, H^2) > 0$, $A_{e_i^{11}, e_i^{11}}^{11}(e_i^{11};\ H^1, H^2) < 0, \forall\, e_i^{11}, H^1, H^2$ $A_{e_i^{12}}^{12}(e_i^{12};\ H^1, H^2) > 0$, $A_{e_i^{12}, e_i^{12}}^{12}(e_i^{12};\ H^1, H^2) < 0, \forall\, e_i^{12}, H^1, H^2$;
- $A^{11}(e_i^{11};\ H^1, H^2)$ increases with H^1 and has positive cross partial derivative with respect to H^1 and e_i^{11}, as higher H^1 implies more unexplored investment opportunities and higher efficiency in using fund industry resources in Country 1, i.e., $A_{H^1}^{11}(e_i^{11};\ H^1, H^2) > 0$, $A_{e_i^{11}, H^1}^{11}(e_i^{11};\ H^1, H^2) > 0, \forall\, e_i^{11} > 0, H^1, H^2$;
- $A^{11}(e_i^{11};\ H^1, H^2)$ increases with H^2 because a higher H^2 implies more unexplored opportunities in Country 2, diverting managerial efforts, leaving more unexplored opportunities in Country 1, and improving effort productivity in Country 1 as well. That is, $A_{H^2}^{11}(e_i^{11};\ H^1, H^2) > 0$, $A_{e_i^{11}, H^2}^{11}(e_i^{11};\ H^1, H^2) > 0, \forall\, e_i^{11}, H^1, H^2$;
- $A^{12}(e_i^{12};\ H^1, H^2)$ increases with H^2 and has positive cross partial derivative with respect to H^2 and e_i^{12}, as higher H^2 implies more unexplored investment opportunities in Country 2, i.e., $A_{H^2}^{12}(e_i^{12};\ H^1, H^2) > 0$, $A_{e_i^{12}, H^2}^{12}(e_i^{12};\ H^1, H^2) > 0$, $\forall\, e_i^{12} > 0, H^1, H^2$;
- $A^{12}(e_i^{12};\ H^1, H^2)$ increases with H^1 because a higher H^1 implies more unexplored opportunities in Country 1, diverting managerial efforts, leaving more unexplored opportunities in Country 2, and improving effort productivity in country 2 as well, i.e., $A_{H^1}^{12}(e_i^{12};\ H^1, H^2) > 0$, $A_{e_i^{12}, H^1}^{12}(e_i^{12};\ H^1, H^2) > 0, \forall\, e_i^{12}, H^1, H^2$;
- no cross partial effects of two countries' concentrations on alpha production, i.e.,

$A^{11}_{H^1,H^2}(e_i^{11};\ H^1,H^2) = 0, A^{12}_{H^1,H^2}(e_i^{12};\ H^1,H^2) = 0, \forall\ e_i^{12}, H^1, H^2.$

Alternatively, we can argue that these cross partial effects are negligible.

From Equation (8) and Definitions (9), manager i's fund expected net alpha is[9]

$$\mathrm{E}(\alpha_i^1 \big| D) = \widehat{a^1} - \widehat{b^1}\frac{S^1}{W^1} + A^{11}(e_i^{11};\ H^1,H^2) + A^{12}(e_i^{12};\ H^1,H^2) - f_i^1. \tag{10}$$

We define the direct benefits of efforts exerted by DAFMI managers in the domestic stock market e_i^{11} and in the foreign stock market e_i^{12}, as follows:

$$B^{11}(e_i^{11};\ H^1,H^2) \triangleq A^{11}(e_i^{11};\ H^1,H^2) - c_2^{11}(e_i^{11};\ H^1,H^2), \tag{11}$$

$$B^{12}(e_i^{12};\ H^1,H^2) \triangleq A^{12}(e_i^{12};\ H^1,H^2) - c_2^{12}(e_i^{12};\ H^1,H^2). \tag{12}$$

These two terms are important for social planners and policy makers, as they capture the direct benefits of e_i^{11} and e_i^{12}, respectively, in terms of increase in gross alpha production minus the corresponding efforts costs.

$B^{11}(e_i^{11};\ H^1,H^2)$ and $B^{12}(e_i^{12};\ H^1,H^2)$ capture the direct benefit from effort exerted in active fund management in terms of increase in gross alpha production minus the effort cost. We interpret *benefits* generally, allowing them to be positive or negative.

Whether manager i's marginal direct benefits of initial effort in each county's stock market are positive [i.e., $B^{11}_{e_i^{11}}(0;\ H^1,H^2) > 0, B^{12}_{e_i^{12}}(0;\ H^1,H^2) > 0, \forall\ H^1,H^2$] is an important condition affecting the equilibrium. If this condition is not met, no effort is exerted, as in PS (see Proposition PS, Section 2.3 in FSX). Whether the sensitivity of manager i's direct benefits at optimal effort is positive or not [i.e., $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1} > 0\ (\leq 0)$ and $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2} > 0\ (\leq 0)$] is also an important condition affecting the equilibrium.[10]

Investors' Problem

---

[9] Following FSX, investors observe the passive benchmark and the AFMI funds' returns. The difference between these returns comes from three components: net alphas, the common risk factor, and idiosyncratic risks. As the distributions of the common risk and idiosyncratic risk are common knowledge, investors know the likelihood function of the net alphas. Given prior beliefs of net alphas, they form posteriors and update their beliefs. In our one-period model, there is no dynamic Bayesian updating, but we suggest that investors reached a fixed-point equilibrium. Further, because investors observe fees, fund sizes, and industry size, they can also infer $A^{11}(e_i^{11};\ H^1,H^2)$ and $A^{12}(e_i^{12};\ H^1,H^2)$. For simplicity and brevity, we depress the notation of $\hat{A}^{11}(e_i^{11};\ H^1,H^2)$ and $\hat{A}^{12}(e_i^{12};\ H^1,H^2)$ in favor of $A^{11}(e_i^{11};\ H^1,H^2)$ and $A^{12}(e_i^{12};\ H^1,H^2)$, as these two functions are deterministic.

[10] See also, Proposition 3 and the "proof intuition" to it.

Country 1's infinitely many mean-variance risk-averse investors invest in M^1 funds, earning returns, $\mathbf{r_F^1}$, a $M^1 \times 1$ vector with elements $r_{F,i}^1$, $i = 1, \dots, M^1$, in excess of the risk-free rate. The model of $\mathbf{r_F^1}$ is

$$\mathbf{r_F^1} = \boldsymbol{\alpha}^\mathbf{1} + \boldsymbol{\beta}^\mathbf{1} r_p + x^1 \boldsymbol{\iota}_{\mathbf{M^1}} + \boldsymbol{\varepsilon}^\mathbf{1}, \tag{13}$$

where $\boldsymbol{\alpha}^\mathbf{1}$ is a $M^1 \times 1$ vector of fund net alphas in Country 1, with each element as α_i^1, $i = 1, \dots, M^1$; and $\boldsymbol{\beta}^\mathbf{1}$ is the beta loading of each fund to an international benchmark portfolio. To simplify the framework, we assume each fund has beta loading equal to one to the international benchmark portfolio,[11] so that $\boldsymbol{\beta}^\mathbf{1}$ is the same as the $M^1 \times 1$ unit vector $\boldsymbol{\iota}_{\mathbf{M^1}}$. r_p is the international benchmark's return in excess of the risk-free rate, with mean μ_p, $\mu_p > 0$, and variance σ_p^2, $\sigma_p^2 > 0$. x^1 is the common risk factor of fund returns in Country 1, with mean 0 and variance $\sigma_{x^1}^2$, $\sigma_{x^1}^2 > 0$. $\boldsymbol{\varepsilon}^\mathbf{1}$ is a $M^1 \times 1$ vector of fund idiosyncratic risk factors in Country 1, and each of its elements is ε_i^1, $i = 1, \dots, M^1$, which has mean 0 and variance $\sigma_{\varepsilon^1}^2$, $\sigma_{\varepsilon^1}^2 > 0$. The parameters μ_p, σ_p^2, $\sigma_{x^1}^2$, and $\sigma_{\varepsilon^1}^2$ are constants, common knowledge to both investors and managers.

Investor j's portfolio return (in excess of the risk-free rate) is

$$r_j^1 = \boldsymbol{\delta}_\mathbf{j}^{\mathbf{1}^\mathbf{T}} \mathbf{r_F^1} + (1 - \boldsymbol{\delta}_\mathbf{j}^{\mathbf{1}^\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}}) r_p = r_p + \boldsymbol{\delta}_\mathbf{j}^{\mathbf{1}^\mathbf{T}} (\boldsymbol{\alpha}^\mathbf{1} + x^1 \boldsymbol{\iota}_{\mathbf{M^1}} + \boldsymbol{\varepsilon}^\mathbf{1}), \tag{14}$$

where $\boldsymbol{\delta}_\mathbf{j}^\mathbf{1}$ is a $M^1 \times 1$ vector of weights that investor j allocates to the M^1 funds, with each element as $\delta_{j,i}^1$, , and superscript T is a transpose operator. Investor j's problem is

$$\max_{\boldsymbol{\delta}_\mathbf{j}^\mathbf{1}} \frac{\mathrm{E}(r_j^1 | D)}{\sqrt{\mathrm{Var}(r_j^1 | D)}}, \tag{15}$$

subject to

$$\delta_{j,i}^1 \geq 0, \qquad \forall i, \tag{16}$$

$$\boldsymbol{\delta}_\mathbf{j}^{\mathbf{1}^\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} \leq 1. \tag{17}$$

Constraints (16) and (17) imply that investors cannot short sell funds, or short sell the international benchmark portfolio. To simplify our analysis, we assume that, in equilibrium, all investors have the same weights allocated to funds (i.e., a symmetric equilibrium), such that

[11] This is a common assumption, as active funds usually have diversified portfolios. See the discussion in Pastor and Stambaugh (2012).

$$\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*} = \boldsymbol{\delta}_{\mathbf{k}}^{\mathbf{1}^*}, \qquad \forall j \neq k. \tag{18}$$

In this case, in equilibrium, the fund industry size in Country 1 is

$$\frac{S^{1\,*}}{W^1} = \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{*\mathbf{T}}} \boldsymbol{\iota}_{\mathbf{M^1}}, \qquad \forall j. \tag{19}$$

We also note that, as in PS and FSX (see PS, pp. 748–750, including Footnote 6, and references therein, and FSX, Footnote 4), DAFMI's and FAFMI's active search for net alphas might have indirect effects not modeled here. It might drive security prices toward their true values; it might induce firms to improve governance and performance and to reduce agency costs. It might induce transfer of wealth from less productive firms or investors to more productive ones. As discussed in PS, FSX, and elsewhere in the literature, gross alphas are zero-sum. We note that this is the case regardless of whether any manager's direct and or indirect benefits are non-zero or zero.

We are now ready to characterize, in the following propositions, lemma, and corollaries, the IAFMI equilibrium, induced by managers choosing optimal effort levels in each country, and optimal fees. That is, we characterize DAFMI equilibrium expected net alphas, Sharpe ratios, effort levels, fee rates, direct benefits of effort, DAFMI size, and DAFMI funds' market shares. In Proposition 0, we formally state the DAFMI Nash equilibrium; in Proposition 1, we describe the qualitative properties of this equilibrium; and in Lemma 1, we describe technical properties of the DAFMI equilibrium used to prove Proposition 0 and 1. We present the two propositions and lemma in a sequence, and then provide the proofs intuition.

We first define DAFMI equilibrium optimal allocations.[12] Let

- $\mathbf{e^{11}}^*$ be an $M^1 \times 1$ vector with Country 1 managers' optimal effort allocations to Country 1 stocks, $e_i^{11^*}$,
- $\mathbf{e^{12}}^*$ be an $M^1 \times 1$ vector with Country 1 managers' optimal effort allocations to Country 2 stocks, with components, $e_i^{12^*}$,
- $\mathbf{f^1}^*$ be an $M^1 \times 1$ vector with Country 1 managers' optimal fee allocations, $f_i^{1^*}$, and
- $\boldsymbol{\delta}^{\mathbf{1}^*}$ be an $M^1 \times N^1$ matrix with vectors of Country 1 investors' optimal wealth weights allocations to funds, $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*}$.

PROPOSITION 0. Unique Nash Equilibrium.

There exists a unique Nash equilibrium, $\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}$.

[12] This is sufficient for describing the IAFMI equilibrium because of the DAFMI–FAFMI symmetry.

Proof of Proposition 0. See the Appendix.

Before providing the proof intuition below, we state the following proposition that characterizes the equilibrium.

PROPOSITION 1. For manager i, $i = 1, \dots, M^1$, if initial effort inputs generate positive direct benefits of effort, then in the DAFMI equilibrium induced by managers choosing optimal effort-fee combinations, $(e_i^{11^*}, e_i^{12^*}, f_i^{1^*})$, DAFMI size, $\frac{S^{1^*}}{W^1}$, and DAFMI fund market shares, $\frac{s_i^{1^*}}{S^{1^*}}$, $\forall i$, adjust such that the following properties are satisfied.

1. Competition drives managers' economic profits to zero, so they can only charge break-even fees.
2. Higher managers' aggregate skill results in higher DAFMI size.
3. Higher manager's relative skill results in higher DAFMI fund market share (relative fund size).
4. Managers offer the same market competitive expected net alphas.
5. Managers offer the same market competitive Sharpe ratios.
6. Investors hold the same DAFMI portfolio weights (which are proportional to DAFMI fund sizes).
7. Equilibrium effort levels and fees are the same across funds.
8. Equilibrium DAFMI direct benefits of effort are the same across funds.

Proof of Proposition 1. See the Appendix. The proof intuition is below.

To prove Proposition 1, we use the seven results of the following Lemma 1, which characterize properties of the IAFMI equilibrium.

LEMMA 1. For every manager i, $i = 1, \dots, M^1$, if initial effort inputs generate positive direct benefits of effort [i.e., $B_{e_i^{11}}^{11}(0;\ H^1, H^2) > 0, B_{e_i^{12}}^{12}(0;\ H^1, H^2) > 0, \forall\ H^1, H^2$], the DAFMI equilibrium induced by managers choosing optimal effort-fee combinations, $(e_i^{11^*}, e_i^{12^*}, f_i^{1^*})$, has the following properties.

1. Fees are equal to costs:

$$f_i^{1^*} - C_i^1\left(e_i^{11^*}, e_i^{12^*};\ s_i^{1^*}, H^1, H^2\right) = 0. \tag{20}$$

2. The impact of marginal effort, in either country, on gross alpha is set to be equal to the marginal average costs of effort in the respective country, thus manager i's marginal direct benefits of effort (in either country) under the optimal effort level are zero:

$$A^{11}_{e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) = B^{11}_{e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) = 0,$$
$$A^{12}_{e_i^{12}}\left(e_i^{12^*};\ H^1,H^2\right) - c^{12}_{2\ e_i^{12}}\left(e_i^{12^*};\ H^1,H^2\right) = B^{12}_{e_i^{12}}\left(e_i^{12^*};\ H^1,H^2\right) = 0. \tag{21}$$

3. When either country's concentration is higher, DAFMI equilibrium optimal effort levels in either country are higher (lower) if and only if higher concentration induces a larger (smaller) marginal effort impact on gross alphas than on costs in the respective country. Or,

$$de_i^{11^*}/dH^1 \geq 0(<0) \text{ iff } A^{11}_{e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right) \geq 0(<0),$$
$$de_i^{12^*}/dH^1 \geq 0(<0) \text{ iff } A^{12}_{e_i^{12},H^1}\left(e_i^{12^*};\ H^1,H^2\right) - c^{12}_{2\ e_i^{12},H^1}\left(e_i^{12^*};\ H^1,H^2\right) \geq 0(<0),$$
$$de_i^{11^*}/dH^2 \geq 0(<0) \text{ iff } A^{11}_{e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right) \geq 0(<0),$$
$$de_i^{12^*}/dH^2 \geq 0(<0) \text{ iff } A^{12}_{e_i^{12},H^2}\left(e_i^{12^*};\ H^1,H^2\right) - c^{12}_{2\ e_i^{12},H^2}\left(e_i^{12^*};\ H^1,H^2\right) \geq 0(<0). \tag{22}$$

4. Whether each country's higher concentrations induce higher equilibrium optimal fees depends on whether they induce changes in equilibrium DAFMI sizes and in equilibrium optimal effort levels in each country that are aggregately positive. Or,

$$\frac{df_i^{1^*}}{dH^1} = \left[\sum_{j=1}^{M^1}\left(c^1_{1,j}\right)^{-1}\right]^{-1} W^1\frac{d\left(\frac{S^1}{W^1}\right)^*}{dH^1} + c^{11}_{2\ e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right)\frac{de_i^{11^*}}{dH^1}$$
$$+c^{12}_{2\ e_i^{12}}\left(e_i^{12^*};\ H^1,H^2\right)\frac{de_i^{12^*}}{dH^1},$$
$$\frac{df_i^{1^*}}{dH^2} = \left[\sum_{j=1}^{M^1}\left(c^1_{1,j}\right)^{-1}\right]^{-1} W^1\frac{d\left(\frac{S^1}{W^1}\right)^*}{dH^2} + c^{11}_{2\ e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right)\frac{de_i^{11^*}}{dH^2}$$
$$+c^{12}_{2\ e_i^{12}}\left(e_i^{12^*};\ H^1,H^2\right)\frac{de_i^{12^*}}{dH^2}. \tag{23}$$

5. When either country's concentrations are higher, equilibrium manager i's direct benefits of effort in the respective country are higher (lower) if and only if higher

concentrations induce, in the respective country, a larger (smaller) impact on gross alphas than on costs. Or,

$$\frac{dB^{11}\left(e_i^{11^*}; H^1,H^2\right)}{dH^1} \geq 0\ (< 0) \text{ iff } A^{11}_{H^1}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\,H^1}\left(e_i^{11^*};\ H^1,H^2\right) \geq 0\ (< 0),$$

$$\frac{dB^{12}\left(e_i^{12^*}; H^1,H^2\right)}{dH^1} \geq 0\ (< 0) \text{ iff } A^{12}_{H^1}\left(e_i^{12^*};\ H^1,H^2\right) - c^{12}_{2\,H^1}\left(e_i^{12^*};\ H^1,H^2\right) \geq 0\ (< 0),$$

$$\frac{dB^{11}\left(e_i^{11^*}; H^1,H^2\right)}{dH^2} \geq 0\ (< 0) \text{ iff } A^{11}_{H^2}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\,H^2}\left(e_i^{11^*};\ H^1,H^2\right) \geq 0\ (< 0),$$

$$\frac{dB^{12}\left(e_i^{12^*}; H^1,H^2\right)}{dH^2} \geq 0\ (< 0) \text{ iff } A^{12}_{H^2}\left(e_i^{12^*};\ H^1,H^2\right) - c^{12}_{2\,H^2}\left(e_i^{12^*};\ H^1,H^2\right) \geq 0\ (< 0). \tag{24}$$

6. Pairwise relative DAFMI fund sizes, $s_i^{1^*}/s_j^{1^*}$, are inversely proportional to their corresponding cost coefficients, $c^1_{1,j}/c^1_{1,i}$ (where $c^1_{1,i}$ is the intensity of fund-level decreasing returns to scale in gross alpha production).
7. DAFMI fund market shares, $s_i^{1^*}/S^{1^*}$s are $s_i^{1^*}/S^{1^*} = \left[c^1_{1,i}\sum_{j=1}^{M^1}\left(c^1_{1,i}\right)^{-1}\right]^{-1}$, $\forall i$.

Proof of Lemma 1. See the Appendix. The proof intuition is below.

The proof of the existence and uniqueness of the Nash equilibrium is similar to the single-country one in FSX. Competing for investments, DAFMI managers maximize fund expected net alphas by choosing optimal effort levels and fees in each country, earning zero economic profits (break-even fees) in equilibrium. The reason for the latter is the following. If DAFMI managers increase fees, they would lower fund expected net alphas and lose all investments. If DAFMI managers decrease fees, they would become insolvent – incurring negative cash flows (costs higher than fees). Deviating from equilibrium effort level would also induce a loss of investments (if decreasing effort) or insolvency (if increasing effort). Therefore, DAFMI managers have no incentive to deviate.

Also, as there are no diversification benefits across funds, DAFMI managers who attempt to provide higher expected net alphas attract investments. Consequently, due to decreasing returns to scale in performance, on the one hand, and increasing fund costs, on the other hand, “alpha gains” are more than mitigated by a (break-even) fee increase, resulting in an overall decrease in expected net alpha. Thus, in equilibrium, the allocation of investments, or fund sizes, sets expected net alphas to be equal across funds. If DAFMI fund managers cannot produce the DAFMI highest expected net alpha, even for an infinitesimal fund size, they lose all investments and go out of the market.

In addition, as DAFMI funds have the same expected net alphas, they have the same expected returns. As the source of DAFMI fund returns' variance is the same across funds, the DAFMI fund return variance is the same across funds. Therefore, DAFMI managers offer the same competitive Sharpe ratio. Because investors cannot obtain a higher Sharpe ratio, they have no incentive to deviate.

These conditions result in a DAFMI unique Nash equilibrium in which neither DAFMI investors nor DAFMI managers have incentives to deviate from their chosen strategies.

If they are higher, either country's concentrations induce a DAFMI higher (lower) marginal effort impact on gross alphas than a marginal effort impact on DAFMI costs, in the respective country. DAFMI managers optimally choose, in each country, higher (lower) effort levels in producing fund net alphas. If they are higher, either country's concentrations induce higher DAFMI equilibrium optimal effort levels in the respective country and DAFMI managers' costs are driven higher, resulting in higher break-even fees.

Higher concentrations in each country have two effects on manager i's direct benefits of effort in the respective country. First, in the respective country, they directly affect the levels of gross alphas production function, $A^{11}(e_i^{11};\ H^1, H^2)$ and $A^{12}(e_i^{12};\ H^1, H^2)$, and the levels of costs, $c_2^{11}(e_i^{11};\ H^1, H^2)$ and $c_2^{12}(e_i^{12};\ H^1, H^2)$, being parameters of each of these functions. Second, in the respective country, they affect DAFMI equilibrium optimal effort levels, consequently changing the respective country's levels of gross alphas and costs. In equilibrium, the latter (net) effect is zero because managers keep increasing DAFMI effort levels in each country until, in each country, the marginal effort impact on gross alphas is equal to the marginal effort impact on costs. Thus, the effect of higher concentration through effort on gross alphas, in each country, is cancelled out by its effects on costs. Therefore, in DAFMI equilibrium (as the net second effect is zero), changes in either country's concentrations affect gross alphas and costs through the (direct) first effect only. Consequently, if higher, either country's concentrations induce higher direct impacts on gross alphas than on costs in the respective country. DAFMI manager i's direct benefits of effort, in this country, increase in the respective concentration level.

DAFMI managers' different costs of producing gross alphas (skills) induce different fund sizes in equilibrium. There is a separation between the determination processes of DAFMI size (that is, DAFMI weight in total wealth, $\frac{{S^1}^*}{W^1}$) and DAFMI fund market shares (that is, relative fund sizes within DAFMI). The former is driven by DAFMI managers' aggregate skill (cost), and the latter by DAFMI managers' relative skills (costs). In other words, how DAFMI investors weight the funds inside DAFMI, or investors' "optimal DAFMI portfolio," could be unaffected by how DAFMI investors weight the DAFMI as a whole relative to the passive benchmark. This separation property facilitates later results.

For convenience in describing the equilibrium in the following propositions, we define the equilibrium optimal expected net alphas of an initial marginal investment in the DAFMI (i.e., where $S^1 = 0$) as $X\left({e_i^{11}}^*, {e_i^{12}}^*;\ H^1, H^2\right)$. Quantitatively,

$$X\left({e_i^{11}}^*, {e_i^{12}}^*;\ H^1, H^2\right) \triangleq \widehat{a^1} + A^{11}(e_i^{11};\ H^1, H^2) + A^{12}(e_i^{12};\ H^1, H^2) \\ -c_0^1 - c_2^{11}(e_i^{11};\ H^1, H^2) - c_2^{12}(e_i^{12};\ H^1, H^2). \tag{25}$$

For DAFMI to exist, we must have positive expected net alphas for initial infinitesimal investments into it, or[13]

$$X\left({e_i^{11}}^*, {e_i^{12}}^*;\ H^1, H^2\right) > 0,\ \forall\, H^1, H^2. \tag{26}$$

If Inequality (26) is violated, investors receive no advantage in diverting funds from the passive index to the DAFMI. Also, to offer meaningful results, we assume that initial marginal allocations of effort generate positive AFMI direct benefits of effort; that is,

$$B_{e_i^{11}}^{11}(0;\ H^1, H^2) > 0,\ B_{e_i^{12}}^{12}(0;\ H^1, H^2) > 0,\ \forall i, H^1, H^2, \tag{27}$$

such that the optimal effort, $({e_i^{11}}^*, {e_i^{12}}^*)$, is positive, finite, and attainable, i.e., $B_{e_i^{11}}^{11}\left({e_i^{11}}^*;\ H^1, H^2\right) = 0,\ B_{e_i^{12}}^{12}\left({e_i^{12}}^*;\ H^1, H^2\right) = 0,\ \ {e_i^{11}}^*, {e_i^{12}}^* < K,\ \ \forall i, H^1, H^2$, for some positive constant K. We focus on the case where the optimal effort is positive.

As in PS (see their Proposition 2) and FSX (see their Proposition RA2), the explicit analytic solutions for $\frac{{S^1}^*}{W^1}$ are solutions of a cubic equation and are cumbersome. The following proposition presents the cubic equation, and its corollary presents properties of its solution.

PROPOSITION 2. Equilibrium Optimal Allocations.

For manager i, $i = 1, \ldots, M^1$, we have:

---

[13] The condition in Inequality (26) here is equivalent to the condition that $a > 0$ in PS. See PS, p. 747, for further discussion and insights.

1. $\mathrm{E}(\alpha_i^1 | D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}} > 0$; and

2. the equilibrium optimal $\frac{S^{1^*}}{W^1}$ is either 1 or a real positive solution (smaller than 1), of the following first-order condition (a cubic equation) of investors' problem [Equations (15)–(17)]. After substituting $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{*\mathbf{T}}} \boldsymbol{\iota}_{\mathbf{M}^1} = \frac{S^{1^*}}{W^1}$, $\forall j$,

$$-\gamma\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^3 - \left\{\gamma\sigma_{a^1}^2 + \gamma\sigma_{x^1}^2 + \widehat{b^1} + \left[\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{S^{1^*}}{W^1}$$
$$+X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right) = 0, \tag{28}$$

where $\gamma \triangleq \mu_p/\sigma_p^2$.

Proof of Proposition 2. See the Appendix.

The intuition of Proposition 2 is as follows. DAFMI investors allocate investments to funds based on their risk-return tradeoffs. Investing wealth in the DAFMI increases portfolios' risk, so they choose to limit these investments, leaving $\mathrm{E}(\alpha_i^1 | D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}} > 0$. The risk-return tradeoff of potentially investing the last dollar, the dollar that would drive DAFMI fund expected net alphas to zero, is "in the variance favor." That is, the marginal cost of risk, of investing this last dollar, is higher than the marginal benefit of the gained net alpha. This prevents optimizing DAFMI risk-averse investors from allocating it to the DAFMI, leaving DAFMI fund expected net alphas to be positive. The properties of the cubic equation guarantee exactly one real positive root. If the positive root is larger than 1, then $\frac{S^{1^*}}{W^1} = 1$.

We can now write the following corollary, characterizing DAFMI equilibrium relations between performance and size, and between the rate of returns to scale decrease and size.

COROLLARY TO PROPOSITION 2. For large enough W^1, such that $\frac{S^{1^*}}{W^1} < 1$, we have

1. Higher equilibrium optimal expected net alphas of an initial marginal investment in the DAFMI induce a larger equilibrium DAFMI size relative to total DAFMI wealth, or

$$\frac{d\frac{S^{1^*}}{W^1}}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)} = \frac{1}{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2 + \sigma_{a^1}^2 + \sigma_{x^1}^2\right] + \widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1} > 0. \tag{29}$$

2. A higher rate of decrease in aggregate DAFMI returns to scale [fund level and industry

level; $\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1$] induces a smaller equilibrium DAFMI size, or

$$\frac{d\frac{S^{1^*}}{W^1}}{d\left\{\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}} = \frac{-\frac{S^{1^*}}{W^1}}{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2+\sigma_{a^1}^2+\sigma_{x^1}^2\right]+\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1} < 0. \tag{30}$$

Proof of Corollary to Proposition 2. See the Appendix.

The intuition of this corollary is as follows. A higher level of DAFMI equilibrium optimal expected net alpha of an initial marginal investment, $X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$, attracts more investments to the DAFMI. Also, we can see that $\widehat{b^1}$ is the industry-level expected decreasing returns to scale rate coming from the alpha production function, based on current information, whereas $\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1$ may be regarded as the equilibrium decreasing returns to scale factor coming from DAFMI managers' costs of alpha production (calculated by aggregating all the fund average cost sensitivities to size, $c_{1,i}^1$'s). The latter decreasing returns to scale factor, $\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1$, is inversely proportional to DAFMI aggregate skill. Thus, the factor $\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1$ may be regarded as the combined decreasing returns to scale factor in DAMI.

The next proposition offers comparative statics that underlie our main empirical analysis.

PROPOSITION 3. DAFMI Size and Expected Net Alphas Sensitivities to Concentrations levels.

Where $\frac{S^{1^*}}{W^1} < 1$, we have the following.[14]

1. Higher concentrations, in either country, induce larger (smaller) DAFMI equilibrium size and higher (lower) DAFMI equilibrium expected net alphas if and only if higher concentrations induce a larger (smaller) aggregate (over the two countries) impacts of induced optimal effort-level changes on gross alphas than on costs.

[14] When $\frac{S^{1^*}}{W^1} = 1$, it is the case that, 1. $\frac{S^{1^*}}{W^1}$ is unrelated to DAFMI and FAFMI concentrations; 2. higher DAMFI/FAFMI concentrations induce higher (lower) DAMFI/FAFMI equilibrium expected net alphas if and only if higher concentrations induces a larger (smaller) impact on gross alphas than on costs; and 3. DAMFI/FAFMI equilibrium expected net alphas are concave (convex), in DAMFI/FAFMI concentrations, if and only if the DAMFI/FAFMI equilibrium direct benefit function is concave (convex), in concentrations.

The analytical statements of the verbal statements are as follows. Regarding DAFMI equilibrium size sensitivity to DAFMI concentration, we have

$$\frac{d\frac{S^1}{W^1}^*}{dH^1} = \frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1}\right]; \tag{31}$$

thus,

$$\frac{d\frac{S^1}{W^1}^*}{dH^1} \geq 0\ (< 0) \text{ iff } \frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1} \geq 0\ (< 0).$$

The analytical statements regarding DAFMI equilibrium size sensitivity to foreign concentration are

$$\frac{d\frac{S^1}{W^1}^*}{dH^2} = \frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^2}\right]; \tag{32}$$

thus,

$$\frac{d\frac{S^1}{W^1}^*}{dH^2} \geq 0\ (< 0) \text{ iff } \frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^2} \geq 0\ (< 0).$$

The analytical statements regarding DAFMI equilibrium expected net alpha sensitivity to DAFMI concentration are

$$\begin{aligned} \left.\frac{d\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}} &= \left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1}\right] \\ &\times \left\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} \frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\right\}; \end{aligned} \tag{33}$$

thus,

$$\left.\frac{d\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \geq 0\ (< 0) \text{ iff } \frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1} \geq 0\ (< 0).$$

The analytical statements regarding DAFMI equilibrium expected net alpha sensitivity to foreign concentration are

$$\begin{aligned} \left.\frac{d\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^2}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}} &= \left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^2}\right] \\ &\times \left\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} \frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\right\}; \end{aligned} \tag{34}$$

thus,

$$\frac{d\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^2}\Bigg|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta^1}^*\right\}} \geq 0\ (< 0) \text{ iff } \frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^2} \geq 0\ (< 0).$$

2. If concave in either country's concentration, DAFMI equilibrium direct benefits of efforts function indicates concave DAFMI equilibrium size in the respective concentration. (If convex in either country's concentration, DAFMI equilibrium size indicates convex, DAFMI equilibrium direct benefits of efforts function in the respective concentration.) The sensitivity of equilibrium DAFMI size to the cross partial derivative of DAFMI and FAFMI concentrations depend on signs and sizes of several terms, including the sum of the sensitivities of DAFMI direct benefits due to efforts exerted in the domestic and foreign stock markets, to the cross partial derivative of DAFMI and FAFMI concentrations, and the product of the sums of DAFMI direct benefits sensitivities, due to efforts exerted in the domestic and foreign stock markets, to DAFMI and FAFMI concentrations, respectively.

The analytical statements of the above verbal statements regarding second-order sensitivity of equilibrium DAFMI size to DAFMI concentration are

$$\frac{d^2 \frac{S^1}{W^1}^*}{d{H^1}^2} = \frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\left[\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{d{H^1}^2}\right] - 6\gamma^1\sigma_{b^1}^2\frac{S^1}{W^1}^*\left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1}\right]^2\left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\right]^3; \tag{35}$$

thus,

if $\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{d{H^1}^2} \leq 0$ then $\frac{d^2\left(S^1/W^1\right)^*}{d{H^1}^2} \leq 0$, and if $\frac{d^2\left(S^1/W^1\right)^*}{d{H^1}^2} \geq 0$, then $\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{d{H^1}^2} \geq 0$.

The analytical statements regarding second-order sensitivity of DAFMI size to FAFMI concentration are

$$\frac{d^2 \frac{S^1}{W^1}^*}{d{H^2}^2} = \frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\left[\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{d{H^2}^2} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{d{H^2}^2}\right] - 6\gamma^1\sigma_{b^1}^2\frac{S^1}{W^1}^*\left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^2}\right]^2\left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\right]^3; \tag{36}$$

thus,

if $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{2^2}}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{2^2}}\le 0$ then $\frac{d^2(S^1/W^1)^*}{dH^{2^2}}\le 0$, and if $\frac{d^2(S^1/W^1)^*}{dH^{2^2}}\ge 0$, then $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{2^2}}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{2^2}}\ge 0$.

The analytical statements regarding the cross partial derivative sensitivity of equilibrium DAFMI size to DAFMI and FAFMI concentrations are

$$\frac{d^2\frac{S^1}{W^1}^*}{dH^1dH^2}=\frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1dH^2}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1dH^2}\right]$$
$$-6\gamma^1\sigma_{b^1}^2\frac{S^1}{W^1}^*\left[\frac{d\frac{S^1}{W^1}^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\right]^3 \qquad (37)$$
$$\times\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\right]\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2}\right];$$

thus, the sign of the cross partial derivative of $\frac{S^1}{W^1}^*$ with respect to H^1 and H^2 depends on the signs and magnitudes of $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1dH^2}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1dH^2}$, $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}$, and $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2}$.

3. Concave equilibrium expected net alphas in either country’s concentration, indicates concave, in concentration, equilibrium direct benefit function. (Convex, in concentration, equilibrium direct benefit function indicates convex, in concentration, equilibrium expected net alphas.)

Similar to the case of equilibrium DAFMI size, the sensitivity of DAFMI equilibrium expected net alpha dependency on the cross partial derivative of DAFMI and AFMI concentrations depends on signs and sizes of several terms, including the sum of the sensitivities of DAFMI direct benefits due to efforts exerted in the domestic and foreign stock markets, to the cross partial derivative of DAFMI and FAFMI concentrations, and the product of the sums of DAFMI direct benefits sensitivities due to efforts exerted in the domestic and foreign stock markets, to DAFMI and FAFMI concentrations, respectively.

The analytical statements of the verbal statements regarding second-order sensitivity of equilibrium DAFMI expected net alpha to DAFMI concentration are

$$\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{1^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} = \left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{1^2}}\right]\Bigg\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} \frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\Bigg\} + 6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* \Bigg\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\Bigg\} \left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\right]^2 \left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\right]^3; \tag{38}$$

thus,

if $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{1^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \leq 0$, then $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{1^2}} \leq 0$ and

if $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{1^2}} \geq 0$, then $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{1^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \geq 0$.

(The fact that equilibrium expected net alpha is concave in H^1 indicates that the sum of the second-order derivatives of $B^{11}\left(e_i^{11^*};\ H^1,H^2\right)$ and $B^{12}\left(e_i^{12^*};\ H^1,H^2\right)$ with respect to H^1 is negative, and the fact that the sum of the second-order derivatives of $B^{11}\left(e_i^{11^*};\ H^1,H^2\right)$ and $B^{12}\left(e_i^{12^*};\ H^1,H^2\right)$ with respect to H^1 is positive indicates that equilibrium expected net alpha is convex in H^1.)

The analytical statements regarding second-order sensitivity of equilibrium DAFMI expected net alpha to FAFMI concentration are

$$\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{2^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} = \left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{2^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{2^2}}\right]\Bigg\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} \frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\Bigg\} + 6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* \Bigg\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\Bigg\} \left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2}\right]^2 \left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\right]^3; \tag{39}$$

thus,

if $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{2^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \leq 0$, then $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{2^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{2^2}} \leq 0$ and

if $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^{2^2}} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^{2^2}} \geq 0$, then $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1 \middle| D\right)}{dH^{2^2}}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \geq 0$.

(The fact that equilibrium expected net alpha is concave in H^2 indicates that the sum of the second-order derivatives of $B^{11}\left(e_i^{11^*};\ H^1, H^2\right)$ and $B^{12}\left(e_i^{12^*};\ H^1, H^2\right)$ with respect to H^2 is negative, and the fact that the sum of the second-order derivatives of $B^{11}\left(e_i^{11^*};\ H^1, H^2\right)$ and $B^{12}\left(e_i^{12^*};\ H^1, H^2\right)$ with respect to H^2 is positive indicates that equilibrium expected net alpha is convex in H^2.)

The analytical statements regarding the cross partial derivative sensitivity of equilibrium DAFMI expected net alpha to DAFMI and FAFMI concentrations are

$$\left.\frac{d^2\mathrm{E}\left(\alpha_i^1\middle|D\right)}{dH^1dH^2}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} = \left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1dH^2} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1dH^2}\right]\left\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}\frac{d\left(\frac{S^1}{W^1}\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\right\} + 6\gamma\sigma_{b^1}^2\frac{S^1}{W^1}^*\left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\right]\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2}\right]\left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\right]^3, \tag{40}$$

thus, the sign of $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1\middle|D\right)}{dH^1dH^2}\right|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$ depends on the signs and magnitudes of $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1dH^2} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1dH^2}$, $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}$, and $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^2} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^2}$.

Proof of Proposition 3. See the Appendix.

The intuition behind Proposition 3 is as follows. Changes of H^1 affect both DAFMI cost and productivity of efforts exerted in alpha production in both domestic and foreign stock markets. In turn, such changes affect equilibrium DAFMI expected net alpha, $\mathrm{E}(\alpha_i^1|D)\big|_{\left\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$, in two stages. In the first stage, if a higher H^1 induces a larger (smaller) aggregate, across the domestic and foreign stock markets, impact on gross alphas than on costs, it increases (decreases) DAFMI managers' ability to produce expected net alphas, thereby increasing (decreasing) DAFMI level of expected net alphas produced. In the second stage, DAFMI investors react to the increase (decrease) in DAFMI fund expected net alphas by increasing (decreasing) investment levels in funds, consequently decreasing (increasing) DAFMI expected net alphas, due to decreasing returns to scale. The risk-return tradeoff of

DAFMI risk-averse investors makes their reaction to changes in DAFMI fund expected net alphas less intense. That is, they subdue their additional investments to funds when inferring higher fund expected net alphas due to risk increase, and they limit their reduction in investments to funds when observing lower fund expected net alphas due risk decrease.

The first stage and second stage described above, the latter as affected by risk attitudes, result in a change of DAFMI optimal effort levels in both the domestic and foreign stock markets. DAFMI new optimal efforts levels, in turn, affect DAFMI level of alphas productions and the efforts costs producing it in both the domestic and foreign stock markets. The overall outcome depends on the aggregate—across the domestic and foreign stock markets—relative sensitivities to DAFMI concentration of—the domestic and foreign stock markets—alpha production functions, on the one hand, and of the efforts cost functions, on the other. Indeed, we formally show that whether a higher H^1 increases equilibrium DAFMI expected net alpha, $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$, depends on whether it has a larger impact on DAFMI gross alphas than on the costs producing it [i.e., the sign of $\left.\frac{d\mathrm{E}(\alpha_i^1|D)}{dH^1}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$, depends on the sign of $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}=A_{H^1}^{11}\left(e_i^{11^*};\,H^1,H^2\right)+A_{H^1}^{12}\left(e_i^{12^*};\,H^1,H^2\right)-c_{2\,H^1}^{11}\left(e_i^{11^*};\,H^1,H^2\right)-c_{2\,H^1}^{12}\left(e_i^{12^*};\,H^1,H^2\right)$.[15] (as we show in Lemma 1.5 above). Thus, a higher H^1 induces a larger equilibrium DAFMI expected net alpha if and only if it induces higher equilibrium DAFMI direct benefits, $\left[B^{11}\left(e_i^{11^*};\,H^1,H^2\right)+B^{12}\left(e_i^{12^*};\,H^1,H^2\right)\right]$. This explains the expected net alpha part of Proposition 3.1.

If a higher H^1 induces a larger (smaller) impact on gross alphas than on costs, then it attracts more (less) investments to the DAFMI [if investors have additional wealth to allocate to funds (i.e., $\frac{S^{1^*}}{W^1}<1$)]. This explains the size part of Proposition 3.1.

The intuition regarding H^2 in Proposition 3.1 is similar.

Examining the second-order effects of DAFMI concentration on DAFMI size, we first note that changes in H^1 that induce a larger $\frac{S^{1^*}}{W^1}$ result in a larger allocation to DAFMI funds and, in turn, in a higher investors' overall portfolio risk. Mean-variance risk-averse investors facing risk-return tradeoffs respond to an increase in marginal portfolio risks, holding other

[15] This total derivative of DAFMI direct benefits with respect to H^1 is the same as its partial derivative with respect to H^1.

parameters constant, by optimally lowering investment in funds. Thus, how changes in H^1 affect changes in equilibrium $\frac{S^1}{W^1}^*$ depends on how changes in H^1 affect this risk-return tradeoff. The implications for the second-order derivative $\frac{d^2 \frac{S^1}{W^1}^*}{dH^{1^2}}$ are in the proof of Proposition 3, which expresses this tradeoff analytically by identifying $\frac{d^2 \frac{S^1}{W^1}^*}{dH^{1^2}}$ as a sum of two addends. The first addend is negative (positive) if the sum of the direct benefits functions is concave (convex) in H^1, and the second one is always negative. This shows that a concave sum of the direct benefits functions in H^1 implies an $\frac{S^1}{W^1}^*$ concave in H^1; and a convex $\frac{S^1}{W^1}^*$ in H^1 implies a convex sum of the direct benefits functions in H^1.

The intuition regarding H^2 in Proposition 3.2 is similar to that of H^1. The intuition regarding the cross partials in Proposition 3.2 is straightforward, as second-order derivatives become cross partial derivatives and squares of first-order derivatives become products of first-order derivatives with respect to both countries' concentrations.

This explains Proposition 3.2.

Similarly, examining the second-order effects of DAFMI concentration on expected net alphas, we show that as H^1 changes, the change of marginal $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$, i.e., $\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{1^2}}\bigg|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$, is positively proportional to the second-order change in the sum of the direct benefit functions plus an adjustment term that captures the effects of risk. This adjustment term ensures that, holding all other parameters constant, if investors' marginal portfolios risks of investing in funds are higher, investors optimally invest less in funds. In doing so, they exert a smaller negative impact on expected net alphas; thus, a higher H^1 induces a higher marginal $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$. We can see that if the second-order derivative of the sum of the direct benefits functions is positive, $\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{1^2}}\bigg|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$ must be positive, whereas if $\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{1^2}}\bigg|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}^{\mathbf{1}^*}\}}$ is negative, the second-order derivative of the sum of the direct benefits functions must be negative.

The intuition regarding H^2 in Proposition 3.3 is similar to that of H^1. The intuition regarding the cross partials in Proposition 3.3, similar to that in Proposition 3.2.

This explains Proposition 3.3.

When investors have no additional wealth to allocate to funds, i.e., $\frac{S^{1^*}}{W^1} = 1$, they exert no impact on marginal $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}}$, making the marginal equilibrium optimal expected net alphas depend only on the effect of H^1 and H^2 on managers' ability to produce net alphas.

We are now ready for the following proposition.

PROPOSITION 4. Relation between skill, market share, and net alpha.

When $\frac{S^{1^*}}{W^1} < 1$, a decrease (increase) in DAFMI manager i's skill, ${c_{1,i}^1}^{-1}$ while DAFMI manager j's skill, ${c_{1,j}^1}^{-1}$, $\forall j \neq i$, is unchanged induces

1. a decrease (increase) in $\frac{S_i^{1^*}}{S^{1^*}}, \forall i$, and an increase (decrease) in $\frac{S_j^{1^*}}{S^{1^*}}, \forall j \neq i$, and
2. a decrease (increase) in $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}}$ and a decrease (increase) in $\mathrm{E}(\alpha_j^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}}, \forall j \neq i$.

Proof of Proposition 4. See the Appendix.

According to Proposition 4, a decrease in DAFMI manager i's skill leads to a decrease in i's market share, $\frac{S_i^{1^*}}{S^{1^*}}$. Some of the assets that fund i loses are invested in all other funds, thereby increasing the market share of all other funds.

Also, a higher skill (lower $c_{1,i}^1$), affects $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}}$ in two stages. In the first stage, it decreases DAFMI manager i's average cost and, thus, induces higher fund expected net alphas. As DAFMI manager i offers a higher fund expected net alpha, investments shift into DAFMI fund i from other DAFMI funds, making all those fund expected net alphas higher due to decreasing returns to scale at fund level. At the second stage, an increase in DAFMI fund expected net alphas attracts investments into DAFMI, which in turn drives down DAFMI funds' expected net alphas due to decreasing returns to scale at industry level. Where $\frac{S^{1^*}}{W^1} < 1$, DAFMI investors' portfolio risks increase (decrease) when they invest more (less) in DAFMI. Thus, they subdue DAFMI investments increases when observing an increase in DAFMI fund expected net alphas, and they limit investment reductions when observing a decrease in DAFMI fund expected net alphas. Thus, DAFMI investors' risk aversion mitigates the countered effect at the second stage and makes the first stage's effect dominant.

Where $\frac{S^{1^*}}{W^1} = 1$, DAFMI investors have no additional wealth to allocate to funds, so their investments have no impact on DAFMI marginal equilibrium optimal expected net alphas, causing the first stage's effect to dominate.

2.2 Endogenous Market Concentrations

Our model allows for an endogenous measure of DAFMI and FAFMI concentrations. Modeling an endogenous measure of concentration facilitates the use of available and prevalent empirical measures. If we define H^1 and H^2 to be Herfindahl-Hirschman indices (HHI), which is the sum of market shares squared, then H^1 and H^2 are endogenous to our model.[16] Using funds' equilibrium market share, as identified in Lemma 1.7, we can write the equilibrium DAFMI and FAFMI concentrations, H^{1^*} and H^{2^*}, as

$$H^{1^*} \triangleq \sum_{i=1}^{M^1} \left(\frac{s_i^{1^*}}{S^{1^*}}\right)^2 = \sum_{i=1}^{M^1} \left[c_{1,i}^1 \sum_{j=1}^{M^1} \left(c_{1,i}^1\right)^{-1}\right]^{-2}, \tag{41}$$

$$H^{2^*} \triangleq \sum_{i=1}^{M^2} \left(\frac{s_i^{2^*}}{S^{2^*}}\right)^2 = \sum_{i=1}^{M^2} \left[c_{1,i}^2 \sum_{j=1}^{M^2} \left(c_{1,i}^2\right)^{-1}\right]^{-2}. \tag{42}$$

We can see that H^{1^*} and H^{2^*} are determined by $c_{1,i}^1$s and $c_{1,i}^2$s. Specifically, depending on the size of $c_{1,i}^1$ relative to that $c_{1,j}^1$, $\forall j \neq i$, an increase in $c_{1,i}^1$, holding $c_{1,j}^1$, $\forall j \neq i$ constant, increases or decreases H^{1^*}.

For simplicity and brevity, we focus our discussion on DAFMI (similar results hold for FAFMI). When the DAFMI concentration is defined as the HHI, Propositions 3 and 4 imply that the relation between the $c_{1,i}^1$s, DAFMI equilibrium fund expected net alphas, and DAFMI size is complex. An increase in $c_{1,i}^1$ affects the DAFMI equilibrium fund expected net alphas in two ways: 1. its direct impact leads to lower DAFMI equilibrium fund expected net alphas (Proposition 4) and 2. depending on fund i's size relative to DAFMI rivals, it increases or decreases H^{1^*}, which consequently increases (decreases) DAFMI equilibrium fund expected net alphas if and only if $\frac{dB^{11}\left(e_i^{11^*}; H^{1^*}, H^{2^*}\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^{1^*}, H^{2^*}\right)}{dH^1} \geq 0 \ (< 0)$ (Proposition 3.1). Similarly, an increase in $c_{1,i}^1$ affects the equilibrium DAFMI size in two ways: 1. its direct impact leads to an (inverse direction) DAFMI size change, and 2. it increases or decreases H^{1^*},

[16] In an M^1-fund DAFMI, for example, the HHI could have values between the highest concentration, 1, in which one of the funds captures practically all the market share, and the lowest concentration, $1/M^1$, in which market shares are evenly divided. That is, in an M^1-funds' market HHI $\in \left[\frac{1}{M^1}, 1\right)$.

which consequently increases (decreases) the equilibrium DAFMI size if and only if $\frac{dB^{11}\left(e_i^{11^*};H^{1^*},H^{2^*}\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^{1^*},H^{2^*}\right)}{dH^1}\geq 0\ (<0)$ (Proposition 3.1). Thus, in the endogenous DAFMI concentration measure case, the relation between the $c_{1,i}^1$s, DAFMI equilibrium fund expected net alphas, and DAFMI size depend on fund i's size relative to rivals.[17]

Due to investments in the foreign stock market, DAFMI is also affected by changes in H^{2^*}. An increase in $c_{1,i}^2$ affects DAFMI equilibrium fund expected net alphas in the following way: depending on fund i's size relative to rivals', it increases or decreases H^{2^*}, which consequently increases (decreases) DAFMI equilibrium fund expected net alphas if and only if $\frac{dB^{11}\left(e_i^{11^*};H^{1^*},H^{2^*}\right)}{dH^2}+\frac{dB^{12}\left(e_i^{12^*};H^{1^*},H^{2^*}\right)}{dH^2}\geq 0\ (<0)$ (Proposition 3.1). Also, an increase in $c_{1,i}^2$ affects DAFMI size in the following way: depending on fund i's size relative to rivals', it increases or decreases H^{2^*}, which consequently increases (decreases) equilibrium DAFMI size if and only if $\frac{dB^{11}\left(e_i^{11^*};H^{1^*},H^{2^*}\right)}{dH^2}+\frac{dB^{12}\left(e_i^{12^*};H^{1^*},H^{2^*}\right)}{dH^2}\geq 0\ (<0)$ (Proposition 3.1). Notice that an increase in $c_{1,i}^2$ does not have direct impact on DAFMI fund expected net alphas and size, as that of $c_{1,i}^1$. Its impact on DAFMI is only through its impact on FAFMI concentration, H^{2^*}.

Thus, when the market concentration is endogenous, the relations between DAFMI concentration and DAFMI equilibrium fund expected net alphas and size are more complex. On the other hand, the relations between FAFMI concentration and the DAFMI equilibrium fund expected net alphas and size are similar to those under the exogenous concentrations framework.

Please see the discussion in FSX regarding the industry characteristics affecting equilibrium markets' concentrations levels and why modeling those here would unnecessarily complicate our model. As long as real-world concentration is not exactly determined by the $c_{1,i}^1$s (or any other exogenous parameter of our model), we are back to the case that when concentration is exogenous (that is, has an exogenous component), our predictions remain unaltered regarding the relation between changes in exogenous DAFMI concentration level,

[17] We believe that our cost function, Equation (18), is a concise one that captures essential effects within our model. To assure that all our functional form restrictions of the non-specialized model (exogenous concentration), which we deem basic and simple, hold in the specialized one (endogenous measure of industry concentration); however, we need to impose additional, technical, "second-order," parameter restrictions. For brevity and simplicity, we do not impose these restrictions. We call the parameter values that make the specialized model abide by these restrictions *plausible*. We later confirm that the said technical restrictions are not empirically binding. That is, imposing these restrictions would not change our empirical results. In other words, the empirically estimated parameters fall within the plausible parameters range.

the DAFMI equilibrium fund expected net alphas, and DAFMI size.

Similarly to FSX, we now proceed with an empirical analysis of the benefits and costs of changing concentrations of DAFMI and FAFMI using the version of our model with endogenous concentrations. This version of our model befits available data of empirical market concentrations, such as the HHI. Popular empirical market concentration measures, such as HHI, are functions of rivals' relative sizes. We use empirical techniques to control potential endogeneity of market concentration measures.

Whether DAFMI fund net alphas and DAFMI size move in the same direction as DAFMI (FAFMI) concentration become empirical questions. Further, in cases where active fund management creates value, if fund net alphas and DAFMI size increase with DAFMI (FAFMI) concentration, our model predicts positive marginal direct benefits of efforts, for plausible parameter values. Both signs of benefits sensitivity to changing concentrations are plausible alternatives to a null hypothesis of no benefits of active fund managers' efforts.

In the following empirical analysis, we use alternative empirical measures of concentration to evaluate robustness to issues such as endogeneity. We also control for potential endogeneity of DAFMI size and alpha using lagged measures of concentration and the recursive demeaning estimator of PST.

3 Empirical Study

We analyze the market concentration–net alpha and market concentration–AFMI size relations using international data of active equity mutual funds. We regard the U.S. AFMI as FAFMI, whose concentrations might affect another market's DAFMI net alphas and size. This is because the U.S. has the largest AFMI, which influences global DAFMIs. We analyze how DAFMI concentrations and, more importantly, how the FAFMI concentration influence global DAFMI net alphas and sizes.

3.1 Methodology

We describe our concentration measures, fund net alpha estimates, and our econometric models in this section.

Concentration Measures

Following FSX, and many other empirical papers, we use the following three indices to measure AFMI concentrations:

1. HHI

$$HHI_{j,t} = \sum_{i}^{M_{j,t}} MS_{i,j,t}^2 \tag{43}$$

2. normalized HHI (NHHI)

$$NHHI_{j,t} = \frac{M_{j,t} \times HHI_{j,t} - 1}{M_{j,t} - 1}, \tag{44}$$

3. sum of the first five largest funds' market shares (5FI)

$$5FI_{j,t} = \sum_{i=1}^{5} MS_{i,j,t}, \tag{45}$$

where the indices i, j, and t indicate the fund, the market, and the time, respectively. $MS_{i,j,t}$ is the market share of a fund in its market, measured as the fund's asset under management divided by the total assets under management in its market, and $M_{j,t}$ is the number of funds in the corresponding market. As some markets tend to have a large number of funds and others tend to have a small number of funds in our sample period, we focus on the results of using NHHI as the market concentration measure because it adjusts the effect of the number of funds on market concentrations [Cremers, Nair and Peyer (2008)]. For robustness check, we redo the analyses using HHI and 5FI.

Style-Matching Model and Net Alpha Estimation

Following FSX, we develop our style-matching model to estimate funds' passive benchmarks and then calculate fund net alphas. We use the following return-generating process:

$$R_{i,j,t} = \alpha_{i,j,t} + b_{i,j,t}^1 F_{j,t}^1 + \cdots + b_{i,j,t}^{n_j} F_{j,t}^{n_j}, \tag{46}$$

where $R_{i,j,t}$ is the return net of management fees of an active fund, $\alpha_{i,j,t}$ is the fund net alpha, and $F_{j,t}^1$ through $F_{j,t}^{n_j}$ are the factors constructing the benchmark portfolio returns. We require the benchmark portfolio to be an international passive benchmark portfolio, so $F_{j,t}^1$ through $F_{j,t}^{n_j}$ include returns net of management fees of domestic tradable index funds of different asset classes, a U.S. large-cap equity tradable index fund, and a domestic risk-free asset. We include a U.S. large-cap equity tradable index fund because it can be a potential factor in this international passive benchmark. Coefficients $b_{i,j,t}^1$ through $b_{i,j,t}^{n_j}$ represent the loadings, and n_j is the number of these factors in a particular market. In our algorithm, in each fund market, we minimize the variance of the residual when projecting $R_{i,j,t}$ on $F_{j,t}^1$ through $F_{j,t}^{n_j}$, and we constrain the coefficients $\hat{b}^{1}_{i,j,t}$ through $\hat{b}^{n_j}_{i,j,t}$ to be positive and sum up to one (as we do not allow short selling). We use a rolling window, from months $t-60$ to $t-1$, to estimate $\hat{b}^{1}_{i,j,t}$ through $\hat{b}^{n_j}_{i,j,t}$. The predicted value $\hat{b}^{1}_{i,j,t}F^{1}_{j,t}+\cdots+\hat{b}^{n_j}_{i,j,t}F^{n_j}_{j,t}$ is the international passive benchmark at time t, and we estimate $\alpha_{i,j,t}$ by subtracting $R_{i,j,t}$ from $\hat{b}^{1}_{i,j,t}F^{1}_{j,t}+\cdots+\hat{b}^{n_j}_{i,j,t}F^{n_j}_{j,t}$. We note that our empirical design of identifying passive benchmarks using matching tradable index funds fits our theoretical structure, which assumes the appropriate international passive benchmarks for each fund.

Our style-matching method is similar to the style-matching model developed by Sharpe (1992). Also, as our passive benchmark is tradable, our net alpha estimation is consistent with the Berk and Binsbergen (2015) argument that to measure the value added by a fund, its performance should be compared to the next-best investment opportunity available to investors. Moreover, our style-matching passive benchmark is similar to the characteristic-based benchmark developed by Daniel, Grinblatt, Titman, and Wermers (1997). Our model is similar to the style-matching model of FSX except that ours contains an additional U.S. large-cap equity tradable index fund besides domestic tradable index funds.

Concentrations–Net Alpha Relation

Pastor, Stambaugh, and Taylor (2015) (PST) develop a recursive demeaning (RD) estimator to control endogeneity bias. We adopt their method here to analyze the concentration–net alpha relation. The model we use is

$$\overline{\alpha_{\iota,J,t}} = \beta_1 \overline{NHHI^{D}_{J,t-1}} + \beta_2 \overline{NHHI^{US}_{t-1}} + \overline{Controls} + \overline{\varepsilon_{\iota,J,t}}, \quad (47)$$

where the superscription *D* and *US* represent the domestic and the U.S. concentration measures, respectively. The bar above the variables represents the forward-demeaning operator. The forward-demeaned value of a time-series variable X_t is

$$\overline{X_t} = X_t - \frac{1}{T-t+1}\sum_{s=t}^{T} X_{s}, \quad (48)$$

where T is the total number of observation of this time-series. Control variables include lagged DAFMI size and a time trend.

This model is similar to the model of market concentration–net alpha relation in FSX, except the following: 1. this model includes the U.S. concentration measure as an explanatory variable, so it fits our international model and studies how U.S. market concentration is associated with the fund net alphas in market j; 2. this model does not include the fund market share as a control because in our unreported tests, we find that the fund market share is insignificant and exclude it to reduce noise in the estimations; and 3. we focus on the first-order effect and do not include the second-order terms of the concentration measures.

We also perform (47) and estimate the coefficients at each global market.

Concentration–DAFMI Size Relation

Our panel regression model is

$$DAFMI\_Size_{j,t} = \beta_0 + \beta_1 NHHI^{D}_{j,t-1} + \beta_2 NHHI^{US}_{t-1} + Controls + \varepsilon_{j,t}, \qquad (49)$$

where $DAFMI\_Size_{j,t}$ is the DAFMI size of each global market, and control variables include lagged DAFMI size, a time trend, and market fixed effects. We also perform (49) and estimate the coefficients at each global market without market fixed effects but with Newey-West estimates of standard error.

3.2 Data

We obtain our data from the Global Databases of Morningstar Direct. Our sample contains 30 active equity mutual fund markets. Due to data availability, most of these markets have observations from 1999, so we set our sample period from the beginning of 1999 to the end of 2015 and use monthly data. Our online Data Appendix supplements the data description below.

The active equity mutual fund filter and the sample development method are similar to those in FSX. We use keywords in Morningstar to identify active equity mutual funds. We require the mutual funds to be open-ended and non-restricted. In each mutual fund market dataset, we exclude index funds, enhanced index funds, funds of funds, and in-house funds of funds. Also, we require funds to be classified as "Equity" in the Global Broad Category Group, and we further identify equity funds based on their Morningstar Category. Next, we use the fund identification provided by Morningstar to aggregate fund share class-level information to fund-level information. To have sufficient observations of net alphas for each fund to mitigate measurement error, we require each of our active equity mutual funds to have at least ten years' return observations, as we use a five-year rolling window to estimate fund net alphas.[18]

The index funds used in the style-matching model are also from Morningstar. We require index funds to have no missing observations in our sample period so that the style-

[18] We also omit some rare cases in which there is a gap with more than five years' return observations missing.

matching model is consistent and stable. The information of the risk-free rate of each country is provided by the International Financial Statistics on the official website of International Monetary Fund (IMF).

For each market, the DAFMI size is calculated as total funds' net assets under management divided by stock market capitalization, which is a relative size measure and which is consistent with FSX and PST. Each market's fund net assets under management and stock market capitalization are also provided by the Global Databases of Morningstar Direct.

All the fund returns are net of administrative and management fees and other costs taken out of fund assets; thus, the fund alphas we estimate are net alphas (net of fees). For comparison purpose and to be consistent with our international model, we measure the fund returns, risk-free returns, fund net assets under management, and stock market capitalization in U.S. dollars.

Table 1 reports the summary statistics of these global active equity mutual fund markets. Panel A presents the summary statistics of market-level variables. It shows that the average DAFMI size greatly varies across the global markets, from around 6.5% in Canada to 0.015% in Germany. The market concentration level also greatly varies across the global markets. The average NHHI value ranges from around 0.36 in Austria to around 0.01 in Taiwan. Panel B shows the summary statistics of fund-level variables. The average R-squared of the style-matching model are quite high in each market (ranging from 97% in Chile to 83% in Mexico), with a low standard deviation in each market. This result indicates that our style-matching benchmarks perform well in tracking the style of the active equity mutual funds, so it is unlikely that our style-matching models omit relevant factors in developing the passive benchmarks. Also, most markets' average fund net returns and fund net alphas are positive with large standard deviations.

Table 1 Summary Statistics

Monthly data is used. Panel A reports the summary statistics for market-level data, and Panel B reports those for fund-level data. We report the number of observations, mean, and standard deviation of each variable. DAFMI Size is the sum of funds' net assets under management in a market, divided by this market's stock market capitalization, and it is in decimal. The Style-Matching Model R^2, DAFMI Share, NHHI, HHI, and 5-Fund-Index are in decimals. Net Return and Net Alpha are in percentages, and both are net of administrative and management fees and other costs taken out of fund assets.

Panel A

	DAFMI Size			NHHI			HHI			5FI		
Global Market	Obs	Mean	Sd	Obs	Mean	Sd	Obs	Mean	Sd	Obs	Mean	Sd
Australia	144	0.0374	0.0065	144	0.0180	0.0048	144	0.021	0.006	144	0.2325	0.0404
Austria	144	0.0178	0.0062	144	0.3612	0.2337	144	0.416	0.232	144	0.8918	0.0823
Belgium	144	0.0020	0.0022	144	0.1224	0.1193	144	0.174	0.139	144	0.6914	0.1673
Brazil	144	0.0017	0.0020	144	0.0237	0.0159	144	0.028	0.019	144	0.2785	0.0971
Canada	144	0.0651	0.0049	144	0.0127	0.0007	144	0.015	0.001	144	0.1818	0.0103
Chile	144	0.0090	0.0130	144	0.0248	0.0082	144	0.053	0.017	144	0.3888	0.0771
China (Mainland)	144	0.0026	0.0011	139	0.1039	0.0637	144	0.219	0.203	144	0.7666	0.2222
Denmark	144	0.0128	0.0039	144	0.0507	0.0158	144	0.076	0.022	144	0.5091	0.0780
Finland	144	0.0077	0.0051	144	0.1014	0.0504	144	0.180	0.090	144	0.7198	0.2124
France	144	0.0109	0.0033	144	0.0251	0.0100	144	0.029	0.011	144	0.2908	0.0488
Germany	144	0.0001	0.0001	144	0.0630	0.0321	144	0.073	0.036	144	0.5237	0.1016
Greece	144	0.0180	0.0061	144	0.1530	0.2439	144	0.199	0.237	144	0.7045	0.1416
Hong Kong	144	0.0008	0.0005	130	0.1330	0.1003	144	0.281	0.267	144	0.8170	0.1219
India	144	0.0034	0.0023	144	0.0837	0.1337	144	0.171	0.226	144	0.5272	0.3156
Israel	144	0.0189	0.0137	144	0.0169	0.0070	144	0.026	0.007	144	0.2572	0.0566
Italy	144	0.0063	0.0030	144	0.0268	0.0119	144	0.040	0.018	144	0.3318	0.0944
Japan	144	0.0034	0.0018	144	0.0976	0.1387	144	0.100	0.139	144	0.3622	0.1757
Korea	144	0.0580	0.0295	144	0.0162	0.0044	144	0.019	0.006	144	0.2115	0.0518
Mexico	144	0.0003	0.0001	144	0.1466	0.1146	144	0.179	0.129	144	0.6584	0.1391
Netherlands	144	0.0021	0.0041	144	0.0841	0.0763	144	0.146	0.098	144	0.7077	0.1512
Norway	144	0.0267	0.0099	144	0.0573	0.0247	144	0.077	0.026	144	0.5101	0.0819
Portugal	144	0.0022	0.0016	117	0.0942	0.0364	117	0.148	0.032	117	0.7513	0.0601
Singapore	144	0.0015	0.0009	144	0.1785	0.1053	144	0.297	0.172	144	0.8351	0.1044
South Africa	144	0.0197	0.0039	144	0.0466	0.0111	144	0.056	0.015	144	0.4269	0.0559
Spain	144	0.0027	0.0025	144	0.0237	0.0086	144	0.032	0.008	144	0.2977	0.0400
Sweden	144	0.0203	0.0097	144	0.0441	0.0484	144	0.067	0.082	144	0.4012	0.2286
Switzerland	144	0.0063	0.0030	144	0.0242	0.0190	144	0.032	0.024	144	0.2877	0.1122
Taiwan	144	0.0212	0.0191	144	0.0094	0.0011	144	0.016	0.001	144	0.1841	0.0086
Thailand	144	0.0141	0.0075	144	0.0201	0.0055	144	0.025	0.006	144	0.2713	0.0392
United Kingdom	144	0.0073	0.0054	144	0.0438	0.0605	144	0.053	0.079	144	0.3510	0.2222

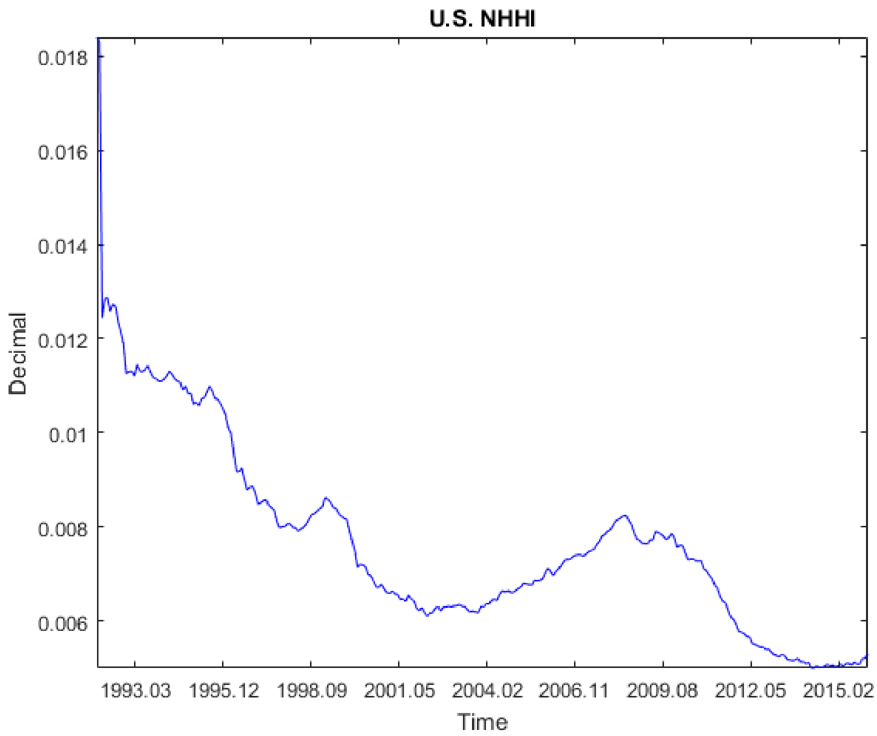
Panel B

	Net Return (%)			Net Alpha (%)			Stlye-Matching Model R^2		
Global Market	Obs	Mean	Sd	Obs	Mean	Sd	Obs	Mean	Sd
Australia	28,197	0.7645	7.1380	28,197	0.2154	1.9923	28,197	0.9285	0.0790
Austria	571	1.1370	8.5657	571	0.1817	2.2183	571	0.9018	0.0416
Belgium	2,690	0.7719	6.1115	2,690	0.2272	1.8962	2,690	0.8959	0.0799
Brazil	11,283	0.3186	10.0539	11,283	0.2377	5.0543	11,283	0.9021	0.0835
Canada	34,810	0.5163	6.1113	34,810	0.0100	1.9951	34,810	0.8877	0.1086
Chile	1,500	1.2362	6.0915	1,500	0.1442	1.0964	1,500	0.9724	0.0273
China (Mainland)	325	0.7791	7.7435	325	0.1558	2.4760	325	0.9479	0.0517
Denmark	2,773	1.2043	6.3738	2,773	0.1216	1.3192	2,773	0.9486	0.0748
Finland	3,356	0.8177	7.0131	3,356	0.0488	1.7025	3,356	0.9293	0.0752
France	22,324	0.5189	6.2557	22,324	0.1273	1.9821	22,324	0.8749	0.1212
Germany	10,630	0.7717	6.8077	10,630	0.1068	2.0592	10,630	0.9066	0.1017
Greece	1,656	-0.4618	9.2689	1,656	0.0328	2.0276	1,656	0.9410	0.0463
Hong Kong	2,134	0.8363	6.3093	2,134	0.1679	2.1306	2,134	0.8640	0.1671
India	11,090	1.2264	8.5167	11,090	0.3846	2.7072	11,090	0.8884	0.0931
Israel	8,466	0.6473	6.9214	8,466	0.1620	2.9836	8,466	0.8287	0.1345
Italy	5,601	0.2624	6.9582	5,601	0.1239	1.2957	5,601	0.9547	0.0569
Japan	35,633	0.3950	5.0012	35,633	0.0656	2.1802	35,633	0.8585	0.1433
Korea	16,819	0.5343	7.4214	16,819	0.0128	1.8854	16,819	0.9491	0.0413
Mexico	3,320	0.5156	6.6944	3,320	0.0493	2.6913	3,320	0.8277	0.1744
Netherlands	1,876	0.6674	6.7878	1,876	0.1517	2.4881	1,876	0.8732	0.1522
Norway	5,738	1.0073	8.4386	5,738	0.0864	2.0225	5,738	0.9316	0.0703
Portugal	1,778	0.1448	7.7343	1,778	0.0488	2.1276	1,778	0.9292	0.0242
Singapore	1,710	0.8435	6.4475	1,710	0.0121	1.3539	1,710	0.9375	0.0551
South Africa	8,995	0.8079	7.2570	8,995	0.1223	2.2113	8,995	0.9137	0.0712
Spain	9,962	0.5168	7.0045	9,962	0.0025	1.2959	9,962	0.9559	0.0758
Sweden	11,197	1.0388	7.2233	11,197	0.0449	1.5148	11,197	0.9468	0.0620
Switzerland	12,788	0.7914	5.2876	12,788	0.0479	1.9316	12,788	0.8940	0.1129
Taiwan	7,308	0.4173	5.5388	7,308	0.1297	2.4830	7,308	0.8365	0.0665
Thailand	14,135	1.0660	6.7499	14,135	0.1447	1.5856	14,135	0.9479	0.0414
United Kingdom	48,939	0.6505	5.4471	48,939	0.0118	1.5585	48,939	0.9206	0.0774

Figure 2 illustrates the monthly NHHI of the U.S. active equity mutual fund market from January 1992 to December 2015. It shows that the concentration level of the U.S. active equity mutual fund market decreases substantially from January 1992 to the end of 2003. It started to increase gradually and decreased again, reaching the lowest point at the current time.

Figure 2 NHHI of the U.S. Active Equity Mutual Fund Market

The NHHI value is in decimals. Sample period is from January 1992 to December 2015.



3.3 Empirical Results

Table 2 reports the empirical results of the concentration–net alpha relation. Panel A shows that, on average, DAFMI net alphas are significantly negatively associated with the DAFMI NHHI and the U.S. NHHI. However, the absolute value of the coefficient of the U.S. NHHI is much larger than that of the DAFMI NHHI, showing that a small change in the concentration in the U.S. AFMI, say 0.01 change in the U.S. NHHI, has a much larger impact on the DAFMI net alphas than the same magnitude change in the DAFMI concentration. Panels B and C use HHI and 5FI as concentration measures, respectively. In model specification (3) of these two panels, we can see that the results are consistent with that in Panel A.

Also, in unreported tests, we test model (47) in Table 2 by using fund fixed-effect regressions instead of the RD method. We also extend the sample period to earlier years, starting from 1997, when quite a few markets do not have observations, and redo the tests in Table 2. We find consistent results in all these robustness checks.

Table 3 reports the empirical results of the concentration–DAFMI size relation. In Panel A, we show that on average, DAFMI size is significantly negatively associated with the U.S. NHHI but insignificantly associated with the DAFMI NHHI. Panels B and C show consistent results in which the concentrations are measured by HHI and 5FI, respectively. In unreported tests using a panel VAR model, regarding DAFMI size and DAFMI concentration measures as endogenous and the U.S. concentration measures as exogenous, we find that DAFMI size has insignificant impact on DAFMI concentration measures. Thus, results in Table 3 are not affected by bias created by the reverse causality between DAFMI size and DAFMI concentration measures.

Results in Table 2 and Table 3 show that, on average, DAFMI markets' fund net alphas and size both decrease with the U.S. NHHI. This finding is consistent with the prediction of our theoretical model under both the exogenous concentration framework and the endogenous concentration framework. Based on our empirical and theoretical results, we conclude that higher concentrations in the U.S. AFMI induce a smaller aggregate impacts of induced optimal effort-level changes on gross alphas than on costs. In other words, higher U.S. AFMI concentration induces lower DAFMI direct benefits.

Table 2 Concentrations and Fund Net Alpha

The dependent variable is fund net alpha and is in percentage. The variables are forward-demeaned. Panels A, B, and C report the results when concentrations are measured by NHHI, HHI, and 5FI, respectively. DAFMI Size is the sum of funds' net assets under management in a market, divided by this market's stock market capitalization, and it is in decimal. The NHHI, HHI, and 5FI are in decimals. For each fund, the time trend variable is equal to one for the first observation and increases by one over each month. Standard errors are clustered by fund and presented in parentheses. The symbols ***, **, and * represent the 1%, 5%, and 10% significant level, respectively, in a two-tail t-test.

Panel A	(1)	(2)	(3)
Lagged DAFMI NHHI	-0.1665*		-0.4504***
	(0.0859)		(0.0886)
Lagged U.S. NHHI		-89.0154***	-97.0355***
		(5.1297)	(5.5256)
Lagged DAFMI Size	-0.0004***	-0.0021***	-0.0026***
	(0.0001)	(0.0002)	(0.0002)
Time Trend	0.6994	0.2269	0.2096
	(0.7668)	(0.7325)	(0.7302)
Obs	325,514	325,514	325,514
R-squared	0.0000	0.0010	0.0011

Panel B	(1)	(2)	(3)
Lagged DAFMI HHI	0.0632		-0.2409***
	(0.0775)		(0.0820)
Lagged U.S. HHI		-89.2356***	-95.2859***
		(5.1572)	(5.7160)
Lagged DAFMI Size	-0.0002	-0.0022***	-0.0025***
	(0.0001)	(0.0002)	(0.0002)
Time Trend	0.7153	0.2275	0.1009
	(0.7687)	(0.7326)	(0.7344)
Obs	325,514	325,514	325,514
R-squared	0.0000	0.0009	0.0010

Panel C	(1)	(2)	(3)
Lagged DAFMI 5FI	0.2317***		-0.0351
	(0.0388)		(0.0458)
Lagged U.S. 5FI		-8.6129***	-8.7985***
		(0.4807)	(0.5815)
Lagged DAFMI Size	0.0001	-0.0027***	-0.0028***
	(0.0002)	(0.0002)	(0.0002)
Time Trend	1.2705	0.2418	0.1442
	(0.7864)	(0.7291)	(0.7577)
Obs	325,514	325,514	325,514
R-squared	0.0002	0.0011	0.0011

Table 3 Concentrations and DAFMI Size

The dependent variable is DAFMI size. Panels A, B, and C report the results when concentrations are measured by NHHI, HHI, and 5FI, respectively. DAFMI Size is the sum of funds' net assets under management in a market, divided by this market's stock market capitalization, and it is in decimal. The NHHI, HHI, and 5FI are in decimals. For each market, the time trend variable is equal to one for the first observation and increases by one over each month. Market fixed effects are controlled. Standard errors are clustered by market and presented in parentheses. The symbols ***, **, and * represent the 1%, 5%, and 10% significant level, respectively, in a two-tail t-test.

Panel A	(1)	(2)	(3)
Lagged DAFMI NHHI	-0.0017		-0.0020
	(0.0015)		(0.0016)
Lagged U.S. NHHI		-0.1066**	-0.1561**
		(0.0393)	(0.0584)
Lagged DAFMI Size	0.9368***	0.9363***	0.9352***
	(0.0128)	(0.0131)	(0.0134)
Time Trend	-0.0000	-0.0000	-0.0000
	(0.0000)	(0.0000)	(0.0000)
Constant	0.0011***	0.0017***	0.0024***
	(0.0003)	(0.0004)	(0.0008)
Obs	4,244	4,290	4,244
R-squared	0.8892	0.8890	0.8894

Panel B	(1)	(2)	(3)
Lagged DAFMI HHI	-0.0011		-0.0015
	(0.0009)		(0.0011)
Lagged U.S. HHI		-0.1070**	-0.1751**
		(0.0394)	(0.0683)
Lagged DAFMI Size	0.9362***	0.9363***	0.9342***
	(0.0132)	(0.0131)	(0.0139)
Time Trend	-0.0000	-0.0000	-0.0000
	(0.0000)	(0.0000)	(0.0000)
Constant	0.0011***	0.0018***	0.0027***
	(0.0004)	(0.0004)	(0.0009)
Obs	4,263	4,290	4,263
R-squared	0.8891	0.8890	0.8893

Panel C	(1)	(2)	(3)
Lagged DAFMI 5FI	-0.0005		-0.0010*
	(0.0004)		(0.0005)
Lagged U.S. 5FI		-0.0105***	-0.0149***
		(0.0038)	(0.0053)
Lagged DAFMI Size	0.9360***	0.9362***	0.9335***
	(0.0133)	(0.0131)	(0.0140)
Time Trend	-0.0000	-0.0000	-0.0000*
	(0.0000)	(0.0000)	(0.0000)
Constant	0.0012***	0.0025***	0.0038***
	(0.0004)	(0.0007)	(0.0012)
Obs	4,263	4,290	4,263
R-squared	0.8890	0.8890	0.8892

Table 4 reports the results of the concentrations–net alpha relations and the concentrations–DAFMI size relations in models (47) and (49), respectively. On average, six (one) DAFMI markets' fund net alphas and size are both significantly negatively (positively) associated with the U.S. NHHI, whereas seven DAFMI markets' fund net alphas and size are both insignificantly associated with the U.S. NHHI. On the other hand, on average, only one (one) DAFMI market's fund net alpha is significantly positively (negatively) associated with the U.S. NHHI, but its size is significantly negatively (positively) associated with the U.S. NHHI. These results show that, in general, DAFMI markets' fund net alphas and size are more likely to move in the same direction as the U.S. NHHI than to move in other directions. This finding is consistent with the prediction of our theoretical model under both the exogenous concentration framework and the endogenous concentration framework.

Regarding the association with the DAFMI NHHI, we find that only one DAFMI markets' fund net alphas and size are both significantly negatively associated with the DAFMI NHHI, and no DAFMI markets' fund net alphas and size are both significantly positively associated with the DAFMI NHHI. Also, five DAFMI markets' fund net alphas and size are both insignificantly associated with the DAFMI NHHI. On the other hand, on average, three (one) DAFMI market's fund net alphas is significantly positively (negatively) associated with the DAFMI NHHI, but its size is significantly negatively (positively) associated with the DAFMI NHHI. Therefore, results of individual markets in Table 4 show that, the relations between DAFMI NHHI and DAFMI fund net alphas and size are more complex, and this finding is consistent with the prediction of our theoretical model under the endogenous concentration framework.

In unreported robustness checks, we replace NHHI by HHI and 5FI, and redo the tests in Table 4. We find consistent results.

The current low and probably decreasing concentration in the U.S. AFMI, given the tradeoff of higher U.S. AFMI concentration is not changed, would benefit (harm) the global DAFMI markets whose fund net alphas and size are, on average, negatively (positively) associated with the U.S. NHHI. Our results show that, on average, the global DAFMI markets in our sample are likely to benefit from the decreasing concentration in the U.S. AFMI.

Table 4 Results of Each Market

Panel A shows the results of the concentrations–net alpha relations for each market, and the dependent variable is fund net alpha. Standard errors are clustered by fund and presented in parentheses. Panel B shows the results of the concentrations–DAFMI size relations for each market, and the dependent variable is DAFMI size. Newey-West estimates of standard error with a maximum lag of 12 months are used and presented in parentheses. In both panels, concentrations are measured by NHHI, and only the coefficients of the DAFMI NHHI and the U.S. NHHI are reported. Other control variables are the same as those in Table 2 and Table 3 and their coefficients are not reported. The symbols ***, **, and * represent the 1%, 5%, and 10% significant level, respectively, in a two-tail t-test.

Panel A

	Lagged DAFMI NHHI		Lagged U.S. NHHI			
Global Market	Coefficient	S.E.	Coefficient	S.E.	Obs	R-squared
Australia	22.4942	(15.0545)	277.3882***	(44.3426)	28,029	0.0404
Austria	4.4711**	(1.2187)	-1,892.0380***	(397.6467)	564	0.0440
Belgium	1.2482*	(0.6324)	-12.9327	(32.3119)	2,668	0.0014
Brazil	-1.2276	(5.4902)	-5.7609	(67.8509)	11,219	0.0052
Canada	-140.9957***	(18.1982)	-62.3097***	(16.6854)	34,675	0.0063
Chile	49.3892***	(15.5754)	-904.6625***	(180.6009)	1,475	0.0418
China (Mainland)	-25.3808	(29.1375)	-354.3659	(524.8610)	321	0.0235
Denmark	6.5880	(3.9022)	-42.3106	(57.9349)	2,763	0.0098
Finland	3.4892***	(0.7299)	-127.0429***	(32.1191)	3,329	0.0142
France	9.9869***	(1.7954)	119.0562***	(17.7270)	22,144	0.0087
Germany	0.9310	(0.6006)	-24.3988	(20.9112)	10,541	0.0013
Greece	-1.3268***	(0.3495)	-109.1678	(64.6555)	1,639	0.0129
Hong Kong	0.4668*	(0.2451)	47.0425	(55.5002)	2,125	0.0111
India	4.7092***	(0.7444)	175.0364***	(46.6073)	10,981	0.0089
Israel	-8.4862**	(3.8349)	-203.9740***	(45.4575)	8,384	0.0038
Italy	-6.9441***	(1.7996)	-141.0607***	(40.9801)	5,554	0.0023
Japan	-0.1759	(0.1941)	-175.2943***	(18.0226)	35,407	0.0071
Korea	100.9002***	(6.6547)	-285.3612***	(32.6749)	16,609	0.0204
Mexico	0.8257	(1.6395)	-354.3416***	(97.6686)	3,291	0.0076
Netherlands	-8.8612**	(3.8478)	-71.5474	(53.3449)	1,861	0.0097
Norway	-21.2901***	(2.9071)	-224.2550***	(39.3722)	5,724	0.0197
Portugal	27.6549***	(4.1280)	-903.2377***	(74.4087)	1,761	0.0492
Singapore	-0.6378	(0.4143)	157.9872*	(75.8613)	1,705	0.0159
South Africa	41.4012***	(6.6139)	-128.2069***	(34.6213)	8,951	0.0198
Spain	-20.4608***	(2.2317)	-107.8012***	(24.6324)	9,877	0.0180
Sweden	2.3351***	(0.6611)	76.4967***	(23.1326)	11,159	0.0013
Switzerland	4.8875***	(1.3860)	16.3939	(26.2527)	12,724	0.0064
Taiwan	144.1748***	(52.8850)	146.9660**	(64.2223)	7,210	0.0072
Thailand	1.8498	(2.4719)	7.7247	(32.4390)	14,045	0.0002
United Kingdom	1.5764***	(0.2118)	-46.0301***	(15.8287)	48,779	0.0044

Panel B

Global Market	Lagged DAFMI NHHI Coefficient	S.E.	Lagged U.S. NHHI Coefficient	S.E.	Obs	R-squared
Australia	0.6249***	(0.2359)	0.7778*	(0.4535)	144	0.8559
Austria	0.0028	(0.0026)	-0.7475*	(0.4508)	144	0.8266
Belgium	0.0021	(0.0021)	-0.0868	(0.0853)	143	0.8852
Brazil	0.0061	(0.0070)	0.2491	(0.1517)	144	0.8845
Canada	0.1890	(0.2570)	-0.1675	(0.2102)	144	0.8559
Chile	0.1389	(0.0843)	-0.8077**	(0.4009)	144	0.9672
China (Mainland)	-0.0007	(0.0008)	0.0854	(0.0557)	138	0.8222
Denmark	0.0074	(0.0181)	0.2887	(0.1745)	144	0.8703
Finland	-0.0010	(0.0020)	0.1541**	(0.0777)	144	0.9817
France	-0.0040	(0.0248)	0.1278	(0.1567)	144	0.9281
Germany	-0.0019***	(0.0005)	-0.0478***	(0.0163)	144	0.2883
Greece	-0.0072***	(0.0021)	-1.7732***	(0.5678)	144	0.7541
Hong Kong	-0.0002**	(0.0001)	-0.0416***	(0.0133)	129	0.9885
India	-0.0006	(0.0009)	0.1122	(0.0733)	143	0.9139
Israel	0.1948	(0.1251)	-4.9103*	(2.7980)	144	0.8292
Italy	0.0086	(0.0094)	-0.1340	(0.1474)	144	0.9417
Japan	0.0006	(0.0007)	-0.0875**	(0.0439)	144	0.9049
Korea	0.3201	(0.3657)	0.1406	(0.7746)	144	0.8945
Mexico	-0.0001*	(0.0001)	-0.0055	(0.0047)	143	0.9780
Netherlands	-0.0001	(0.0028)	0.0811	(0.2892)	144	0.0019
Norway	0.0063	(0.0088)	0.0357	(0.1987)	144	0.9772
Portugal	0.0223	(0.0138)	-0.1995	(0.1405)	116	0.9015
Singapore	-0.0003*	(0.0002)	-0.0426	(0.0411)	144	0.9604
South Africa	-0.0066	(0.0222)	0.0809	(0.0669)	144	0.9682
Spain	0.0051*	(0.0028)	-0.0915***	(0.0309)	144	0.9928
Sweden	-0.0226*	(0.0131)	-0.7917	(0.5138)	144	0.9533
Switzerland	-0.0085	(0.0081)	-0.0764	(0.2098)	144	0.7317
Taiwan	0.7602	(0.6852)	-1.0715*	(0.6254)	144	0.9627
Thailand	0.0161	(0.0236)	-0.2496	(0.2070)	144	0.9701
United Kingdom	-0.0085***	(0.0025)	-0.9730***	(0.2742)	144	0.9197

4 Conclusion

We introduce a theoretical model of IAFMI equilibrium, where we investigate DAFMI performance, size, and managerial efforts under a continuum of DAFMI and FAFMI concentrations. Utilizing PS's and FSX's single-country frameworks, we create a two-country IAFMI framework in which in each country, due to transaction and information costs, investors invest only in DAFMI funds, whereas fund managers invest in both domestic and foreign stock markets. (If both investors and fund managers invest in both countries, the two countries, effectively, become one.) Gross alpha production and managerial efforts' costs depend on concentrations. In particular, higher FAFMI concentration implies more unexplored investment opportunities in the foreign stock market, making effort spent in FAFMI more productive. Moreover, it diverts managerial effort to FAFMI, as it has more unexplored investment opportunities, leaves more unexplored opportunities in DAFMI, and makes effort spent in DAFMI more productive. By symmetry, higher DAFMI concentration induces similar effects.

Our model's comparative statics characterize the association between DAFMI expected net alphas and a continuum of DAFMI and FAFMI concentrations, and that between DAFMI size and a continuum of DAFMI and FAFMI concentrations. In particular, we show that, in equilibrium, if and only if higher FAFMI concentration induces higher (lower) DAFMI direct benefits, it induces higher (lower) DAFMI fund expected net alphas and size. By symmetry, a similar necessary and sufficient condition holds for higher DAFMI concentration.

In addition, the concavity of DAFMI fund expected net alphas in FAFMI concentration indicates that DAFMI direct benefits of effort are concave in FAFMI concentration. This further induces concavity of DAFMI size in FAFMI concentration. On the other hand, equilibrium convex DAFMI size in FAFMI concentration implies convex direct benefits in FAFMI concentration and, consequently, convex DAFMI fund expected net alphas in FAFMI concentration. By symmetry, similar second-order results hold for DAFMI concentration.

We specialize our model to allow for endogenous concentrations, which befits empirical market concentration measures, thus facilitating empirical studies. Although the relation between DAFMI concentration and DAFMI expected net alpha, and that between DAFMI concentration and DAFMI size, become more complex in this framework, we are still able to conclude that DAFMI fund expected net alphas and size, in equilibrium, move in the same direction as FAFMI concentration.

We use the data of 30 active equity mutual fund markets in Morningstar Direct to test our theoretical findings. We find that, on average, DAFMI fund net alphas and size both decrease with the U.S. AFMI concentration.

Our findings provide relevant implications for fund managers, investors, and regulators. If market parameters leading to the current equilibrium persist, the current low, and probably decreasing, concentration in the U.S. AFMI would benefit (harm) the global DAFMI markets whose fund net alphas and size are, on average, negatively (positively) associated with the U.S. AFMI concentration. Our empirical results suggest that, on average, the global DAFMI markets would benefit from the declining U.S. AFMI concentration.

REFERENCES

Berk, J. B. and Binsbergen, J. H., 2015, Measuring Skill in the Mutual Fund Industry, *Journal of Financial Economics* 118, 1–20.

Berk, J. B. and Green, R. C., 2004, Mutual Fund Flows and Performance in Rational Markets, *Journal of Political Economy* 112, 1269–1295.

Chan, K., Covrig, V. and Ng, L., 2005, What Determines the Domestic Bias and Foreign Bias? Evidence from Mutual Fund Equity Allocations Worldwide, *The Journal of Finance* 3, 1495–1534.

Cremers, M., Nair, V. B. and Peyer, U., 2008, Takeover Defenses and Competition: The Role of Stakeholders, *Journal of Empirical Legal Studies* 5, 791–818.

Daniel, K., Grinblatt, M., Titman, S. and Wermers, R., 1997, Measuring Mutual Fund Performance with Characteristic-Based Benchmarks, *Journal of Finance* 52, 1035–1058.

DeFond, M., Hu, X., Hung, M. and Li, S., 2011, The Impact of Mandatory IFRS Adoption on Foreign Mutual Fund Ownership: The Role of Comparability, *Journal of Accounting and Economics* 51, 240–258.

Feldman, D., Saxena, K. and Xu, J., 2019, Is the Active Fund Management Industry Concentrated Enough? *Journal of Financial Economics*, forthcoming.

Ferreira, M. A., Keswani, A., Miguel, A. F. and Ramos, S. B., 2013, The Determinants of Mutual Fund Performance: A Cross-Country Study, *Review of Finance* 17, 483–525.

Ferreira, M. A., Keswani, A., Miguel, A. F. and Ramos, S. B., 2012, The Flow-Performance Relationship Around the World, *Journal of Banking and Finance* 36, 1759–1780.

Goldstein, I. and Pauzner, A., 2004, Contagion of Self-Fulfilling Financial Crises Due to Diversification of Investment Portfolios, *Journal of Economic Theory* 119, 151–183.

Jotikasthira, C., Lundblad, C. and Ramadorai, T., 2012, Asset Fire Sales and Purchases and the International Transmission of Funding Shocks, *The Journal of Finance* 6, 2015–2050.

Khorana, A., Servaes, H. and Tufano, P., 2005, Explaining the Size of the Mutual Fund Industry Around the World, *Journal of Financial Economics* 78, 145–185.

Khorana, A., Servaes, H. and Tufano, P., 2008, Mutual Fund Fees Around the World, *The Review of Financial Studies* 22, 1279–1310.

Pastor, L. and Stambaugh, R. F., 2012, On the Size of the Active Management Industry, *Journal of Political Economy* 120, 740–781.

Pastor, L., Stambaugh, R. F. and Taylor, L. A., 2015, Scale and Skill in Active Management, *Journal of Financial Economics* 116, 23–45.

Sharpe, W. F., 1992, Asset Allocation: Management Style and Performance Measurement, *Journal of Portfolio Management* Winter, 7–19.

Yu, G. and Wahid, A. S., 2014, Accounting Standards and International Portfolio Holdings, *The Accounting Review* 89, 1895–1930.

APPENDIX FOR ONLINE PUBLICATION

Proof of Managers' Maximization Problems Equivalence: Profits and Expected Net Alpha

We prove that when DAFMI managers maximize fund expected net alphas, they maximize profits and must do so in order to survive (that is, have wealth to manage and be solvent).[19] We also show that this maximization leads to a unique Nash equilibrium.

First, we establish that all managers offer the same level of fund expected net alpha. This is the case in PS and FSX, as well, and the rationale here is the same: managers who offer expected net alpha that is lower than the highest offered by some other manager attract no investments, as diversification benefits are irrelevant, negligible to risk-averse investors, and, thus, out of the DAFMI.

Next, we show that DAFMI managers' competition drives the DAFMI (unique) level of expected net alpha to be the highest possible one, in which managers are still solvent; that is, managers charge break-even fees.

Suppose that managers choose profit maximizing optimal effort and fees to set DAFMI funds expected net alpha to be $\bar{\alpha}$. Without loss of generality, we assume that $\bar{\alpha}$ is between zero and the highest expected net alpha that allows solvency. We show that, in equilibrium, $\bar{\alpha}$ is the maximum fund expected net alpha that mangers can produce (while staying solvent). They do that by choosing optimal efforts ${e_i^{11}}^*$ and ${e_i^{12}}^*$ (by fulfilling the condition in Lemma 1.2) and charging a fee f_i^1 such that the fund expected net alpha is exactly $\bar{\alpha}$. Substituting $\bar{\alpha}$ into Equation (10) (our "state" equation that links efforts, alpha production functions, fees, and fund expected net alphas) yields

$$f_i^1 = \widehat{a^1} - \widehat{b^1}\frac{{S^1}^*}{W^1} + A^{11}\left({e_i^{11}}^*;\ H^1, H^2\right) + A^{12}\left({e_i^{12}}^*;\ H^1, H^2\right) - \bar{\alpha}. \tag{50}$$

Denote the profit rate of manager i, as pro_i^1, $pro_i^1 \triangleq f_i^1 - C_i^1\left({e_i^{11}}^*, {e_i^{12}}^*;\ s_i^1, H^1, H^2\right)$. Then, from the last definition and equation (50), we have

$$\begin{gathered}\bar{\alpha} = \widehat{a^1} - \widehat{b^1}\frac{{S^1}^*}{W^1} + A^{11}\left({e_i^{11}}^*;\ H^1, H^2\right) + A^{12}\left({e_i^{12}}^*;\ H^1, H^2\right) - pro_i^1 - \\ c_0^1 - c_{1,i}^1 s_i^1 - c_2^{11}\left({e_i^{11}}^*;\ H^1, H^2\right) - c_2^{12}\left({e_i^{12}}^*;\ H^1, H^2\right).\end{gathered} \tag{51}$$

As all managers produce the same level of expected net alphas, Equation (51) implies an equilibrium condition,

[19] By our model assumptions, insolvent managers are out of the DAFMI.

$$pro_i^1 + c_{1,i}^1 s_i^1 = pro_j^1 + c_{1,j}^1 s_j^1,\ \forall i, j. \tag{52}$$

Next, we consider manager i's total dollar profit function (size in dollars times the per dollar profit rate):

$$s_i^1 pro_i^1 = s_i^1[f_i^1 - c_0^1 - c_{1,i}^1 s_i^1 - c_2^{11}(e_i^{11^*};\ H^1, H^2) - c_2^{12}(e_i^{12^*};\ H^1, H^2)], \tag{53}$$

and by the first-order condition, the optimal fund size given manager i's profit level is

$$s_i^{1^{opt}} = \frac{f_i^1 - c_0^1 - c_2^{11}(e_i^{11^*};\ H^1, H^2) - c_2^{12}(e_i^{12^*};\ H^1, H^2)}{2c_{1,i}^1} = \frac{pro_i^1}{2c_{1,i}^1} + \frac{s_i^1}{2}. \tag{54}$$

The latter equality is useful in presenting the optimal size relative to current size. Note that if manager i maximizes her fund's expected net alpha, the profit rate $pro_i^1 = 0$, and the condition in Equation (54) for $s_i^{1^{opt}}$ does not exist. For some manager j, $j \neq i$, it is possible that pro_j^1 is so high that $s_j^1 < s_j^{1^{opt}}$. In other words, it might be possible that some manager j, $j \neq i$, increases his (dollar) profits by increasing his fund expected net alpha, reducing profit rates and increasing (his fund) size. As manager i does not observe other managers' cost functions,[20] she must consider the above possibility [to avoid losing (all) the wealth she manages].

We now demonstrate that the possible scenario described above indeed occurs. We analyze a simple game between manager i and all other managers, denoted "$-i$". The actions of this game are to either maintain expected net alpha or improve it by an infinitesimal amount. Throughout, we assume that the diversification benefits of investing in both manager i and manager $-i$ are negligible. The payoffs are the profits of the two managers.

If manager i improves her fund expected net alpha infinitesimally and manager $-i$ does not follow, then manager i's profit change by an infinitesimal amount, say ε_i^1, and manager $-i$ receives no investments and earns no profits. If, on the other hand, manager i does not follow manager $-i$ when increasing her fund's expected net alpha infinitesimally, then manager $-i$ profits change by ε_{-i}^1, and manager i receives no investments and earns no profits. Suppose that manager i believes that manager $-i$'s strategy is to improve his or her fund expected net alpha, $\bar{\alpha}$ with (nontrivial) probability p and to maintain $\bar{\alpha}$ with probability $1 - p$. Suppose that manager i's strategy is to improve her fund expected net alpha with probability θ and maintain

[20] If cost functions were common knowledge, each manager could have calculated the DAFMI equilibrium independently.

$\bar{\alpha}$ with probability $1-\theta$.

The payoffs of such a game are illustrated in the following table, with the row (column) representing manager i's ($-i$'s) action, and with manager i's ($-i$'s) payoffs in the first (second) figures in the brackets.[21]

		Maintain $\bar{\alpha}$	Improve Infinitesimally
		$1-p$	p
Maintain $\bar{\alpha}$	$1-\theta$	$(pro_i^1 s_i^1, pro_{-i}^1 s_{-i}^1)$	$(0, pro_{-i}^1 s_{-i}^1 + \varepsilon_{-i}^1)$
Improve Infinitesimally	θ	$(pro_i^1 s_i^1 + \varepsilon_i^1, 0)$	$(pro_i^1 s_i^1 + \varepsilon_i^1, pro_{-i}^1 s_{-i}^1 + \varepsilon_{-i}^1)$

We show that in this game, manager i optimally chooses $\theta = 1$, until reaching the highest level of fund expected net alphas. (This is the break-even/zero-profits point, beyond which the manager becomes insolvent.) As manager i is a generic manager, this implies that all mangers do that. We also show that once managers reach the point of producing the highest level of fund expected net alphas, they are in (a Nash) equilibrium.

The expected payoff of manager i is[22]

$$\pi_i^1 = (1-p)[(1-\theta)pro_i^1 s_i^1 + \theta(pro_i^1 s_i^1 + \varepsilon_i^1)] + p\theta(pro_i^1 s_i^1 + \varepsilon_i^1). \qquad (55)$$

The first-order condition is

$$\frac{d\pi_i^1}{d\theta} = \varepsilon_i^1 + p * pro_i^1 s_i^1. \qquad (56)$$

Equation (56) shows that $\varepsilon_i^1 \rightarrow 0$ implies that $d\pi_i^1/d\theta > 0$. Thus, manager i's optimal choice to maximize π_i^1 is $\theta = 1$. That is, increasing fund expected net alphas increases profits.

As managers keep increasing fund expected net alphas, they reach a level of fund expected net alpha where $\bar{\alpha}$ is the maximum fund expected net alpha. At this point, managers' profit rates must be zero (otherwise managers could use profits to increase fund expected net alphas). Moreover, further increases of fund expected net alphas (by increasing effort levels or decreasing fees) make managers insolvent. Thus, at this point, when $\bar{\alpha}$ is the optimal fund expected net alpha, ε_i^1 and ε_{-i}^1 are negative. Managers are, then, in a Nash equilibrium

---

[21] For simplicity and brevity, we do not introduce new notation to differentiate the infinitesimal profit changes when one or two players move. We use ε_i^1 and ε_{-i}^1 in both cases.

[22] Generally, ε_i^1 and ε_{-i}^1 may be positive or negative, which does not affect our results as they approach zero. If the infinitesimal profit change for manager i, when both players move, was denoted δ_i^1, Equation (56) would have been $\frac{d\pi_i}{d\theta} = \varepsilon_i^1 + p(\delta_i^1 - \varepsilon_i^1) + p \times pro_i s_i$, yielding the same result as ε_i^1 and δ_i^1 approach zero.

(Maintain $\bar{\alpha}$, Maintain $\bar{\alpha}$).

Therefore, each manager will improve his or her fund expected net alpha as long as it is below the maximum fund expected net alpha. Thus, managers' problems of maximizing profits is equivalent to maximizing their funds' expected net alphas.

Next, we show that that managers' optimization leads to a unique DAFMI equilibrium. Because at any fund expected net alpha level below the maximizing level, managers attract no investments and have incentives to increase fund expected net alphas. Because further increasing fund expected net alpha above the maximizing level drives managers to insolvency, this Nash equilibrium is unique.

The proof for FAFMI managers is similar.

Q.E.D.

Proof of Proposition 0

$\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\}$ is a Nash Equilibrium for the following reasons.

1. Given other DAFMI managers' optimal choices, a manager has incentives to not deviate from $\mathbf{e}^{\mathbf{11}^*}$, $\mathbf{e}^{\mathbf{12}^*}$, and $\mathbf{f}^{\mathbf{1}^*}$. If a DAFMI manager deviates from $\mathbf{e}^{\mathbf{11}^*}$, $\mathbf{e}^{\mathbf{12}^*}$, and $\mathbf{f}^{\mathbf{1}^*}$, this manager decreases the fund expected net alpha, either losing all investment or becoming insolvent. Also, DAFMI managers cannot deviate from both in offsetting ways and gain. This is because effort increases do not sufficiently improve performance to justify costs and fee increases, and effort reductions cause too great a loss of performance that cannot be returned to investors through fee reductions. The reason is that a manager's optimal effort and fee together determine his or her fund expected net alpha. If the DAFMI manager deviates from the equilibrium and produces a higher fund expected net alpha, he or she incurs a loss; and if the DAFMI manager deviates and produces a lower fund expected net alpha, he or she receives no investments. We proved these results in the previous proof of maximization problem equivalence.
2. Given DAFMI managers' and other DAFMI investors' optimal choices, a DAFMI investor has no incentive to deviate from $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*}$. This is because, when there are infinitely many small mean-variance risk-averse investors, each investor's choice does not affect fund sizes and, thus, DAFMI size. Changing allocations across funds does not improve an DAFMI investor's portfolio Sharpe ratio, whereas changing allocations between the DAFMI and the passive benchmark decreases the portfolio Sharpe ratio.

$\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\}$ is unique because

$\mathbf{e^{11}}^*$ is unique because, for each DAFMI fund, ${e_i^{11}}^*$ is the unique solution of $B^{11}_{e_i^{11}}\left({e_i^{11}}^*;\ H^1, H^2\right) = 0$;

$\mathbf{e^{12}}^*$ is unique because, for each DAFMI fund, ${e_i^{12}}^*$ is the unique solution of $B^{12}_{e_i^{12}}\left({e_i^{12}}^*;\ H^1, H^2\right) = 0$;

$\mathbf{f^1}^*$ is unique because, for each DAFMI fund, ${f_i^1}^* - C_i^1\left({e_i^{11}}^*, {e_i^{12}}^*;\ {s_i^1}^*, H^1, H^2\right) = 0$, where $C_i^1\left({e_i^{11}}^*, {e_i^{12}}^*;\ {s_i^1}^*, H^1, H^2\right)$ is a deterministic function of ${e_i^{11}}^*$ and ${e_i^{12}}^*$**,** and $\mathbf{e^{11}}^*$ and $\mathbf{e^{12}}^*$ are unique;

$\boldsymbol{\delta^1}^*$ is unique because allocations to DAFMI funds maximize DAFMI investor portfolios' Sharpe ratios, driving fund expected net alphas to the same values. Deviating, thus, cannot help and to the extent that large deviation would affect fund sizes, they will decrease Sharpe ratios. Moreover, the uniqueness of $\mathbf{e^{11}}^*, \mathbf{e^{12}}^*$and $\mathbf{f^1}^*$ rules out the existence of additional equilibrium allocations. We show below (Proposition 2) that each $\boldsymbol{\delta_j^1}^*$ is the weights vector of DAFMI funds' "market portfolio."

The proof for FAFMI managers is similar.

Q.E.D.

Proof of Proposition 1 and Lemma 1.

The proof of Proposition 1.1 is in the Proof of 0.1.

To maximize $\mathrm{E}(\alpha_i^1 | D)\big|_{\left\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta^1}^*\right\}}$, manager i chooses the breakeven management fee. This is because choosing higher fee would decrease expected net alpha and choosing lower fee would induce insolvency. Moreover, changing both fees and effort levels would move managers away from optimal effort levels. Thus,

$$ {f_i^1}^* - C_i^1\left({e_i^{11}}^*, {e_i^{12}}^*;\ {s_i^1}^*, H^1, H^2\right) = 0. \tag{57} $$

This proves Lemma 1.1.

If direct benefit of DAFMI manager i exerted to domestic stock market $B^{11}(e_i^{11};\ H^1, H^2)$ has a partial derivative with respect to effort, at zero effort, is positive, i.e., $B^{11}_{e_i^{11}}(0;\ H^1, H^2) = A^{11}_{e_i^{11}}(0;\ H^1, H^2) - c^{11}_{2\, e_i^{11}}(0;\ H^1, H^2) > 0$, then it pays to exert this effort, and the optimal level is positive, i.e., ${e_i^{11}}^* > 0$. The first-order condition, with respect to effort, to maximize $\mathrm{E}(\alpha_i^1 | D)$ becomes

$$ A^{11}_{e_i^{11}}\left({e_i^{11}}^*;\ H^1, H^2\right) - c^{11}_{2\, e_i^{11}}\left({e_i^{11}}^*;\ H^1, H^2\right) = B^{11}_{e_i^{11}}\left({e_i^{11}}^*;\ H^1, H^2\right) = 0. \tag{58} $$

The related second-order condition, $A^{11}_{e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) < 0$, is satisfied by assumptions. (This is because we assume that productivity effort decreases in scale, i.e., $A^{11}_{e_i^{11},e_i^{11}}(e_i^{11};\ H^1,H^2) < 0, \forall e_i^{11}$, and that the costs of effort increase in scale, i.e., $c^{11}_{2\ e_i^{11},e_i^{11}}(e_i^{11};\ H^1,H^2) > 0,, \forall e_i^{11}$). Thus, $e_i^{11^*}$ is a maximum. (We assume that functional forms of effort productivities and effort costs induce a finite $e_i^{11^*}$.)

By symmetries, the proof of the results regarding $e_i^{12^*}$ is similar to the one regarding $e_i^{11^*}$. This proves Lemma 1.2.

Fully differentiating (58) with respect to H^1 and H^2, we have

$$\frac{de_i^{11^*}}{dH^1} = -\frac{A^{11}_{e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right)}{A^{11}_{e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right)} \tag{59}$$

$$\frac{de_i^{11^*}}{dH^2} = -\frac{A^{11}_{e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right)}{A^{11}_{e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},e_i^{11}}\left(e_i^{11^*};\ H^1,H^2\right)}. \tag{60}$$

Thus, the sign of $de_i^{11^*}/dH^1$ ($de_i^{11^*}/dH^2$) depends on the sign of $A^{11}_{e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^1}\left(e_i^{11^*};\ H^1,H^2\right)$ ($A^{11}_{e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right) - c^{11}_{2\ e_i^{11},H^2}\left(e_i^{11^*};\ H^1,H^2\right)$), because we have shown that the denominator of $de_i^{11^*}/dH^1$ ($de_i^{11^*}/dH^2$) is negative.

By symmetries, the proof of the results regarding $e_i^{12^*}$ is similar to the one regarding $e_i^{11^*}$. This proves Lemma 1.3.

The optimal DAFMI manager effort $e_i^{11^*}$ is determined only by the functions $A^{11}(e_i^{11};\ H^1,H^2)$ and $c_2^{11}(e_i^{11};\ H^1,H^2)$, which are the same across DAFMI funds. Thus, we have $e_i^{11^*} = e_j^{11^*}$ and $B^{11}\left(e_i^{11^*};\ H^1,H^2\right) = B^{11}\left(e_j^{11^*};\ H^1,H^2\right), \forall i,j$. By symmetries, the proof of the results regarding $e_i^{12^*}$ is similar to the one regarding $e_i^{11^*}$. Because, in equilibrium, managers produce the (same) level of fund expected net alphas (Proposition 1.3, which we proved above in the Manager's Equivalence Problems theorem) and, as we just showed, exert the same optimal effort levels (i.e., $e_i^{11^*} = e_j^{11^*}$ and $e_i^{12^*} = e_j^{12^*}, \forall i,j$), from the definition of fund net alpha in (8), we have that $f_i^{1^*} = f_j^{1^*}, \forall i,j$.

These prove Proposition 1.7.

As $e_i^{11^*}$, $e_i^{12^*}$, and $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11^*}},\mathbf{e^{12^*}},\mathbf{f^{1^*}},\boldsymbol{\delta}^{1^*}\}}$ are the same across DAFMI funds, we further have $C_i^1\left(e_i^{11^*}, e_i^{12^*};\ s_i^{1^*}, H^1, H^2\right) = C_j^1\left(e_j^{11^*}, e_j^{12^*};\ s_j^{1^*}, H^1, H^2\right), \forall i, j$, and recall that $C_i^1\left(e_i^{11^*}, e_i^{12^*};\ s_i^{1^*}, H^1, H^2\right) = c_0^1 + c_{1,i}^1 s_i^{1^*} + c_2^{11}\left(e_i^{11^*};\ H^1, H^2\right) + c_2^{12}\left(e_i^{12^*};\ H^1, H^2\right)$. As c_0^1, $e_i^{11^*}$, $e_i^{12^*}$ and $C_i^1\left(e_i^{11^*}, e_i^{12^*};\ s_i^{1^*}, H^1, H^2\right)$ are the same across DAFMI funds, we have the following relationship between different funds' sizes and costs:

$$c_{1,i}^1 s_i^{1^*} = c_{1,j}^1 s_j^{1^*}, \forall i, j, \tag{61}$$

or $s_i^{1^*}/s_j^{1^*} = c_{1,j}^1/c_{1,i}^1, \forall i, j$.

This proves Lemma 1.6.

Summing $s_i^{1^*}/s_j^{1^*}$ with respect to i, $i = 1,2, \ldots, M^1$, we have $\sum_{i=1}^{M^1} \frac{s_i^{1^*}}{s_j^{1^*}} = \frac{S^{1^*}}{s_j^{1^*}} = \sum_{i=1}^{M^1} \frac{c_{1,j}^1}{c_{1,i}^1}$. Inversing the second equality and exchanging the subscripts j and i gives

$$\frac{s_j^{1^*}}{S^{1^*}} = \left[c_{1,i}^1 \sum_{j=1}^{M^1} \left(c_{1,i}^1\right)^{-1}\right]^{-1}, \forall i. \tag{62}$$

This proves Lemma 1.7.

Using the break-even fee condition and Equation, we can write

$$\begin{aligned} f_i^{1^*} &= C_i^1\left(e_i^{11^*}, e_i^{12^*};\ s_i^{1^*}, H^1, H^2\right) \\ &= c_0^1 + c_{1,i}^1 s_i^{1^*} + c_2^{11}\left(e_i^{11^*};\ H^1, H^2\right) + c_2^{12}\left(e_i^{12^*};\ H^1, H^2\right) \\ &= c_0^1 + c_{1,i}^1 \frac{s_i^{1^*}}{S^{1^*}} \left(\frac{S^{1\,*}}{W^1}\right) W^1 + c_2^{11}\left(e_i^{11^*};\ H^1, H^2\right) + c_2^{12}\left(e_i^{12^*};\ H^1, H^2\right) \\ &= c_0^1 + \left[\sum_{j=1}^{M^1} \left(c_{1,j}^1\right)^{-1}\right]^{-1} W^1 \left(\frac{S^{1\,*}}{W^1}\right) + c_2^{11}\left(e_i^{11^*};\ H^1, H^2\right) \\ &\quad + c_2^{12}\left(e_i^{12^*};\ H^1, H^2\right). \end{aligned} \tag{63}$$

Fully differentiate $f_i^{1^*}$ with respect to H^1 and H^2, we have

$$\begin{aligned} \frac{df_i^{1^*}}{dH^1} &= \left[\sum_{j=1}^{M^1} \left(c_{1,j}^1\right)^{-1}\right]^{-1} W^1 \frac{d(S^1/W^1)^*}{dH^1} + c_{2\,e_i^{11}}^{11}\left(e_i^{11^*};\ H^1, H^2\right) \frac{de_i^{11^*}}{dH^1} \\ &\quad + c_{2\,e_i^{12}}^{12}\left(e_i^{12^*};\ H^1, H^2\right) \frac{de_i^{12^*}}{dH^1} \end{aligned} \tag{64}$$

$$\frac{df_i^{1*}}{dH^2} = \left[\sum_{j=1}^{M^1} \left(c_{1,j}^1\right)^{-1}\right]^{-1} W^1 \frac{d(S^1/W^1)^*}{dH^2} + c_{2\,e_i^{11}}^{11}\left(e_i^{11^*};\, H^1, H^2\right)\frac{de_i^{11^*}}{dH^2} + c_{2\,e_i^{12}}^{12}\left(e_i^{12^*};\, H^1, H^2\right)\frac{de_i^{12^*}}{dH^2}. \tag{65}$$

Thus, whether each country's higher concentrations induce higher equilibrium optimal fees depends on whether they induce changes in equilibrium DAFMI sizes and in equilibrium optimal effort levels in each country that are aggregately positive.

This proves Lemma 1.4.

Fully differentiate $B^{11}\left(e_i^{11^*};\, H^1, H^2\right)$ with respect to H^1 and H^2, and use the result $B_{e_i^{11}}^{11}\left(e_i^{11^*};\, H^1, H^2\right) = 0$, and we have

$$\begin{aligned}\frac{dB^{11}\left(e_i^{11^*};\, H^1, H^2\right)}{dH^1} &= B_{e_i^{11}}^{11}\left(e_i^{11^*};\, H^1, H^2\right)\frac{de_i^{11^*}}{dH^1} + A_{H^1}^{11}\left(e_i^{11^*};\, H^1, H^2\right) - c_{2\,H^1}^{11}\left(e_i^{11^*};\, H^1, H^2\right) \\ &= A_{H^1}^{11}\left(e_i^{11^*};\, H^1, H^2\right) - c_{2\,H^1}^{11}\left(e_i^{11^*};\, H^1, H^2\right)\end{aligned} \tag{66}$$

$$\begin{aligned}\frac{dB^{11}\left(e_i^{11^*};\, H^1, H^2\right)}{dH^2} &= B_{e_i^{11}}^{11}\left(e_i^{11^*};\, H^1, H^2\right)\frac{de_i^{11^*}}{dH^2} + A_{H^2}^{11}\left(e_i^{11^*};\, H^1, H^2\right) - c_{2\,H^2}^{11}\left(e_i^{11^*};\, H^1, H^2\right) \\ &= A_{H^2}^{11}\left(e_i^{11^*};\, H^1, H^2\right) - c_{2\,H^2}^{11}\left(e_i^{11^*};\, H^1, H^2\right).\end{aligned} \tag{67}$$

By symmetries, the proof of the results regarding $e_i^{12^*}$ is similar to the one regarding $e_i^{11^*}$. Thus, where either country concentrations are higher, equilibrium manager i's direct benefits of effort in the respective country are higher (lower) if and only if higher concentrations induce, in the respective country, a larger (smaller) impact on gross alphas than on costs.

This proves Lemma 1.5.

In equilibrium, all DAFMI funds' expected alphas are the same, i.e., $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11^*}},\mathbf{e^{12^*}},\mathbf{f^{1^*}},\boldsymbol{\delta}^{1^*}\}}$ is the same across for all funds. Consequently, DAFMI fund expected returns $\mathrm{E}\left(r_{F,i}^1|D\right)\big|_{\{\mathbf{e^{11^*}},\mathbf{e^{12^*}},\mathbf{f^{1^*}},\boldsymbol{\delta}^{1^*}\}} = \mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e^{11^*}},\mathbf{e^{12^*}},\mathbf{f^{1^*}},\boldsymbol{\delta}^{1^*}\}} + \mu_p$ are the same in equilibrium.

In addition, as DAFMI funds have the same expected alphas, they have the same expected returns. The source of DAFMI fund returns' variance is the same across funds, and $\mathrm{Var}\left(r_{F,i}^{1}|D\right)\big|_{\left\{\mathbf{e}^{\mathbf{11}^*},\mathbf{e}^{\mathbf{12}^*},\mathbf{f}^{\mathbf{1}^*},\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} = \sigma_p^2 + \sigma_{a^1}^2 + \left(\frac{S^{1^*}}{W^1}\right)^2 \sigma_{b^1}^2 + \sigma_x^2 + \sigma_\varepsilon^2,\ \forall i$. That is, the DAFMI fund return variance is the same across funds. Combining these results, we conclude that all managers offer the same competitive Sharpe ratio.

This proves Proposition 1.4 and 1.5.

We note that Proposition 1.3 is a direct consequence of Lemma 1.6 and 1.7.

Finally, to prove Proposition 1.2, recalling that aggregate skill is $\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}$, we differentiate $\frac{S^{1^*}}{W^1}$ by parts to get

$$\frac{d\frac{S^{1^*}}{W^1}}{d\left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}} \frac{d\left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}}{d\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}} > 0. \tag{68}$$

The inequality is correct because, from Equation (76), the first multiplicand of the LHS is negative, and because the variables in the second multiplicand of the LHS are positive, the second multiplicand is negative.

This proves Proposition 1.2.

This proves Proposition 1 except for Proposition 1.6, which is proved in the next section.

Q.E.D.

Proof of Proposition 1.6, Proposition 2, and Corollary to Proposition 2

DAFMI investor j's portfolio Sharpe ratio is

$$\begin{aligned}
&\frac{\mathrm{E}\left(r_j^1|D\right)}{\sqrt{\mathrm{Var}\left(r_j^1|D\right)}} \\
&= \frac{\mu_p + \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\iota}_{\mathbf{M^1}}\left[\widehat{a^1} - \widehat{b^1}\frac{S^{1^*}}{W^1} + A^{11}\left(e_i^{11^*};\ H^1,H^2\right) + A^{12}\left(e_i^{12^*};\ H^1,H^2\right) - f_i^{1^*}\right]}{\sqrt{\sigma_p^2 + \left[\sigma_{a^1}^2 + \sigma_{x^1}^2 + \left(\frac{S^{1^*}}{W^1}\right)^2\sigma_{b^1}^2\right]\left(\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\iota}_{\mathbf{M^1}}\right)^2 + \sigma_{\varepsilon^1}^2\left(\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}}\right)}} \qquad (69) \\
&= \frac{\mu_p + \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\iota}_{\mathbf{M^1}}\left\{-\frac{S^{1^*}}{W^1}\left\{\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1 + \widehat{b^1}\right\} + X\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)\right\}}{\sqrt{\sigma_p^2 + \left[\sigma_{a^1}^2 + \sigma_{x^1}^2 + \left(\frac{S^{1^*}}{W^1}\right)^2\sigma_{b^1}^2\right]\left(\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\iota}_{\mathbf{M^1}}\right)^2 + \sigma_{\varepsilon^1}^2\left(\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^{\mathbf{T}}}\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}}\right)}}.
\end{aligned}$$

The second equality holds because $f_i^{1^*} - C_i^1\left(e_i^{11^*},e_i^{12^*};\ s_i^{1^*},H^1,H^2\right) = 0$, the definition of

$X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$, $c_{1,i}^1 s_i^{1^*} = c_{1,i}^1 \frac{s_i^{1^*}}{S^{1^*}} \left(\frac{S^{1^*}}{W^1}\right) W^1$, and Equation (62). We assume that marginal diversification benefits of investing in one more fund is trivial, so we set $\sigma_{\varepsilon^1}^2 \left(\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}\,\mathbf{T}} \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}}\right) \to \infty$ when solving the problem. When maximizing DAFMI investor j's portfolio Sharpe ratio, we take the first-order condition with respect to $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}}$. We have

$$\frac{\mu_p}{\sigma_p^2}\left[\sigma_{a^1}^2 + \sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2 + \sigma_{x^1}^2\right] \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} - \left\{\left[\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1 + \widehat{b^1}\right\} \frac{S^{1^*}}{W^1} + X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right) = 0. \tag{70}$$

Notice that each small investor regards $\frac{S^{1^*}}{W^1}$ as given since each of them cannot affect this ratio. Substitute $\gamma \triangleq \mu_p/\sigma_p^2, (\gamma > 0)$ and symmetric equilibrium condition $\frac{S^{1^*}}{W^1} = \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}}$ into (70), we have

$$-\gamma\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^3 - \left\{\gamma\sigma_{a^1}^2 + \gamma\sigma_{x^1}^2 + \widehat{b^1} + \left[\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} \frac{S^{1^*}}{W^1} + X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right) = 0. \tag{71}$$

If the constraint $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} \leq 1$ is not binding (i.e., $\frac{S^{1^*}}{W^1} < 1$), the equilibrium optimal $\frac{S^{1^*}}{W^1}$ is a real positive solution of this cubic equation. This is because the condition $X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right) > 0, \forall\, H^1, H^2$ (positivity of the lowest order polynomial coefficient) and the negativity of the two higher order polynomial coefficients $-\gamma\sigma_{b^1}^2$, and $-\left\{\gamma\sigma_{a^1}^2 + \gamma\sigma_{x^1}^2 + \widehat{b^1} + \left[\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}$, (i.e., $-\gamma\sigma_{b^1}^2 < 0$, and $-\left\{\gamma\sigma_{a^1}^2 + \gamma\sigma_{x^1}^2 + \widehat{b^1} + \left[\sum_{i=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\} < 0$) guarantee the existence of exactly one positive real solution for $\frac{S^{1^*}}{W^1}$ (and two imaginary ones). Also, as each DAFMI investor cannot affect the value of $\frac{S^{1^*}}{W^1}$, the cubic equation above shows that the solution for $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} = \frac{S^{1^*}}{W^1}$ is unique given the parameter values and the market $\frac{S^{1^*}}{W^1}$.

If the constraint $\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} \leq 1$ is binding, (i.e., $\frac{S^{1^*}}{W^1} = 1$), there is an obviously unique solution where DAFMI investors maximize their portfolio Sharpe ratios by allocating all their wealth to the DAFMI (no international passive index holdings).

We, thus, demonstrated that $\boldsymbol{\delta}_{\mathbf{i}}^{\mathbf{1}^*} = \boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*}, \forall i,j$, such that ${\boldsymbol{\delta}_{\mathbf{j}}^{\mathbf{1}^*}}^{\mathbf{T}} \boldsymbol{\iota}_{\mathbf{M^1}} = \frac{S^{1^*}}{W^1}$, induces a unique equilibrium.

This proves Proposition 1.6 and Proposition 2.2.

In addition, from the proofs above, we have shown that

$$\mathrm{E}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} = -\frac{S^{1^*}}{W^1}\left\{\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1 + \widehat{b^1}\right\} + X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right). \tag{72}$$

Also, by taking variance on both sides of the alpha production function, we have

$$\mathrm{Var}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} = \sigma_{a^1}^2 + \left(\frac{S^{1^*}}{W^1}\right)^2 \sigma_{b^1}^2. \tag{73}$$

By substituting (72) into (71) and rearranging, we have

$$\mathrm{E}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} = \frac{S^{1^*}}{W^1}\,\gamma\left[\sigma_{x^1}^2 + \mathrm{Var}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}}\right]. \tag{74}$$

Because all the components on the right-hand side of the equation above are positive, we have $\mathrm{E}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} > 0$. The intuition is that a portfolio with allocations to DAFMI funds and the international passive benchmark is always riskier (i.e., higher portfolio return variance) than a portfolio with allocations only to the international passive benchmark. If $\mathrm{E}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} = 0$, because of a sufficiently large amount of investment in DAFMI funds, DAFMI investors can always improve their portfolio Sharpe ratios (in particular, reduce their portfolios risk) by shifting wealth from DAFMI to the international passive benchmark. Thus, we should have $\mathrm{E}(\alpha_i^1 | D)\Big|_{\{\mathbf{e^{11}}^*, \mathbf{e^{12}}^*, \mathbf{f^1}^*, \boldsymbol{\delta}^{\mathbf{1}^*}\}} > 0$ to induce investments to DAFMI funds.

This proves Proposition 2.1.

This proves Proposition 2.

Where $\frac{S^{1^*}}{W^1} < 1$, fully differentiating (71) with respect to $X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$ and $\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1$, respectively, we have

$$\frac{d\frac{S^{1^*}}{W^1}}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)} = \frac{1}{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2 + \sigma_{a^1}^2 + \sigma_{x^1}^2\right] + \widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1} > 0, \tag{75}$$

$$\frac{d\frac{{S^1}^*}{W^1}}{d\left\{\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}}=\frac{-\frac{{S^1}^*}{W^1}}{\gamma\left[3\sigma_{b^1}^2\left(\frac{{S^1}^*}{W^1}\right)^2+\sigma_{a^1}^2+\sigma_{x^1}^2\right]+\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1}<0. \tag{76}$$

The inequalities above hold because the parameters are positive.

This proves the Corollary of Proposition 2.

Q.E.D.

Proof of Proposition 3

Also, where $\frac{{S^1}^*}{W^1}<1$, by the chain rule, we have

$$\frac{d\frac{{S^1}^*}{W^1}}{dH^1}=\frac{d\frac{{S^1}^*}{W^1}}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\frac{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}{dH^1}$$

$$=\frac{d\frac{{S^1}^*}{W^1}}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\left[A_{H^1}^{11}\left(e_i^{11^*};\ H^1,H^2\right)+A_{H^1}^{12}\left(e_i^{12^*};\ H^1,H^2\right)\right.$$
$$\left.-c_{2\,H^1}^{11}\left(e_i^{11^*};\ H^1,H^2\right)-c_{2\,H^1}^{12}\left(e_i^{12^*};\ H^1,H^2\right)\right] \tag{77}$$

$$=\frac{d\frac{{S^1}^*}{W^1}}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\left[\frac{dB^{11}\left(e_i^{11^*};\ H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};\ H^1,H^2\right)}{dH^1}\right].$$

Recall that $\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}>0$. Thus, we have that $\frac{d\left(S^1/W^1\right)^*}{dH^1}\geq 0\ (<0)$ if and only if $\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\geq 0\ (<0)$.

Fully differentiating $\frac{d\frac{{S^1}^*}{W^1}}{dH^1}$ with respect to H^1 again, we have

$$\frac{d^2\left(S^1/W^1\right)^*}{d{H^1}^2}=\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{d{H^1}^2}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{d{H^1}^2}\right]+$$
$$\frac{d^2\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)^2}\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1}+\frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\right] \tag{78}$$

$$=\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)}\left[\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{d{H^1}^2}+\frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{d{H^1}^2}\right]-$$

$$6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* \left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1}\right]^2 \left[\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)}\right]^3.$$

The second equality holds because, by differentiating (75) with respective to $X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$ again, we have

$$\frac{d^2(S^1/W^1)^*}{dX\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)^2} = -6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* \left[\frac{d(S^1/W^1)^*}{dX\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)}\right]^3 \tag{79}$$

and then substitute the result.

Notice that $6\gamma^1\sigma_{b^1}^2 \frac{S^1}{W^1}^* > 0$, $\left[\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1}\right]^2 > 0$, and $\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*}; H^1, H^2\right)} > 0$. Thus, if $\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^{1^2}} \leq 0$, then $\frac{d^2\left(S^1/W^1\right)^*}{dH^{1^2}} \leq 0$, and if $\frac{d^2\left(S^1/W^1\right)^*}{dH^{1^2}} \geq 0$, then $\frac{d^2B^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^{1^2}} \geq 0$.

Following similar mathematics, we can prove the results of $\frac{d\left(S^1/W^1\right)^*}{dH^2}$, $\frac{d^2\left(S^1/W^1\right)^*}{dH^{2^2}}$, and $\frac{d^2\left(S^1/W^1\right)^*}{dH^1 dH^2}$ where $\frac{S^1}{W^1}^* < 1$.

Where $\frac{S^1}{W^1}^* = 1$, $\frac{S^1}{W^1}^*$ does not depend on H^1 or H^2.

Moreover, where $\frac{S^1}{W^1}^* < 1$, fully differentiating (72) with respect to H^1, we have

$$\left.\frac{d\mathrm{E}(\alpha_i^1 | D)}{dH^1}\right|_{\left\{\mathbf{e^{11^*}}, \mathbf{e^{12^*}}, \mathbf{f^{1^*}}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$$

$$= -\frac{d(S^1/W^1)^*}{dH^1}\left\{\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1 + \widehat{b^1}\right\} \tag{80}$$

$$+ \left[\frac{dB^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^1}\right]$$

$$= \left[\frac{dB^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^1}\right]\left\{1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{d(S^1/W^1)^*}{dX\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)}\right\}.$$

By (75), we have

$$1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{d\left(\frac{S^1}{W^1}\right)^*}{dX\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)} = \frac{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2 + \sigma_{a^1}^2 + \sigma_{x^1}^2\right]}{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^{1^*}}{W^1}\right)^2 + \sigma_{a^1}^2 + \sigma_{x^1}^2\right] + \widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1} > 0. \tag{81}$$

The last inequality holds because the values of all the parameters and variables in the equation are positive. Then, from the result of this inequality, Equation (80) implies that $\left.\frac{d\mathrm{E}\left(\alpha_i^1 | D\right)}{dH^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}} \geq 0\ (< 0)$ if and only if $\frac{dB^{11}\left(e_i^{11^*}; H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*}; H^1, H^2\right)}{dH^1} \geq 0\ (< 0)$.

Also, differentiate $\left.\frac{d\mathrm{E}\left(\alpha_i^1 | D\right)}{dH^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*}, \mathbf{e}^{\mathbf{12}^*}, \mathbf{f}^{\mathbf{1}^*}, \boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$ again with respect to H^1. Using the result of (79), we have

$$\left.\frac{d^2\mathrm{E}(\alpha_i^1|D)}{d{H^1}^2}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta^1}^*\}}$$

$$= \left[\frac{d^2B^{11}\left(e_i^{11^*};\ H^1,H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*};\ H^1,H^2\right)}{d{H^1}^2}\right]\Bigg\{1$$

$$-\left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{d(S^1/W^1)^*}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\Bigg\}$$

$$-\left[\frac{dB^{11}\left(e_i^{11^*};\ H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};\ H^1,H^2\right)}{dH^1}\right]\Bigg\{\widehat{b^1}$$

$$+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\Bigg\}\frac{d^2(S^1/W^1)^*}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)^2} \qquad (82)$$

$$= \left[\frac{d^2B^{11}\left(e_i^{11^*};\ H^1,H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*};\ H^1,H^2\right)}{d{H^1}^2}\right]\Bigg\{1$$

$$-\left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{d(S^1/W^1)^*}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\Bigg\}$$

$$+ 6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* \Bigg[\frac{dB^{11}\left(e_i^{11^*};\ H^1,H^2\right)}{dH^1}$$

$$+\frac{dB^{12}\left(e_i^{12^*};\ H^1,H^2\right)}{dH^1}\Bigg]^2 \left[\frac{d(S^1/W^1)^*}{dX\left(e_i^{11^*},e_i^{12^*};\ H^1,H^2\right)}\right]^3.$$

Notice that $6\gamma\sigma_{b^1}^2 \frac{S^1}{W^1}^* > 0$, $\left[\frac{dB^{11}\left(e_i^{11^*};H^1,H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};H^1,H^2\right)}{dH^1}\right]^2 > 0$, $\frac{d\left(S^1/W^1\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)} > 0$, and $1 - \left\{\widehat{b^1} + \left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1} W^1\right\}\frac{d\left(\frac{S^1}{W^1}\right)^*}{dX\left(e_i^{11^*},e_i^{12^*};H^1,H^2\right)} > 0$. Thus, if $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1|D\right)}{d{H^1}^2}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta^1}^*\}} \leq 0$, then $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{d{H^1}^2} \leq 0$, and if $\frac{d^2B^{11}\left(e_i^{11^*};H^1,H^2\right)}{d{H^1}^2} + \frac{d^2B^{12}\left(e_i^{12^*};H^1,H^2\right)}{d{H^1}^2} \geq 0$, then $\left.\frac{d^2\mathrm{E}\left(\alpha_i^1|D\right)}{d{H^1}^2}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta^1}^*\}} \geq 0$.

Where $\frac{S^1}{W^1}^* = 1$, fully differentiating (72) with respect to H^1, we have

$$\left.\frac{d\mathrm{E}(\alpha_i^1|D)}{dH^1}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}} = \frac{dX\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)}{dH^1} = \\ = \frac{dB^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^1} \tag{83}$$

$$\left.\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{1^2}}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}} \\ = \frac{d^2B^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^{1^2}}. \tag{84}$$

Following similar mathematics, we can prove the results of $\left.\frac{d\mathrm{E}(\alpha_i^1|D)}{dH^2}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}}$, $\left.\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{2^2}}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}}$, and $\left.\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^1dH^2}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}}$.

Putting the results in this section together, where $\frac{S^1}{W^1}^* < 1$, we have

$$\frac{d\frac{S^1}{W^1}^*}{dH^1} \geq 0(< 0) \Leftrightarrow = \frac{dB^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^1} + \frac{dB^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^1} \geq 0(\\ < 0) \Leftrightarrow \left.\frac{d\mathrm{E}(\alpha_i^1|D)}{dH^1}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}} \geq 0(< 0) \tag{85}$$

$$\frac{d^2(S^1/W^1)^*}{dH^{1^2}} > 0 \Rightarrow \frac{d^2B^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^{1^2}} > 0 \\ \Rightarrow \left.\frac{d^2\mathrm{E}(\alpha_i^1|D)}{dH^{1^2}}\right|_{\{\mathbf{e^{11}}^*,\mathbf{e^{12}}^*,\mathbf{f^1}^*,\boldsymbol{\delta}\mathbf{^1}^*\}} > 0 \tag{86}$$

$$\left.\frac{d^2E(\alpha_i^1|D)}{dH^{1^2}}\right|_{\{e^{11^*},e^{12^*},f^{1^*},\delta^{1^*}\}} < 0 \\ \Rightarrow \frac{d^2B^{11}\left(e_i^{11^*};\ H^1, H^2\right)}{dH^{1^2}} + \frac{d^2B^{12}\left(e_i^{12^*};\ H^1, H^2\right)}{dH^{1^2}} < 0 \\ \Rightarrow \frac{d^2(S^1/W^1)^*}{dH^{1^2}} < 0. \tag{87}$$

This proves Proposition 3.

Q.E.D.

Proof of Proposition 4

Where $\frac{S^1}{W^1}^* < 1$, if we fully differentiate $\frac{S^1}{W^1}^*$ with respect to $c_{1,i}^1$, we have

$$\frac{d\frac{S^1}{W^1}^*}{dc_{1,i}^1} = \frac{-\frac{S^1}{W^1}^* W^1}{\left\{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^1}{W^1}^*\right)^2+\sigma_{a^1}^2+\sigma_{x^1}^2\right]+\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}\left(\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right)^2\left(c_{1,i}^1\right)^2} < 0. \tag{88}$$

The first equality holds because the derivative of $X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$ with respect to $c_{1,i}^1$ is 0, and the last inequality holds because all the parameter and variable values in the equation above are positive and we have a negative sign in the numerator.

Also, fully differentiating $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}}$ with respect to $c_{1,i}^1$, and substituting the result of $\frac{d\frac{S^1}{W^1}^*}{dc_{1,i}^1}$ above, we have

$$\frac{d\mathrm{E}(\alpha_i^1|D)}{dc_{1,i}^1}\Bigg|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}} = -\left\{\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}\frac{d\left(\frac{S^1}{W^1}\right)^*}{dc_{1,i}^1} - \frac{-\left(\frac{S^1}{W^1}\right)^* W^1}{\left(\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right)^2\left(c_{1,i}^1\right)^2} \tag{89}$$

$$= \frac{-\gamma\left[3\sigma_{b^1}^2\left(\frac{S^1}{W^1}^*\right)^2+\sigma_{a^1}^2+\sigma_{x^1}^2\right]\frac{S^1}{W^1}^* W^1}{\left\{\gamma\left[3\sigma_{b^1}^2\left(\frac{S^1}{W^1}^*\right)^2+\sigma_{a^1}^2+\sigma_{x^1}^2\right]+\widehat{b^1}+\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1\right\}\left(\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right)^2\left(c_{1,i}^1\right)^2} < 0.$$

The last inequality holds because all the parameter and variable values in the equation above are positive and we have a negative sign in the numerator.

Where $\frac{S^1}{W^1}^* = 1$, $\frac{d\left(S^1/W^1\right)^*}{dc_{1,i}^1} = 0$, and as $\mathrm{E}(\alpha_i^1|D)\big|_{\{\mathbf{e}^{11^*},\mathbf{e}^{12^*},\mathbf{f}^{1^*},\boldsymbol{\delta}^{1^*}\}} = -\left\{\left[\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right]^{-1}W^1+\widehat{b^1}\right\}+X\left(e_i^{11^*}, e_i^{12^*};\ H^1, H^2\right)$, we have

$$\left.\frac{d\mathrm{E}\left(\alpha_i^1\middle|D\right)}{dc_{1,i}^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*},\mathbf{e}^{\mathbf{12}^*},\mathbf{f}^{\mathbf{1}^*},\boldsymbol{\delta}^{\mathbf{1}^*}\right\}} = -\frac{-W^1}{\left(\sum_{j=1}^{M^1}\left(c_{1,i}^1\right)^{-1}\right)^2\left(c_{1,i}^1\right)^2} < 0. \tag{90}$$

The last inequality holds all the parameter and variable values are positive and we have a negative sign in the numerators. The result of $\left.\frac{d\mathrm{E}\left(\alpha_j^1\middle|D\right)}{dc_{1,i}^1}\right|_{\left\{\mathbf{e}^{\mathbf{11}^*},\mathbf{e}^{\mathbf{12}^*},\mathbf{f}^{\mathbf{1}^*},\boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$, $\forall j \neq i$ are similar.

This proves Proposition 4.2.

As $\frac{s_i^{1^*}}{S^{1^*}}$ decreases in $c_{1,i}^1$ whereas $\frac{s_j^{1^*}}{S^{1^*}}, \forall i \neq j$, increases in $c_{1,i}^1$, from results above, we find that $\mathrm{E}(\alpha_i^1|D)|_{\left\{\mathbf{e}^{\mathbf{11}^*},\mathbf{e}^{\mathbf{12}^*},\mathbf{f}^{\mathbf{1}^*},\boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$ and $\frac{s_i^{1^*}}{S^{1^*}}$ are increasing/decreasing in the same direction due to changes in $c_{1,i}^1$, and that $\mathrm{E}\left(\alpha_j^1\middle|D\right)|_{\left\{\mathbf{e}^{\mathbf{11}^*},\mathbf{e}^{\mathbf{12}^*},\mathbf{f}^{\mathbf{1}^*},\boldsymbol{\delta}^{\mathbf{1}^*}\right\}}$ and $\frac{s_j^{1^*}}{S^{1^*}}, \forall i \neq j$ are increasing/decreasing inversely due to changes in $c_{1,i}^1$, whether $\frac{S^{1^*}}{W^1} < 1$ or $\frac{S^{1^*}}{W^1} = 1$.

This proves Proposition 4.1.

This proves Proposition 4.

Q.E.D.