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Tutorial: How to estimate and measure electrical noise

Contents

1 Introduction

Any electronic measurement will inevitably be contaminated by noise. Understanding and minimizing this noise is a large part of the experimentalist's skill. Often you want to answer questions such as:

1. What noise level do I expect, and how do I quantify it?
2. How do I measure my noise level?
3. Given my noise level, how long will it take me to resolve a signal of known magnitude?

This tutorial will guide you in answering these questions, starting with the basic principles but including specific instructions[1].

2 Quantifying noise

2.1 Noise variance and spectral density

The two simplest ways to specify noise are in terms of variance and spectral density. Suppose we have a noise voltage $V_\mathrm{N}(t)$ that varies with time t. Its **variance** is

$$\begin{aligned} \operatorname{var}(V_\mathrm{N}) \equiv \sigma^2(V_\mathrm{N}) &\equiv \left\langle \left(V_\mathrm{N}(t) - \overline{V_\mathrm{N}}\right)^2 \right\rangle \\ &= \langle V_\mathrm{N}^2(t) \rangle \end{aligned} \tag{1}$$

where as usual $\sigma(V_\mathrm{N})$ is the standard deviation of V_N, $\langle \dots \rangle$ denotes the expectation value, and $\overline{V_\mathrm{N}}$ denotes the average of V_N, which I will henceforth assume to be zero. (If $\overline{V_\mathrm{N}}$ isn't zero, then you should measure it and subtract it from the signal.) The expectation value in Eq. (1) can be calculated from a series of individual measurements, or from a trace.

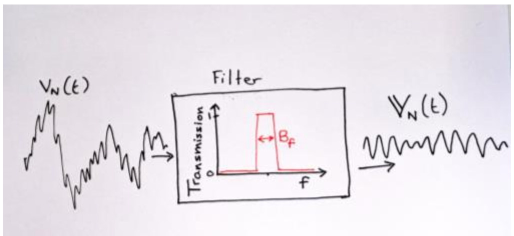

Figure 1: The conceptual filter that defines the spectral density of $V_N(t)$.

Clearly, the larger the variance, the larger the noise. In many situations the variance of a measurement is what matters, because it enters the formula for standard error[2]. However, often we want a different measure of noise: the **spectral density**. To define it, imagine that we filter $V_\mathrm{N}(t)$ to pass only components within a bandwidth B_f that includes frequency f (Figure 1). If the filtered signal is $\mathbb{V}_\mathrm{N}(t)$, then the spectral density of V_N is[3]

$$S_{VV}^\mathrm{N}[f] \equiv \lim_{B_f \to 0} \frac{\langle \mathbb{V}_\mathrm{N}^2 \rangle}{B_f}. \tag{2}$$

The spectral density is related to the variance by

$$\operatorname{var}(V_\mathrm{N}) = \int_0^\infty S_{VV}^\mathrm{N}[f]\, df, \tag{3}$$

[1] These instructions are based on my memory, rather than actually checking them; if you find a mistake, please let me know.

[2] As you should know, the formula for the standard error in an average over n independent measurements is[2]

$$\sigma_V = \frac{\sigma(V_\mathrm{N})}{\sqrt{n}}$$

[3] This tutorial uses square brackets to denote quantities in frequency space, and follows the single-sided definition of spectral density, which is dominant in electrical engineering. Beware of alternative definitions; Supplementary Box 1 of Ref. [1] explains how to convert between them.

but tells you more than the variance does; not only how much noise there is in total, but also which frequencies contribute most.
These definitions can be obviously modified to apply to other measured quantities instead of voltage (for example current or displacement), but for simplicity these notes only describe voltage measurements.

2.2 Input-referred noise, added noise, and system noise

To make fair comparisons, it is essential to specify the noise level at the same point in a circuit. To see why, suppose we insert an ideal amplifier[4] with voltage gain G, such that its output and input voltages are related by

$$V_{\mathrm{out}}(t) = GV_{\mathrm{in}}(t). \tag{4}$$

If we feed this amplifier a noisy voltage, then the spectral density at its output will be larger than at its input, by a factor G^2. For consistency, we usually specify the value at the input to the measurement. This is the **input-referred noise**.
Now let's consider a real amplifier, for which the analogue of Eq. (4) is[5]

$$V_{\mathrm{out}}(t) = GV_{\mathrm{in}}(t) + V_{\mathrm{N@out}}(t). \tag{5}$$

Here $V_{\mathrm{N@out}}(t)$ is the extra noise at the output due to the non-ideality of the amplifier. To specify the input-referred noise, we rewrite this equation as

$$V_{\mathrm{out}}(t) = G\left(V_{\mathrm{in}}(t) + V_{\mathrm{N,A}}(t)\right) \tag{6}$$

where $V_{\mathrm{N,A}}(t) = V_{\mathrm{N@out}}(t)/G$ is the **added noise** due to the amplifier. Obviously, other components in the measurement chain can also add noise, which is calculated in the same way.
Often we want to specify the noise in an experiment including both the noise at the input $V_{\mathrm{N@input}}(t)$ (due to shot noise in the device under test, for example), and the noise added by the measurement chain. The combination of these is the **system noise**, given by:

$$V_{\mathrm{N,system}}(t) = V_{\mathrm{N@input}}(t) + V_{\mathrm{N,A}}(t) \tag{7}$$

In general

$$\text{System noise} = \text{device noise} + \text{added noise}. \tag{8}$$

2.3 Other ways to quantify noise

The noise spectral density can be expressed in other useful ways:

1. The sensitivity is

$$\text{Sensitivity} \equiv \sqrt{S_{VV}^{\mathrm{N}}[f]}. \tag{9}$$

   The reason for this name is given in Footnote 13.
2. The noise power density is defined as

$$p_{\mathrm{N}}[f] = \frac{S_{VV}^{\mathrm{N}}[f]}{Z_0}. \tag{10}$$

3. It is the power per unit bandwidth dissipated in a resistance Z_0 when the noise voltage across it has spectral density $S_{VV}^{\mathrm{N}}[f]$. Here Z_0 is the matched resistance (usually 50 Ω). This is useful because power is what a spectrum analyser measures.

[4] Strictly speaking, an ideal classical amplifier; quantum mechanics makes some level of added noise unavoidable.
[5] To avoid ambiguity between input-referred and output-referred signals, I use the subscript @X when I want to be explicit that a voltage is measured at point X.

Table 1: Conversion between different measures of noise. The constants are $k = 1.38 \times 10^{-23}\ J/K$, $T_0 \equiv 290\ K$, and $Z_0 = 50\ \Omega$ (usually).

	Spectral density	**Noise power density**	**Noise temperature**	**Noise figure**
Symbol→	S_{VV}^{N}	p_{N}	T_{N}	F_{dB}
Units→	V²/Hz	W/Hz	K	dB
Spectral density		$S_{VV}^{\mathrm{N}} = Z_0 p_{\mathrm{N}}$	$S_{VV}^{\mathrm{N}} = Z_0 k T_{\mathrm{N}}$	$S_{VV}^{\mathrm{N}} = Z_0 k T_0 \left(10^{F_{\mathrm{dB}}/10} - 1\right)$
Noise power density	$p_{\mathrm{N}} = \frac{S_{VV}^{\mathrm{N}}}{Z_0}$		$p_{\mathrm{N}} = k T_{\mathrm{N}}$	$p_{\mathrm{N}} = k T_0 \left(10^{F_{\mathrm{dB}}/10} - 1\right)$
Noise temperature	$T_{\mathrm{N}} = \frac{S_{VV}^{\mathrm{N}}}{k Z_0}$	$T_{\mathrm{N}} = \frac{p_{\mathrm{N}}}{k}$		$T_{\mathrm{N}} = T_0 \left(10^{F_{\mathrm{dB}}/10} - 1\right)$
Noise figure	$F_{\mathrm{dB}} = 10 \log\left(1 + \frac{S_{VV}^{\mathrm{N}}}{k T_0 Z_0}\right)$	$F_{\mathrm{dB}} = 10 \log\left(1 + \frac{p_{\mathrm{N}}}{k T_0}\right)$	$F_{\mathrm{dB}} = 10 \log\left(1 + \frac{T_{\mathrm{N}}}{T_0}\right)$	

4. The **noise temperature** $T_{\mathrm{N}}[f]$ is defined as
$$T_{\mathrm{N}}[f] = \frac{S_{VV}^{\mathrm{N}}[f]}{k Z_0} \tag{11}$$
where k is Boltzmann's constant. It is the temperature of a resistor, matched to the input impedance Z_0 of the measuring circuit, whose Johnson noise would contribute a spectral density S_{VV}^{N} to the input-referred noise[6]. It is useful because it makes it easy to compare other noise sources to the Johnson noise.
5. The **noise figure** is a way to characterise added noise, defined in linear units by
$$F = 1 + \frac{T_{\mathrm{N}}}{T_0} \tag{12}$$
where $T_0 \equiv 290$ K. In dB units (which are more common),
$$F_{\mathrm{dB}} = 10 \log F. \tag{13}$$
6. The noise figure F is the factor by which an amplifier with added noise temperature T_{N} degrades the power signal-to-noise ratio, compared to Johnson noise at T_0. For example, a noise figure of $F_{\mathrm{dB}} = 3$ dB corresponds to an added noise temperature close to 290 K; a noiseless amplifier has $F_{\mathrm{dB}} = 0$ dB.

Table 1 summarises how to convert between these quantities.

3 How to calculate the expected noise

3.1 A chain of amplifiers: Friis' formula

Nearly all experiments use a chain of linear amplifiers ending in a recording device such as a digitiser (similar to Figure 2). Linear in this case means that the output of the ith amplifier is proportional to its input (plus noise):
$$V_{\mathrm{out},i}(t) = G_i V_{\mathrm{in},i}(t) + \mathrm{noise} \tag{14}$$
where G_i is the voltage gain. The chain is therefore equivalent to single amplifier with gain
$$G = G_1 G_2 G_3 \ldots . \tag{15}$$
The added noise temperature of the chain is given by **Friis' formula**:

[6] This sentence is true in the classical limit only, but Eq. (10) is generally accepted to define T_{N} in both classical and quantum situations.

$$T_{\mathrm{N,A}} = T_{\mathrm{N,A1}} + \frac{T_{\mathrm{N,A2}}}{G_1^2} + \frac{T_{\mathrm{N,A3}}}{(G_1 G_2)^2} + \cdots \quad (16)$$

where $T_{\mathrm{N,A}i}$ is the added noise temperature of the ith amplifier. A useful consequence of Friis' formula is that the noise temperature of a chain is mostly determined by its first few components.

Other linear components can be used in Friis' formula as if they were amplifiers as follows:

a. An attenuator is equivalent to an amplifier with gain $G_i = 1/L_i$ and added noise temperature

$$T_{\mathrm{N,A}i} = (L_i^2 - 1)T_i, \quad (17)$$

where L_i is the voltage attenuation factor and T_i is the ambient temperature.

b. A single impedance step can be modelled by an amplifier with gain

$$G_{\mathrm{mismatch}} = \left|\frac{2Z}{Z_0 + Z}\right| \quad (18)$$

and no added noise. Here Z_0 and Z are the impedances before and after the mismatch. The most common example is between an amplifier with output impedance $Z_0 = 50\ \Omega$ and an oscilloscope with input impedance $Z \gg Z_0$, in which case $G_{\mathrm{mismatch}} = 2$.

c. If you have multiple impedance changes, you can still apply Eq. (18) but it may not be accurate because it does not account for multiple reflections. In this situation you should accept that your estimate for the added noise may not be very good.

d. A mixer is equivalent (for the purpose of Friis' formula; see Appendix A) to an amplifier with gain

$$G_{\mathrm{mixer}} = \begin{cases} \dfrac{1}{L_{\mathrm{C}}} & \text{(in homodyne configuration)} \\ \dfrac{1}{\sqrt{2}\,L_{\mathrm{C}}} & \text{(in heterodyne configuration)} \end{cases} \quad (19)$$

The added noise is often not specified but can be taken as zero if there is plenty of pre-amplification.

Knowing this, a procedure for calculating the expected noise of a chain is:

1. Calculate the expected gain at each stage of the chain in voltage units. (Table 2 reminds you how to do this if the gain is specified in dB.)
2. Calculate the expected added noise temperature at each stage, if necessary using Table 1 to convert from other units.
3. Use Eq. (16) to calculate the combined added noise temperature $T_{\mathrm{N,A}}$.
4. Calculate the system noise, if you need it, by adding the input noise to the added noise.
5. If you need the noise in units other than temperature, use Table 1 to perform the conversion.

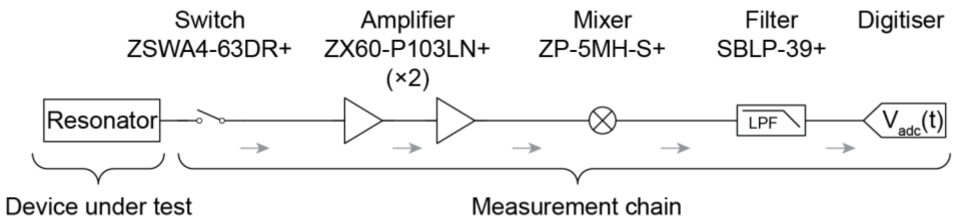

Figure 2: Measurement chain analysed in the text. Thie example is a simplification of a circuit used for pulsed spin resonance (see Matthew Green's thesis). All part numbers are from Minicircuits.

3.2 Example 1: Predicting the noise in Figure 2

Suppose we are measuring a tank circuit via the series of components shown in Figure 2. What should we expect for the input-referred added noise and system noise?

Let's calculate the noise up to the input of the mixer; as we know, added noise is dominated by the first few components so this should be a good approximation. Following Section 3.1:

1. We need the gain for three components: The switch and two identical amplifiers. Looking at the datasheets, and taking our working frequency as 100 MHz, the gains are:
   - Switch: This gain is the inverse of the insertion loss, so we have $G_{1,\mathrm{dB}} = -0.9$ dB, which converts to a voltage gain of $G_1 = 0.90$.
   - Amplifiers: Both have $G_{\mathrm{dB}} = 25$ dB, which converts to $G_2 = G_3 = 17.8$.

   The combined gain is therefore
   $$G = G_1 G_2 G_3 = 285. \tag{20}$$
2. We calculate the noise temperatures as follows:
   - The switch is effectively an attenuator, and is at room temperature of 300 K. Equation (17) therefore gives:
   $$T_{\mathrm{N,A1}} = 70\ \mathrm{K}. \tag{21}$$
   - The amplifiers each have a specified noise figure of 0.6 dB, which converts to
   $$T_{\mathrm{N,A2}} = T_{\mathrm{N,A3}} = 44\ \mathrm{K}. \tag{22}$$
3. The total added noise, by Eq. (16), is therefore
   $$T_{\mathrm{N,A}} = 120\ \mathrm{K} \tag{23}$$
4. To calculate the system noise, we need to know the noise at the output of the resonator. We expect this is determined by Johnson noise, which means that the input noise temperature is the same as room temperature. The system noise temperature is therefore:
   $$\begin{aligned} T_{\mathrm{N,system}} &= 300\ \mathrm{K} + T_{\mathrm{N,A}} \\ &= 420\ \mathrm{K} \end{aligned} \tag{24}$$
5. Let's suppose the quantity we want to know is the voltage sensitivity. Equation (24) converts to
   $$\sqrt{S_{VV}^{\mathrm{N}}} = 0.54\ \mathrm{nV}/\sqrt{\mathrm{Hz}}\,. \tag{25}$$

Table 2: Conversion between gains in dB and ratios in linear units. Any quantity which is a ratio of voltages converts in the same way as voltage gain; any quantity which is a ratio of powers converts in the same way as power gain.

	Voltage gain	**Power gain**	**Gain in dB**
Symbol→	G	G_{P}	G_{dB}
Units→	-	-	dB
Voltage gain		$G = \sqrt{G_P}$	$G = 10^{G_{\mathrm{dB}}/20}$
Power gain	$G_{\mathrm{P}} = G^2$		$G_{\mathrm{P}} = 10^{G_{\mathrm{dB}}/10}$
Gain in dB	$G_{\mathrm{dB}} = 20 \log G$	$G_{\mathrm{dB}} = 10 \log G_{\mathrm{P}}$	

4 How to measure system noise

There three main ways to measure noise, summarised below.

- A spectrum analyser is easiest and works well above about 100 kHz.
- An oscilloscope or digitiser works well at low frequencies (up to its analogue bandwidth) and is convenient if you are using it already to record your signal.
- A lock-in works well at intermediate frequency, and is also convenient if it is what you are using already.

In the example of Figure 2, you would probably use a spectrum analyser just before the mixer to measure the noise including the input amplifiers, and then check this number using the final digitiser.

All these instruments measure system noise, since of course they can't tell whether the noise they are fed originates from the device or the measurement chain. If you want to know the added noise, then you should measure the system noise and then subtract your best estimate of the device noise[7].

4.1 Using a spectrum analyser

A spectrum analyser measures spectral power

$$P_{\text{spectral}}[f] \equiv R_f \, p[f] \tag{26}$$

where R_f is the instrument's **resolution bandwidth** and $p[f]$ is the power density defined similarly to Eq. (10)[8]. From Eq. (26), it is straightforward to measure the system noise.

The procedure is:

1. Use a network analyser to measure the total gain G from the device under test to the spectrum analyser input.
2. Set up the spectrum analyser: choose the frequency range, make sure that the acquisition is set to V_{rms} (not V_{max}), and make sure the video bandwidth is not larger than the resolution bandwidth (for example by setting it to auto).
3. Measure the spectrum. Pick a frequency where it is dominated by noise, not signal. If necessary, convert $P_{\text{spectral}}[f]$ to SI units using Table 3.
4. Measure the noise floor P_{term} with a terminator connected, in order to account for the instrument's own noise (You can omit this step if you are sure this contribution is negligible, or if you want to include the spectrum analyser as part of the system noise.)

Table 3: Conversion between dBm and SI units of power.

	Power	**Power in dBm**
Symbol→	P	P_{dBm}
Units→	W	dBm
Power		$P = 10^{-3}\ \text{W} \times 10^{P_{\text{dBm}}/10}$
Power in dBm	$P_{\text{dBm}} = 10 \times \log \dfrac{P}{10^{-3}\ \text{W}}$	

[7] There are ways to distinguish system and added noise, such as the Y-factor method; if you really need this, look it up.

[8] I have dropped the N subscript because of course the spectral analyser doesn't distinguish between the signal and noise contributions to its input.

5. Using Eq. (26) and the known gain and noise floor, convert the measured power to an input-referred system noise as follows:
$$p_{\mathrm{N}}[f] = \frac{1}{G^2}\left(\frac{P_{\mathrm{spectral}}[f] - P_{\mathrm{term}}[f]}{R_f} + kT\right) \tag{27}$$
where $P[f]$ is the spectral power with the analyser connected to the measurement chain, $P_{term}[f]$ is the spectral power with it connected to a terminator, and T the temperature of the terminator.
If you have omitted step 4, the calculation simplifies to
$$p_{\mathrm{N}}[f] = \frac{P_{\mathrm{spectral}}[f]}{G^2 R_f} \tag{28}$$
6. Convert the input-referred system noise to your desired units using Table 1.

4.2 Using an oscilloscope or digitiser

This method relies on Eq. (3), which implies that filtered noise satisfies:
$$\langle \mathbb{V}_{\mathrm{N}}^2 \rangle = \int_0^\infty S_{\mathbb{V}\mathbb{V}}^{\mathrm{N}}[f]\, df \tag{29}$$
Therefore, if we take a noisy trace $V_{\mathrm{N}}(t)$, filter it with amplitude transmission function $F[f]$, and measure the variance, we can infer the spectral density.
More precisely, we rewrite Eq. (29) as
$$\begin{aligned}\langle \mathbb{V}_{\mathrm{N}}^2 \rangle &= \int_0^\infty |F[f]|^2\, S_{VV}^{\mathrm{N}}[f]\, df \\ &\approx B_f S_{VV}^{\mathrm{N}}[f_{\mathrm{c}}]\end{aligned} \tag{30}$$
where f_{c} is the centre frequency of the filter and B_f is its **noise equivalent bandwidth**, defined as
$$B_f \equiv \int_0^\infty |F[f]|^2\, df. \tag{31}$$
Then the noise spectral density is
$$S_{VV}^{\mathrm{N}}[f_{\mathrm{c}}] \approx \frac{\langle \mathbb{V}_{\mathrm{N}}^2 \rangle}{B_f} \tag{32}$$
Equation (32) holds provided S_{VV}^{N} is approximately constant across the filter band, and applies whether the filter is implemented in hardware or software.
The procedure to estimate noise is therefore:

1. Set your digitiser gain (and any other parameters) as in the real experiment. Call the input voltage at the digitiser $\mathbb{V}_{@\mathrm{D}}$
2. Connect the digitiser to a terminator. Acquire a trace $\mathbb{V}_{\mathrm{term@D}}(t)$, and use this to check the variance due to the digitiser noise. (You can omit this step if you want to measure the noise including the digitiser, or if are sure that its noise is negligible.)
3. Connect the digitiser to the measurement chain, and measure the gain G from the device under test to the digitiser by injecting a known probe signal and measuring the change in amplitude between probe on and probe off.
4. Reconnect the device under test, acquire a trace $\mathbb{V}_{\mathrm{N@D}}(t)$ and calculate its variance.
5. Calculate the input-referred system noise, which is
$$S_{VV}^{\mathrm{N}}[f_c] = \frac{\mathrm{var}\big(\mathbb{V}_{\mathrm{N@D}}(t)\big) - \mathrm{var}\big(\mathbb{V}_{\mathrm{term@D}}(t)\big)}{G^2\, B_f} \tag{33}$$
(If you want to include the digitiser's noise in the system noise, then omit the subtraction here.)

6. Convert to your desired units using Table 1.

4.3 Using a lock-in amplifier

A lock-in amplifier carries out homodyne demodulation, which means that noise at the signal frequency is converted into a dc output. The lock-in demodulates its input $V(t)$ into two filtered quadratures:

$$\begin{aligned}\mathbb{X}(t) &= \left[\sqrt{2}\, V(t)\cos 2\pi f_{\mathrm{r}} t\right] \\ \mathbb{Y}(t) &= \left[\sqrt{2}\, V(t)\sin 2\pi f_{\mathrm{r}} t\right]\end{aligned} \tag{34}$$

where $[\cdot]$ denotes a low-pass filter and f_{r} is the demodulation frequency[9]. Some maths shows that

$$S_{\mathbb{XX}}[0] = S_{\mathbb{YY}}[0] = S_{VV@\mathrm{L}}[f_{\mathrm{r}}]. \tag{35}$$

where $S_{VV@\mathrm{L}}$ is the voltage spectral density at the lock-in input. Therefore

$$\mathrm{var}(\mathbb{X}(t)) = S^{\mathrm{N}}_{VV@\mathrm{L}}[f_{\mathrm{r}}]\, B_f \tag{36}$$

and so

$$S^{\mathrm{N}}_{VV@\mathrm{L}}[f_{\mathrm{r}}] = \frac{\mathrm{var}\big(\mathbb{X}(t)\big)}{B_f} \tag{37}$$

where B_f is the noise equivalent bandwidth, which is determined by the lock-in filter order and time constant.

So the procedure to measure your system noise using a lock-in is:

1. Set your lock-in frequency, gain, and filter, and any other parameters, as in the real experiment. Find out the noise-equivalent bandwidth. (In Zurich's LabOne interface, this is one of the filter settings.
2. Connect the lock-in to the experiment. Measure $\mathrm{var}(\mathbb{X}(t))$ and apply Eq. (37) to calculate $S^{\mathrm{N}}_{VV@\mathrm{L}}[f_{\mathrm{r}}]$ referenced to the lock-in input.
3. Divide through by G^2, where G is the gain before the lock-in, to refer $S^{\mathrm{N}}_{VV}[f_{\mathrm{r}}]$ to the device under test.
4. Convert to your desired units using Table 1.

4.4 Example 2: Measuring the system noise in Figure 2

There are two ways we might measure the noise in this situation. The simpler one is to measure up to the input of the mixer. This is the first part of the chain and should therefore account for most of the noise; however, if we want to make a more careful measurement we should measure all the way up to the digitiser. I'll give examples of both[10].

4.4.1 Measuring up to the mixer input

We are measuring at rf frequency, so a good choice is to use a spectrum analyser. Using only the part of Figure 2 up to the mixer input, and following the steps of Section 4.1:

1. We use a network analyser to measure from the switch input to the second amplifier output, finding $G_{\mathrm{dB}} = 48\ \mathrm{dB}$ and therefore
$$G = 251. \tag{38}$$
2. We disconnect the network analyser, reconnect the resonator, and connect the spectrum analyser to the second amplifier output.
3. With a resolution bandwidth of $R_f = 10\ \mathrm{kHz}$, we measure noise power of $P_{\mathrm{dBm}}[f = 100\ \mathrm{MHz}] = -84\ \mathrm{dBm}$ and therefore

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[9] These equations hold for Zurich lock-ins; see https://www.zhinst.com/europe/en/resources/principles-of-lock-in-detection. Stanford lock-ins apparently multiply by a square wave; I haven't done the maths for this.

[10] All numbers in Example 2 are made up; I didn't actually do the experiment (but I should; if I have made a mistake, please tell me!).

$$P[f] = 3.98 \times 10^{-12}\ \mathrm{W}. \tag{39}$$

4. The spectrum analyser is not part of the final experiment, so we shouldn't include it in the system noise. We therefore measure the noise floor when it is connected to a terminator, finding $P_{\mathrm{term,dB}}[f] = -100\ \mathrm{dBm}$ and therefore
$$P_{\mathrm{term}}[f] = 1.0 \times 10^{-13}\ \mathrm{W}. \tag{40}$$
5. Applying Eq. (27) therefore gives for the input-referred noise power density (assuming a terminator temperature of $T = 300\ \mathrm{K}$):
$$p_{\mathrm{N}} = 6.2 \times 10^{-21}\ \mathrm{W/Hz}. \tag{41}$$
6. This is an input-referred system sensitivity of
$$\sqrt{S_{VV}^{\mathrm{N}}} = 0.55\ \mathrm{nV}/\sqrt{\mathrm{Hz}}\,. \tag{42}$$
This is close to our prediction in Eq. (25), and confirms that there is no unexpected noise source.

4.4.2 Measuring up to the digitiser

Using the entire chain in Figure 2, and following the steps of Section 4.2 (For consistent notation with Appendix A, I'm calling the digitiser input voltage $\mathbb{U}_{@\mathrm{D}}(t)$ instead of $\mathbb{V}_{@\mathrm{D}}(t)$ as in Section 4.2.):

1. We set up the digitiser …
2. … and skip step 2, because the digitiser noise is part of the system noise.
3. To measure the gain up to the digitiser input, we inject a probe tone of power 1 nW (i.e. $P_{\mathrm{dBm}} = -60\ \mathrm{dBm}$) into the measurement chain, and measure an amplitude change between probe on and probe off of $\Delta\mathbb{U}_{@\mathrm{D}} = 44\ \mathrm{mV}$. Taking heed of Eq. (66), this means the total gain of the chain is[11]
$$G = \frac{\Delta\mathbb{U}_{@\mathrm{D}}}{V_{\mathrm{rms}}^{\mathrm{probe}}} = 197 \tag{43}$$
where $V_{\mathrm{rms}}^{\mathrm{probe}}$ is the rms of the probe tone.
4. With the resonator reconnected, we measure a variance at the digitiser of $\mathrm{var}(\mathbb{U}_{\mathrm{N}@\mathrm{D}}) = (0.90\ \mathrm{mV})^2$.
5. We now apply Eq. (33). The digitiser is part of the system, so we should omit the subtraction step; thus
$$S_{VV}^{\mathrm{N}} = \frac{\mathrm{var}(\mathbb{U}_{\mathrm{N}@\mathrm{D}})}{G^2\, B_f}. \tag{44}$$
The bandwidth B_f should be the noise-equivalent bandwidth of the final filter. The datasheet doesn't specify this, but it does specify the 3 dB bandwidth, which ought to be about the same. We therefore take $B_f = 39\ \mathrm{MHz}$, which leads to input-referred noise
$$S_{VV}^{\mathrm{N}} = 1.1 \times 10^{-16}\ \mathrm{V^2/Hz}. \tag{45}$$
6. The corresponding input-referred system sensitivity is
$$\sqrt{S_{VV}^{\mathrm{N}}} = 0.73\ \mathrm{nV}/\sqrt{\mathrm{Hz}}\,, \tag{46}$$
which is close to the value measured at the mixer input, and confirms that the demodulation and digitising steps do not badly degrade the sensitivity.

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[11] I can check this against my expectation: Minicircuits specifies the conversion loss of this mixer as $L_{\mathrm{dB}} = 5\ \mathrm{dB}$, which by Footnote 15 means $L_{\mathrm{C}} = 1.3$, which by Eq. (19) is a mixer gain of $G_{\mathrm{mixer}} = 0.80$. We already know (see Eq. (38)) that the gain up to the mixer is 251, so the gain we expect is Eq. (43) is $G = 251 \times 0.80 = 200$, which is close.

5 What is the uncertainty in my measurement?

The usual benefit of knowing the system noise is that it determines the uncertainty in a measurement of voltage (or of anything that can be transduced into it). As usual, let's define the uncertainty in a single measurement as the standard deviation if it were to be repeated many times.

To be specific, suppose we are measuring a voltage

$$V(t) = V_{\mathrm{S}}(t) + V_{\mathrm{N}}(t) \tag{47}$$

or equivalently

$$V[f] = V_{\mathrm{S}}[f] + V_{\mathrm{N}}[f]. \tag{48}$$

Then the uncertainty we need is

$$\text{Uncertainty in measuring } \mathrm{V}_0 \equiv \sigma(\overline{V_0}) \tag{49}$$

where V_0 is the amplitude of the signal V_{S} and $\overline{V_0}$ is our estimate of it after a single measurement[12]. Knowing $S_{VV}^{\mathrm{N}}[f]$ and some details of our measurement, we can calculate this uncertainty. This calculation is presented in the Supplementary to Ref. [1]; here I will simply state the results.

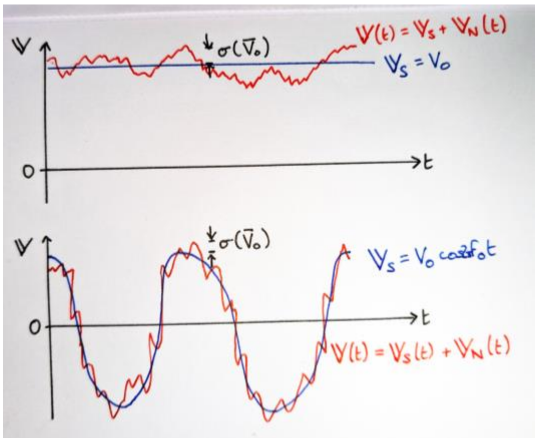

Figure 3 Estimating the amplitude of a signal (blue) from a noisy voltage (red), for a constant signal $\mathbb{V}_S(t) = V_0$ (top) and for an oscillating signal $\mathbb{V}_S(t) = V_0 \cos 2\pi f_S t$ (bottom).

5.1 When measuring over a fixed bandwidth

Let's suppose I measure $V(t)$ in a fixed bandwidth; in other words, I filter it with bandwidth B_f to output a voltage $\mathbb{V}(t)$. I use one instant in the trace of $\mathbb{V}(t)$ to estimate the amplitude of the signal – any instant if the signal is a constant, or an instant at the crest if the signal is oscillating (as in Figure 3). The expected error is[13]

$$\sigma(\overline{V_0}) = \sqrt{B_f\, S_{VV}^{\mathrm{N}}[f_{\mathrm{S}}]} \tag{50}$$

where f_{S} is the centre frequency of the signal (with $f_{\mathrm{S}} = 0$ for a constant signal, of course).

5.2 When measuring over a fixed duration

5.2.1 When measuring a voltage

Another common situation is a measurement in fixed time; in other words, I record my voltage $V(t)$ over a duration τ and use this record to estimate its amplitude. In this case, the uncertainty depends on whether I am measuring a constant or oscillating signal.

- If my signal is constant, i.e. $V_{\mathrm{S}}(t) = V_0$, then my best estimate of V_0 is the average over duration τ; the uncertainty of my estimate is

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[12] If I repeat this measurement many times, then of course I can average the results to get the smaller uncertainty given in Footnote 2.

[13] Equation (50) is the reason that $\sqrt{S_{VV}^{\mathrm{N}}}$ is called the sensitivity; a sensitivity of 1 nV/√Hz means that after a 1 Hz filter, a 1 nV signal is the same size as the rms noise fluctuations – which is a good threshold for saying that it's detectable.

$$\sigma(\overline{V_0}) = \sqrt{\frac{S_{VV}^{\mathrm{N}}[0]}{2\tau}}. \tag{51}$$

- If my signal is oscillating, i.e. $V_S(t) = V_0 \cos(2\pi f_S t + \phi_S)$, then my best estimate of V_0 comes from a least-squares fit; the uncertainty in my estimate is

$$\sigma(\overline{V_0}) = \sqrt{\frac{S_{VV}^{\mathrm{N}}[f_S]}{\tau}}. \tag{52}$$

5.2.2 When measuring the power from an incoherent source

To measure power from a coherent source (i.e. one whose phase and amplitude stay fixed during one measurement), then you should measure the amplitude and convert it to a power. However, often you want to measure power from an incoherent source; for example in noise thermometry, intensity mapping, or an axion haloscope. In this case, it is the system noise itself that is what you want to quantify.

If the noise is white, then the best possible uncertainty is given by the Dicke radiometer equation; it is

$$\begin{aligned}\sigma(\overline{P}) &= \frac{P}{\sqrt{B_f \tau}} \\ &= \sqrt{\frac{B_f}{\tau}} \frac{S_{VV}^{\mathrm{N}}[f_S]}{Z_0}\end{aligned} \tag{53}$$

where B_f is the bandwidth over which your power detector is sensitive, τ is the measurement time (assumed to satisfy $\tau B_f \gg 1$), f_S is the centre frequency at which you are detecting, and the power is obviously related to the spectral density by

$$P = B_f \frac{S_{VV}^{\mathrm{N}}}{Z_0} \tag{54}$$

where S_{VV}^{N} is the spectral density of the system noise.

5.3 What is the smallest signal I can resolve?

Suppose the signal you want to measure is an input-referred voltage of known amplitude, say 1 nV. Can you detect it? The answer is yes, if your noise is low enough and you can measure for long enough.

We call a signal **resolved** if it is larger than the uncertainty in the measurement, i.e.

$$V_0 \geq \sigma(\overline{V_0}). \tag{55}$$

A signal is barely resolved if the signal-to-uncertainty ratio is unity, which means it will be larger than most of the random fluctuations due to noise[14]. One way to quantify how hard a signal is to measure is to state the minimum time $\tau_{\min}$ (or, equivalently, the maximum bandwidth $B_{f,\max}$) for it to become barely resolved. Using Eqs. (50) to (53), we can see that these are (approximating white noise near the relevant frequencies):

- To resolve a constant voltage V_0:

$$\begin{aligned}\tau > \tau_{\min} &= \frac{S_{VV}^{\mathrm{N}}[0]}{2V_0^2} \\ B_f < B_{f,\max} &= \frac{V_0^2}{S_{VV}^{\mathrm{N}}[0]}\end{aligned} \tag{56}$$

[14] To be precise: if the noise is gaussian-distributed (which it usually is), then a barely resolved signal is larger than about 84% of measurement outcomes when there is zero signal.

- To resolve an oscillating voltage $V_0 \cos 2\pi f_S t$:

$$\tau > \tau_{\min} = \frac{S_{VV}^{\mathrm{N}}[f_S]}{V_0^2}$$
$$B_f < B_{f,\max} = \frac{V_0^2}{S_{VV}^{\mathrm{N}}[f_S]} \tag{57}$$

- To resolve an incoherent power P_0:

$$\tau > \tau_{\min} = B_f \left(\frac{S_{VV}^{\mathrm{N}}[f_S]}{P_0 Z_0}\right)^2 \tag{58}$$

These minimum resolution times are useful benchmarks for comparing different experiments and deciding whether a particular measurement is feasible. A trace with each point at the resolution threshold will usually look OK – but for nice-looking data, you will want to measure for longer.

6 Appendix A: Chains including a mixer

Introducing mixers to the measurement chain violates Eq. (4); it both scales the output noise and shifts it from one frequency to another. However, if you are careful about your definitions, you can still use Friis' formula (Eq. (16)) as in the main text.

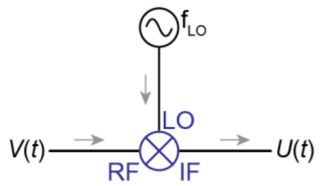

Figure 4: Inputs and output of a mixer.

6.1 What a mixer does

If a signal $V(t)$ enters a mixer through its RF port while a local oscillator at approximately the specified level enters through the LO port (as in Figure 4), then the IF output will be

$$U(t) = \frac{\sqrt{2}}{L_{\mathrm{C}}} \cos(2\pi f_{\mathrm{LO}} t + \phi) V(t) \tag{59}$$

where f_{LO} and ϕ are the frequency and phase of the local oscillator. Here L_{C} is the **mixer conversion loss** in voltage units[15]. If the spectral density of the voltage entering the mixer is $S_{VV}^{\mathrm{N}}[f]$, it follows from Eq. (59) that the spectral density at the output will be

$$S_{UU}^{\mathrm{N}}[f] = \frac{1}{2L_{\mathrm{C}}^2}\left(S_{VV}^{\mathrm{N}}[f - f_{\mathrm{LO}}] + S_{VV}^{\mathrm{N}}[f + f_{\mathrm{LO}}]\right). \tag{60}$$

6.2 Demodulation using a mixer

Let's call the signal at the RF port

$$V(t) = A \cos 2\pi f_S t + \text{noise}, \tag{61}$$

where the signal amplitude A varies slowly[16]. Let's also suppose that $S_{VV}^{\mathrm{N}}[f]$ is roughly independent of frequency close to f_S (which is usually true). Then we can calculate the amplitude and gain of the mixer's RF output for each of the two configurations used in demodulation (Figure 5):

- In a homodyne measurement configuration, we:

[15] Conversion loss is usually specified in dB units as L_{dB}, which regrettably has two definitions: Pozar's classic textbook[3] defines L_{dB} so that

$$L_{\mathrm{C}} = 10^{L_{\mathrm{dB}}/20} \qquad \text{(Pozar convention)}$$

while Minicircuits[4] defines L_{dB} so that

$$L_{\mathrm{C}} = \frac{1}{\sqrt{2}} \times 10^{L_{\mathrm{dB}}/20}. \qquad \text{(Minicircuits convention)}$$

[16] I've ignored a possible sine contribution to $V(t)$, which is equivalent to shifting the zero of t.

- Set f_{LO} equal to the signal frequency f_{S}.
- Low-pass filter $U(t)$ to keep only the components $\mathbb{U}(t)$ near zero frequency.

Applying Eqs. (59) and (60) shows that the output is

$$\mathbb{U} = \frac{\cos\phi}{\sqrt{2}L_{\mathrm{C}}} A \tag{62}$$

plus noise with spectral density (within the post-filter bandwidth)

$$\begin{aligned} S^{\mathrm{N}}_{\mathbb{UU}}[f] &\approx S^{\mathrm{N}}_{\mathbb{UU}}[0] \\ &= \frac{1}{L_{\mathrm{C}}^2} S^{\mathrm{N}}_{VV}[f_{\mathrm{LO}}] \end{aligned} \tag{63}$$

where the first line follows because we assumed that $S^{\mathrm{N}}_{VV}[f]$ is roughly flat, and the second follows from Eq. (60), using that $S[-f] \equiv S[f]$.

- In a heterodyne measurement configuration, we:
  - Before the mixer, pre-filter $V(t)$ around the signal frequency f_S.
  - Set f_{LO} to be different from f_S.
  - Band-pass filter $U(t)$ around $|f_{\mathrm{S}} - f_{\mathrm{LO}}|$.

The filtered output is then

$$\mathbb{U}(t) = \frac{1}{\sqrt{2}L_{\mathrm{C}}} \cos(2\pi(f_{\mathrm{LO}} - f_{\mathrm{S}})t + \phi)\, A \tag{64}$$

plus noise with spectral density (within the post-filter bandwidth)

$$\begin{aligned} S^{\mathrm{N}}_{\mathbb{UU}}[f] &\approx S^{\mathrm{N}}_{\mathbb{UU}}[f_{\mathrm{S}} - f_{\mathrm{LO}}] \\ &= \frac{1}{2L_{\mathrm{C}}^2}\left(S^{\mathrm{N}}_{VV}[f_{\mathrm{S}}] + S^{\mathrm{N}}_{VV}[2f_{\mathrm{LO}} - f_{\mathrm{S}}]\right) \\ &\approx \frac{1}{2L_{\mathrm{C}}^2} S^{\mathrm{N}}_{VV}[f_{\mathrm{S}}]. \end{aligned} \tag{65}$$

where the last step follows because the pre-filter has removed noise components away from f_{S}.

6.3 Friis' law including a demodulation stage

The results of Section 6.2 mean is that we can apply Friis' formula to a chain containing a mixer, provided that we define the gain to be

$$G_{\mathrm{mixer}} \equiv \frac{\mathbb{U}_{\mathrm{rms}}}{V_{\mathrm{rms}}}. \tag{66}$$

By doing this, we have generalised Eq. (4); we are treating a demodulator – for the purpose of noise calculations - as an amplifier that also changes the frequency.
With this definition, the gain and added noise of an ideal mixer are

$$G_{\mathrm{mixer}} = \begin{cases} \dfrac{\cos\phi}{L_{\mathrm{C}}} & \text{(homodyne configuration)} \\ \dfrac{1}{\sqrt{2}\,L_{\mathrm{C}}} & \text{(heterodyne configuration)} \end{cases} \tag{67}$$

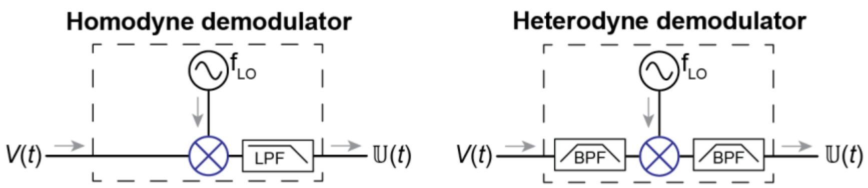

Figure 5: Homodyne and heterodyne demodulation configurations.

$$T_{\mathrm{N,A}} = \begin{cases} \tan^2\phi \times T_{\mathrm{N@in}} & \text{(homodyne configuration)} \\ 0 & \text{(heterodyne configuration)} \end{cases} \tag{68}$$

where $T_{\mathrm{N@in}}$ is the noise temperature at the mixer's RF input, referred to that input. The conventional choice is to set $\phi = 0$, in which case this simplifies to

$$G_{\mathrm{mixer}} = \begin{cases} \dfrac{1}{L_{\mathrm{C}}} & \text{(homodyne configuration)} \\ \dfrac{1}{\sqrt{2}\,L_{\mathrm{C}}} & \text{(heterodyne configuration)} \end{cases} \tag{69}$$

and

$$T_{\mathrm{N,A}} = 0. \tag{70}$$

Equations (69) and (70) are the values to use in Friis' equation.

6.4 How to measure mixer conversion loss

If you want to measure your conversion loss, for example to check the value in the datasheet, this is the procedure:

1. Connect a signal generator to the LO port. Set the signal generator to your desired LO frequency and power specified in the datasheet (e.g. 13 dBm for a Level 13 mixer).
2. If you plan to operate the mixer in homodyne configuration:
    a. Connect a low-pass filter and a voltmeter to the IF port.
    b. Connect a terminator to the RF port and measure the voltmeter reading $\mathbb{U}_0$. This is the output offset.
    c. Instead of the terminator, connect a second signal generator to the RF port and clock it from the 10 MHz reference of the first one. Set its frequency to f_{LO}, and its power P_{RF} well below the LO power. This generator supplies $V(t)$.
    d. Adjust the phase of the first signal generator (you can use a phase shifter, but if so you must account for its insertion loss when setting the LO level) until $\mathbb{U}$ is maximal.
    e. Calculate $\Delta\mathbb{U} \equiv \mathbb{U} - \mathbb{U}_0$. If your voltmeter has input impedance much greater than 50 Ω, then divide $\Delta\mathbb{U}$ by 2, to account for the impedance mismatch of Eq. (18).
    f. The conversion loss is then

$$L_{\mathrm{C}} = \frac{A}{\sqrt{2}\,\Delta\mathbb{U}} = \frac{\sqrt{Z_0 P_{\mathrm{RF}}}}{\Delta\mathbb{U}}. \tag{71}$$

3. If you plan to operate the mixer in heterodyne configuration:
    a. Connect a second generator to the RF port. Set its frequency close to your expected f_S, and its power P_{RF} well below the LO power. This generator supplies $V(t)$.
    b. Connect a spectrum analyser to the IF port. A peak should appear at frequency $|f_{\mathrm{S}} - f_{\mathrm{LO}}|$.
    c. Call the power in this peak P_{IF}. The conversion loss is then

$$L_{\mathrm{C}} = \sqrt{\frac{P_{\mathrm{RF}}}{2\,P_{\mathrm{IF}}}}. \tag{72}$$

7 References

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2. Hughes, I. G. & Hase, T. P. A. *Measurements and Their Uncertainties: A Practical Guide to Modern Error Analysis*. (Oxford University Press, 2010).
3. Pozar, D. M. *Microwave Engineering*. (John Wiley & Sons, 2012).
4. Understanding Mixers-Terms Defined, and Measuring Performance Mixer Terms Defined. https://www.minicircuits.com/app/AN00-009.pdf.