On the Role of Uncertainty in Timing Environmental Policies

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Abstract

We develop an analytical real options framework for the optimal timing of climate policy when both economic cost and stocks of environmental pollutants evolve stochastically and are correlated. By modelling environmental damage as a mean-reverting pollution stock jointly driven with uncertain economic losses, we derive a two-dimensional optimal stopping boundary, moving beyond the single-state or simplified threshold rules commonly used in earlier work, including Pindyck (2000, 2002).

A key theoretical insight is that the option value of waiting depends non-linearly on ecological conditions, reflected in a strictly positive coefficient on pollution in the continuation value, correcting prior simplifications that implied independence from environmental risk. Our analytical characterisation of the exercise boundary shows it is convex and downward-sloping: higher ecological degradation lowers the economic trigger for action.

Most importantly, we find that the correlation between state variables governs how rising volatility affects optimal policy timing. When economic and ecological risks are positively correlated, greater volatility accelerates policy adoption, directly contradicting the classical "uncertainty-delay" effect argued by Pindyck, while negative correlation restores the traditional delay incentive.

These results underscore that treating uncertainties jointly rather than separately can fundamentally change policy recommendations, with profound implications for climate risk governance and investment under transition and physical risk.

Keywords: Real options; Climate risk; Uncertainty; Environmental policy; Irreversibility.

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1 Introduction

The growing threat of climate change entails potentially catastrophic economic consequences, compelling social planners to confront a fundamental dynamic optimisation problem: not only whether to implement emission reduction policies, but crucially when to do so. Standard decision frameworks such as Cost-Benefit Analysis (CBA) and Net Present Value (NPV) overlook the interplay between uncertainty and irreversibility, and therefore, as emphasised by Pindyck (2000), can lead to inefficiently delayed action. The real options approach overcomes these shortcomings by valuing flexibility under uncertainty. However, existing analytical models typically rely on restrictive assumptions, such as treating some state variables as deterministic or assuming independence between economic cost and stocks of environmental pollutants, in order to preserve tractability. This paper relaxes these limitations by deriving an analytical solution for the optimal policy threshold under a structure where the two stochastic state variables are explicitly correlated, and without further simplifications, such as setting the decay rate of the pollutant to zero, as in Pindyck (2002).

Developed through the foundational work of Myers (1977), McDonald and Siegel (1986), Myers and Majd (1990), and Dixit and Pindyck (1994), the framework of real options provides a more comprehensive approach to operation optimisation, confirmed by both theoretical and empirical literature (Truong et al., 2018; Araya et al., 2021; Fan et al., 2023; Truong et al., 2024; Imset et al., 2025). Consequently, we adopt the real options framework as our fundamental model for analysing policy implementation.

In real-world applications of real options theory, multiple state variables are naturally embedded within the analytical framework, capturing future uncertainties and creating additional option value through the ability to delay irreversible decisions. Consequently, real option models should explicitly incorporate multiple sources of uncertainty to better reflect the complexity of real-world optimisation problems. For instance, Maier et al. (2020) value interdependent real options under four uncertainties and demonstrate that neglecting these risk dimensions leads to substantial misvaluation. Oh and Yoon (2020) analyse sequential irreversible investment in the 2002–2011 housing boom–bust cycle under three interacting factors—prices, uncertainty, and construction bottlenecks—and show that delays during the boom were driven primarily by bottlenecks, whereas heightened uncertainty dominated during the bust. Ewald and Taub (2022) derive the optimal exercise rule for a benchmark portfolio in an incomplete market, distinguishing the roles of systematic and idiosyncratic risk, and show that systematic risk may accelerate execution while reducing option value, contrary to standard intuition. A broader body of research similarly evaluates multiple uncertainties—such as price and convenience yield, technological and market demand risk, construction risk and perpetual cash flow uncertainty, or payoff shocks and profitability dynamics (Ewald et al., 2017; Fleten et al., 2020; Thijssen, 2022; Liu et al., 2025).

However, despite this extensive literature, the majority of existing studies concentrate on firm-level decision problems and corporate investment behaviour. Far less attention has been devoted to optimisation problems faced by a social planner, particularly from a macroeconomic and welfare-oriented perspective. In addition, only a limited number of contributions examine climate policy adoption within a real options framework, even though uncertainty, irreversibility, and optimal timing are central features of environmental policy design. This gap underscores the need for analytical models that explicitly incorporate these characteristics when evaluating climate policy interventions.

Given the urgency of climate risk mitigation and the analytical advantages of the real options method, extensive research has addressed economic issues in climate change, including sea level rise, carbon dioxide, electricity, and clean energy (Yao et al., 2020; Matthäus et al., 2021; Flora and Tankov, 2023; Truong et al., 2024; Aghajani et al., 2025; De Weerdt, 2025). Specifically, Guthrie (2019) modify a binomial tree model to incorporate gradual learning through a Bayesian updating process driven by new observations of extreme climate events, finding that the real option to delay investment consistently provides additional value. Deeney et al. (2021) provide a compound real option design using a Poisson process to simulate the discrete progress typical of research and development advancements, demonstrating its application in evaluating CO_2 recycling technology. Basei et al. (2024) propose a real option framework capturing both the uncertainty of the socioeconomic impact of pollution and the unpredictable future social and economic costs, termed the "uncertainty over uncertainty," showing that optimal timing occurs when the learning process becomes decisive. Figuerola-Ferretti et al. (2025) develop a real options valuation model for hydropower plants, considering the uncertainties in prices of electricity and water, water inflows, and drought severity. They highlight the strong seasonality and uncertainty of price dynamics and found that revenues are highly sensitive to the climate conditions. These comprehensive but sophisticated structures for climate risk analysis, however, typically lack analytical solutions and rely on numerical methods, making them difficult to apply in real-world policy scenarios.

To obtain analytically tractable solutions to the climate policy adoption problem, researchers have typically restricted the number of stochastic state variables and sources of uncertainty in real options models. Following the tradition of Conrad (1992), much of the literature achieves mathematical simplification by focusing on a single dominant uncertainty dimension—such as economic damages (Pindyck, 2000; Ferrari and Koch, 2019; Guthrie, 2023), the stock of environmental pollutants (Saphores and Carr, 2000; Saphores, 2004; Sims and Finnoff, 2012), or technological progress (Zeng et al., 2020)—sometimes combined with additional structural assumptions such as barriers or deterministic components. A smaller set of contributions introduces multiple uncertainties, yet commonly maintains independence between them to preserve analytical solvability (Pindyck, 2002). These uncertainties are generally modelled using standard stochastic processes such as Geometric Brownian Motion or mean reversion.

However, such separations are difficult to reconcile with the reality of climate risk. Pollution accumulation and economic damages evolve jointly and are intrinsically connected: ecological deterioration directly amplifies future economic losses, while climate policy is designed precisely to break this link. Ignoring this interdependence for mathematical convenience risks mischaracterising the timing incentives and policy trade-offs. The central message of this paper is therefore that modelling correlated ecological and economic uncertainties is not merely a technical refinement, but essential for understanding the true option value of waiting and the structure of the optimal exercise boundary in climate policy adoption.

In light of these considerations, this paper examines the operational optimisation problem faced by a social planner: whether, and critically when, to implement climate policy aimed at reducing emissions when both the economic cost factor (θ_t) and the stock of environmental pollutants (M_t) evolve stochastically and are correlated. We contrast our findings with Pindyck (2000), who employs a deterministic framework with simplifying assumptions, and Pindyck (2002), who considers two stochastic state variables but assumes zero correlation. In addition, we conduct sensitivity analysis with respect to key parameters, including the volatilities of both state variables ($\sigma_\theta, \sigma_\mathrm{M}$) and the correlation coefficient (ρ).

Our analysis yields several important results. First, the option value of waiting depends not only on economic uncertainty but also non-linearly on ecological conditions, as reflected in the strictly non-zero coefficient ($\mathrm{A}_1 \neq 0$) in the general solution. This challenges the common simplification in earlier studies that effectively removes the influence of pollution dynamics on continuation value. Second, we derive an explicit optimal free boundary that characterises the policy adoption threshold as a trade-off between total benefits and total costs; this boundary is jointly determined by volatility and correlation. Third, when correlation is accounted for, the free boundary becomes a curve that is consistently convex with a negative slope, implying that a higher pollution stock justifies a lower economic cost threshold for policy implementation.

Most importantly, we show that the correlation coefficient plays a decisive role in determining how uncertainty affects policy timing. Under positive correlation ($\rho > 0$), increases in either the correlation coefficient or ecological volatility (σ_{M}) accelerate policy adoption—directly contradicting the classic "uncertainty-delay" conclusion of Pindyck (2000). By contrast, when the state variables are negatively correlated ($\rho < 0$), higher volatility (σ_θ or σ_{M}) or stronger negative correlation delays implementation, consistent with traditional real options intuition. Finally, in the uncorrelated case ($\rho = 0$), ecological volatility (σ_{M}) has no effect on timing, while higher economic volatility (σ_θ) delays policy execution, again yielding dynamics distinct from Pindyck (2000).

Our contribution to the literature can be summarised in three main dimensions. First, we relax the restrictive theoretical assumptions in Pindyck (2000) by modelling climate policy adoption in a framework where both the economic cost factor and the stock of environmental pollutants evolve stochastically and are explicitly correlated. In doing so, we also correct the common simplifying assumption that the option value of waiting is independent of ecological conditions ($\mathrm{A}_1 = 0$), showing that such an assumption is economically implausible when the state variables are correlated. We formally prove that this coefficient is strictly non-zero and derive a general solution involving the imaginary error function, thereby establishing a more rigorous theoretical foundation for climate policy timing. Second, we obtain an analytical expression for the optimal free boundary governed by a genuinely two-dimensional stochastic system, advancing beyond the single trigger point formulations prevalent in earlier work. Third, contrary to the widely held view that rising volatility necessarily delays policy adoption, we demonstrate that higher volatility in both the economic cost factor and the pollution stock can accelerate policy implementation when the correlation is positive, consistent with the insights of Ewald and Taub (2022).

The remainder of the paper is structured as follows. Section 2 presents the theoretical framework, introducing the correlated dynamics of the economic cost factor and the stock of environmental pollutants. Section 3 compares our findings with Pindyck (2000) and Pindyck (2002) and provides a sensitivity analysis based on the derived optimal free boundary. Section 4 summarises the main results and discusses potential directions for future research. The Appendix contains the technical derivations supporting the key analytical results.

2 The model

Building on the framework presented in Pindyck (2000), we model economic cost and stocks of environmental pollutants using correlated stochastic processes and formulate the corresponding objective function for climate policy design. Then, we separate the continuity and adoption regions and derive the associated Hamilton-Jacobi-Bellman equations (hereafter HJB equations), along with their particular and general solutions. After that, we analyse the economic and mathematical behaviour when these state variables reach their limits. Finally, we obtain an analytical expression for the optimal free boundary by applying the value-matching and smooth-pasting conditions.

2.1 Model setup

Pindyck (2000) and Pindyck (2002) assume that the flow of social costs due to emissions is determined through a function $\mathrm{B}(\theta_t, \mathrm{M}_t)$, where M_t represents the quantity of greenhouse gases in the atmosphere and θ_t is a cost factor. Pindyck (2000) assumes that at least one of the two is deterministic. Pindyck (2002) allows both to be stochastic, but requires that the two processes are uncorrelated, and in addition that greenhouse gases in the atmosphere have no rate of decay. However, these assumptions are unrealistic, according to the scientific and economic evidence documented by Nordhaus (2017), Rennert et al. (2022) and IPCC (2021). Pindyck (2000) and Pindyck (2002) do these simplifications in order to solve their models analytically. However, as we will show, these simplifications are unnecessary. A more sophisticated analysis of the underlying HJB equation allows us to still obtain analytical explicit expressions for the value functions and thresholds, even in the more general case. Furthermore, the general conclusions from Pindyck (2000) and Pindyck (2002) change significantly; in particular, increased levels of uncertainty in M_t can lead to accelerated action rather than a delay.

Let us now introduce the modelling framework in more detail. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is taken to be the augmented natural filtration generated by correlated Brownian motions W^1 and W^2. We assume that the economic cost factor θ_t follows a geometric Brownian motion:

$$\mathrm{d}\theta_t/\theta_t = \alpha \mathrm{d}t + \sigma_\theta \mathrm{dW}_t^1.$$

Following Saphores and Carr (2000), we assume that the policy adoption reduces exogenous pollution emissions from an initial level E^1 to a lower level E^2, incurring a positive sunk cost $\mathrm{K}(\mathrm{E}^2)$, with

$$\mathrm{K}(\mathrm{E}^2) = k(\mathrm{E}^1 - \mathrm{E}^2), \tag{1}$$

where E^1 and E^2 are constants satisfying $0 \leq \mathrm{E}^2 < \mathrm{E}^1$. The state variable M_t, representing the evolution of environmental pollution levels over time, evolves in two stages: the pre-policy stage M_t^1 associated with initial emission (E^1), and the post-policy stage M_t^2 associated with the reduced emission (E^2). In each stage, we assume M_t^i follows an Ornstein-Uhlenbeck (OU) process:

$$\mathrm{dM}_t^i = (\beta \mathrm{E}^i - \delta \mathrm{M}_t^i)\mathrm{d}t + \sigma_\mathrm{M} \mathrm{dW}_t^2, \quad i = 1, 2,$$

where β represents the proportion of the emission transition (the transfer rate of pollution emission to the stocks M_t^i), δ is the natural rate of dissipation of M_t^i, and σ_M is a non-negative constant that characterises the scale of stocks of environmental pollutants. We capture the correlation of θ_t and M_t by the correlation coefficient ρ between their Brownian motions:

$$\mathrm{dW}_t^1 \mathrm{dW}_t^2 = \rho \mathrm{d}t.$$

To reflect a more realistic situation and facilitate derivations, we specify the flow of social cost associated with stocks of environmental pollutants M_t as:

$$\mathrm{B}(\theta_t, \mathrm{M}_t) = -\theta_t (\mathrm{M}_t)^2. \tag{2}$$

Here $(\mathrm{M}_t)^2$ is the loss caused by the total amount of pollution, and θ_t represents a stochastic cost parameter that scales the severity of these economic losses. Given the stochastic dynamics of M_t^i and

θ_t with initial values M and θ, respectively, the purpose of a social planner is to determine the optimal stopping time τ to maximise the present value function by permanently reducing the emission from E^1 to E^2 and paying a sunk cost $\mathrm{K}(\mathrm{E}^2)$:

$$\mathrm{W}(\theta,\mathrm{M}) = \sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0\left[\int_0^\tau \mathrm{B}(\theta_t,\mathrm{M}_t^1)e^{-rt}\mathrm{d}t - \mathrm{K}(\mathrm{E}^2)e^{-r\tau} + \int_\tau^\infty \mathrm{B}(\theta_t,\mathrm{M}_t^2)e^{-rt}\mathrm{d}t\right]. \tag{3}$$

Here $\mathcal{T}_0$ denotes the set of stopping times with respect to the filtration $(\mathcal{F}_t)_{0\le t}$ and values in $[0,\infty]$, $\mathrm{E}_0[\cdot] = \mathrm{E}[\cdot]$ denotes the unconditional expectation, and r is the risk-free interest rate.

2.2 Derivation of the Hamilton-Jacobi-Bellman equations

Using the sunk cost (1) and social cost (2), we separate the decision space into the policy non-adoption region, with associated value function $\mathrm{W}^\mathrm{N}(\theta,\mathrm{M})$, and the policy adoption region, with value function $\mathrm{W}^\mathrm{A}(\theta,\mathrm{M})$. Then the option values in these two regions are determined as part of the solution to the optimal stopping problem defined in (3).

In the non-adoption region ($0 < t < \tau$), emissions equal E^1, and running costs follow $\mathrm{B}(\theta_t,\mathrm{M}_t) = -\theta_t(\mathrm{M}_t^1)^2$. The stochastic process M_t follows:

$$\mathrm{dM}_t^1 = (\beta\mathrm{E}^1 - \delta\mathrm{M}_t^1)\mathrm{d}t + \sigma_\mathrm{M}\mathrm{dW}_t^2.$$

Applying the two-dimensional Itô Lemma and the dynamic programming principle (hereafter DPP) formally gives,

$$r\mathrm{W}^\mathrm{N}(\theta,\mathrm{M})\mathrm{d}t = \mathrm{B}(\theta,\mathrm{M})\mathrm{d}t + \mathrm{E}[\mathrm{dW}^\mathrm{N}(\theta,\mathrm{M})],$$

which leads to the following HJB equation for the value function $\mathrm{W}^\mathrm{N}(\theta,\mathrm{M})$ (see Appendix A for the full derivation),

$$r\mathrm{W}^\mathrm{N} = -\theta\mathrm{M}^2 + \Bigg[(\alpha\theta)\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\theta} + (\beta\mathrm{E}^1 - \delta\mathrm{M})\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\mathrm{M}} + \frac{1}{2}(\theta^2\sigma_\theta^2)\frac{\partial^2\mathrm{W}^\mathrm{N}}{\partial\theta^2} + \frac{1}{2}(\sigma_\mathrm{M}^2)\frac{\partial^2\mathrm{W}^\mathrm{N}}{\partial\mathrm{M}^2} + (\theta\sigma_\theta\sigma_\mathrm{M}\rho)\frac{\partial^2\mathrm{W}^\mathrm{N}}{\partial\theta\partial\mathrm{M}}\Bigg]. \tag{4}$$

In Equation (4) and throughout the paper, we suppress explicit argument dependence in partial derivatives for the ease of notation. For example, we denote $\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\theta}(\theta,\mathrm{M})$ and $\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\mathrm{M}}(\theta,\mathrm{M})$ by $\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\theta}$ and $\frac{\partial\mathrm{W}^\mathrm{N}}{\partial\mathrm{M}}$, respectively, and apply the same convention to higher-order derivatives.

Next, in the adoption region ($\tau \le t \le \infty$), emissions are set to E^2, and the running costs here are $\mathrm{B}(\theta_t,\mathrm{M}_t) = -\theta_t(\mathrm{M}_t^2)^2$. Consequently, the dynamics of M_t satisfy:

$$\mathrm{dM}_t^2 = (\beta\mathrm{E}^2 - \delta\mathrm{M}_t^2)\mathrm{d}t + \sigma_\mathrm{M}\mathrm{dW}_t^2.$$

Once again, by applying Itô's Lemma and the DPP, we obtain the following heuristic infinitesimal relation for $\mathrm{W}^\mathrm{A}(\theta,\mathrm{M})$,

$$r\mathrm{W}^\mathrm{A}(\theta,\mathrm{M})\mathrm{d}t = \mathrm{B}(\theta,\mathrm{M})\mathrm{d}t + \mathrm{E}[\mathrm{dW}^\mathrm{A}(\theta,\mathrm{M})],$$

from which we obtain the following HJB equation for the adaptation region (see the detailed steps in Appendix A):

$$rW^{A} = -\theta M^2 + \left[(\alpha\theta)\frac{\partial W^{A}}{\partial\theta} + (\beta E^2 - \delta M)\frac{\partial W^{A}}{\partial M} + \frac{1}{2}(\theta^2\sigma_\theta^2)\frac{\partial^2 W^{A}}{\partial\theta^2} + \frac{1}{2}(\sigma_M^2)\frac{\partial^2 W^{A}}{\partial M^2} + (\theta\sigma_\theta\sigma_M\rho)\frac{\partial^2 W^{A}}{\partial\theta\partial M}\right]. \tag{5}$$

2.3 The particular solution

Before we proceed with the derivation, it is instructive to clarify the structure of the solutions to the HJB equations derived above. Mathematically, equations (4) and (5) are linear non-homogeneous partial differential equations (PDE), where the non-homogeneous term corresponds to the running flow of social costs, $B(\theta_t, M_t) = -\theta_t M_t^2$. Following standard differential equation theory, the complete solution is the sum of a particular solution to the non-homogeneous equation and a general solution to the corresponding homogeneous equation.

Economically, this decomposition has a clear interpretation. The particular solution represents the expected present value of the perpetual social costs assuming the current carbon emission regime (E^1 or E^2) continues indefinitely. The general solution, which solves the homogeneous equation (where the running cost equals zero), captures the option value of waiting, namely the value of the flexibility to switch regimes at an optimal future date. Accordingly, let $P^N(\theta, M)$ and $P^A(\theta, M)$ be the particular solutions. Since $W^A(\theta, M)$ is the value after policy adoption ($t \in [\tau, \infty]$, emission E^2), no further option exists:

$$W^{A}(\theta, M) = P^{A}(\theta, M). \tag{6}$$

Conversely, the value before adoption $W^N(\theta, M)$ ($t \in [0, \tau]$, emission E^1) is the sum of the present value of expected future cost under permanent E^1 ($P^N(\theta, M)$) and the general solution $\varphi(\theta, M)$, which is the value of flexibility from optimal exercising time τ with paying $K(E^2)$ and reducing emissions to E^2:

$$W^{N}(\theta, M) = P^{N}(\theta, M) + \varphi(\theta, M). \tag{7}$$

In the remainder of this section, we will make the assumptions of the form for particular solutions $P^A(\theta, M)$ and $P^N(\theta, M)$, and use the HJB equations above to obtain the analytical results of these solutions.

2.3.1 Derivation of $P^A(\theta, M)$

Since the partial differential equation (5) for $W^A(\theta, M)$ contains a linear term in θ and a quadratic term in M, we assume that its particular solution takes the form:

$$P^{A}(\theta, M) = c_{A,1}\theta M^2 + c_{A,2}\theta M + c_{A,3}\theta + c_{A,4}M^2 + c_{A,5}M + c_{A,6}. \tag{8}$$

The corresponding partial derivatives are:

$$\begin{cases} \frac{\partial P^{A}}{\partial\theta} = c_{A,1}M^2 + c_{A,2}M + c_{A,3} \\ \frac{\partial^2 P^{A}}{\partial\theta^2} = 0 \\ \frac{\partial P^{A}}{\partial M} = 2c_{A,1}\theta M + c_{A,2}\theta + 2c_{A,4}M + c_{A,5} \\ \frac{\partial^2 P^{A}}{\partial M^2} = 2c_{A,1}\theta + 2c_{A,4} \\ \frac{\partial^2 P^{A}}{\partial\theta\partial M} = 2c_{A,1}M + c_{A,2}. \end{cases}$$

Substituting these terms into the PDE and rearranging yields:

$$
\begin{aligned}
r\Big[&c_{\mathrm{A},1}\theta\mathrm{M}^2 + c_{\mathrm{A},2}\theta\mathrm{M} + c_{\mathrm{A},3}\theta + c_{\mathrm{A},4}\mathrm{M}^2 + c_{\mathrm{A},5}\mathrm{M} + c_{\mathrm{A},6}\Big] \\
&= -\theta\mathrm{M}^2 + \Big[(\alpha\theta)(c_{\mathrm{A},1}\mathrm{M}^2 + c_{\mathrm{A},2}\mathrm{M} + c_{\mathrm{A},3}) \\
&\quad + (\beta\mathrm{E}^2 - \delta\mathrm{M})(2c_{\mathrm{A},1}\theta\mathrm{M} + c_{\mathrm{A},2}\theta + 2c_{\mathrm{A},4}\mathrm{M} + c_{\mathrm{A},5}) \\
&\quad + \frac{1}{2}\sigma_{\mathrm{M}}^2(2c_{\mathrm{A},1}\theta + 2c_{\mathrm{A},4}) + (2c_{\mathrm{A},1}\mathrm{M} + c_{\mathrm{A},2})(\theta\sigma_\theta\sigma_{\mathrm{M}}\rho)\Big].
\end{aligned}
$$

We then collect terms by $\theta\mathrm{M}^2$, $\theta\mathrm{M}$, θ, M^2, M, and constant:

$$
\begin{aligned}
&(rc_{\mathrm{A},1} + 1 - c_{\mathrm{A},1}\alpha + 2c_{\mathrm{A},1}\delta)\theta\mathrm{M}^2 + (rc_{\mathrm{A},2} - c_{\mathrm{A},2}\alpha - 2c_{\mathrm{A},1}\beta\mathrm{E}^2 + c_{\mathrm{A},2}\delta - 2c_{\mathrm{A},1}\sigma_\theta\sigma_{\mathrm{M}}\rho)\theta\mathrm{M} \\
&\quad + (rc_{\mathrm{A},3} - c_{\mathrm{A},3}\alpha - c_{\mathrm{A},2}\beta\mathrm{E}^2 - c_{\mathrm{A},1}\sigma_{\mathrm{M}}^2 - c_{\mathrm{A},2}\sigma_\theta\sigma_{\mathrm{M}}\rho)\theta + (rc_{\mathrm{A},4} + 2c_{\mathrm{A},4}\delta)\mathrm{M}^2 \\
&\quad + (rc_{\mathrm{A},5} - 2c_{\mathrm{A},4}\beta\mathrm{E}^2 + c_{\mathrm{A},5}\delta)\mathrm{M} + (rc_{\mathrm{A},6} - c_{\mathrm{A},5}\beta\mathrm{E}^2 - c_{\mathrm{A},4}\sigma_{\mathrm{M}}^2) = 0.
\end{aligned}
$$

For this equation to hold for all values of θ and M, the coefficient of each term must be zero:

$$
\begin{cases}
rc_{\mathrm{A},1} + 1 - c_{\mathrm{A},1}\alpha + 2c_{\mathrm{A},1}\delta = 0 \\
rc_{\mathrm{A},2} - c_{\mathrm{A},2}\alpha - 2c_{\mathrm{A},1}\beta\mathrm{E}^2 + c_{\mathrm{A},2}\delta - 2c_{\mathrm{A},1}\sigma_\theta\sigma_{\mathrm{M}}\rho = 0 \\
rc_{\mathrm{A},3} - c_{\mathrm{A},3}\alpha - c_{\mathrm{A},2}\beta\mathrm{E}^2 - c_{\mathrm{A},1}\sigma_{\mathrm{M}}^2 - c_{\mathrm{A},2}\sigma_\theta\sigma_{\mathrm{M}}\rho = 0 \\
rc_{\mathrm{A},4} + 2c_{\mathrm{A},4}\delta = 0 \\
rc_{\mathrm{A},5} - 2c_{\mathrm{A},4}\beta\mathrm{E}^2 + c_{\mathrm{A},5}\delta = 0 \\
rc_{\mathrm{A},6} - c_{\mathrm{A},5}\beta\mathrm{E}^2 - c_{\mathrm{A},4}\sigma_{\mathrm{M}}^2 = 0.
\end{cases}
$$

Solving this system gives:

$$
\begin{cases}
c_{\mathrm{A},1} = -\frac{1}{r+2\delta-\alpha} \\
c_{\mathrm{A},2} = -\frac{2(\beta\mathrm{E}^2+\sigma_{\mathrm{M}}\sigma_\theta\rho)}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\
c_{\mathrm{A},3} = \frac{1}{r-\alpha}\left[-\frac{\sigma_{\mathrm{M}}^2}{r+2\delta-\alpha} - \frac{2(\beta\mathrm{E}^2+\sigma_\theta\sigma_{\mathrm{M}}\rho)^2}{(r+2\delta-\alpha)(r+\delta-\alpha)}\right] \\
c_{\mathrm{A},4} = c_{\mathrm{A},5} = c_{\mathrm{A},6} = 0.
\end{cases}
\tag{9}
$$

Here we assume $r + 2\delta - \alpha > 0$, $r + \delta - \alpha > 0$, and $r - \alpha > 0$ to ensure the solution $\mathrm{P^A}(\theta, \mathrm{M})$ remains finite. We also assume non-zero denominators: $r + 2\delta \neq 0$, $r + \delta \neq 0$, and $r \neq 0$ for mathematical validity.

Finally, substituting these coefficients back into Equation (8) yields the particular solution:

$$
\begin{aligned}
\mathrm{P^A}(\theta, \mathrm{M}) &= -\frac{\theta\mathrm{M}^2}{r+2\delta-\alpha} - \frac{2\theta\mathrm{M}(\beta\mathrm{E}^2+\sigma_{\mathrm{M}}\sigma_\theta\rho)}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\
&+ \quad \frac{\theta}{r-\alpha}\left[-\frac{\sigma_{\mathrm{M}}^2}{r+2\delta-\alpha} - \frac{2(\beta\mathrm{E}^2+\sigma_\theta\sigma_{\mathrm{M}}\rho)^2}{(r+2\delta-\alpha)(r+\delta-\alpha)}\right].
\end{aligned}
\tag{10}
$$

2.3.2 Derivation of $\mathrm{P^N}(\theta, \mathrm{M})$

Analogous to Equation (8), we assume the particular solution for $\mathrm{P^N}(\theta, \mathrm{M})$ takes the form:

$$
\mathrm{P^N}(\theta, \mathrm{M}) = c_{\mathrm{N},1}\theta\mathrm{M}^2 + c_{\mathrm{N},2}\theta\mathrm{M} + c_{\mathrm{N},3}\theta + c_{\mathrm{N},4}\mathrm{M}^2 + c_{\mathrm{N},5}\mathrm{M} + c_{\mathrm{N},6}.
$$

We also group terms by $\theta\mathrm{M}^2$, $\theta\mathrm{M}$, θ, M^2, M, and constant. Solving the resulting system for the coefficients gives:

$$\begin{cases} c_{\mathrm{N},1} = -\frac{1}{r+2\delta-\alpha} \\ c_{\mathrm{N},2} = -\frac{2(\beta\mathrm{E}^1+\sigma_\mathrm{M}\sigma_\theta\rho)}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\ c_{\mathrm{N},3} = \frac{1}{r-\alpha}\left[-\frac{\sigma_\mathrm{M}^2}{r+2\delta-\alpha} - \frac{2(\beta\mathrm{E}^1+\sigma_\theta\sigma_\mathrm{M}\rho)^2}{(r+2\delta-\alpha)(r+\delta-\alpha)}\right] \\ c_{\mathrm{N},4} = c_{\mathrm{N},5} = c_{\mathrm{N},6} = 0. \end{cases} \tag{11}$$

Consequently, the particular solution $\mathrm{P}^\mathrm{N}(\theta, \mathrm{M})$ is

$$\begin{aligned} \mathrm{P}^\mathrm{N}(\theta,\mathrm{M}) = & -\frac{\theta\mathrm{M}^2}{r+2\delta-\alpha} - \frac{2\theta\mathrm{M}(\beta\mathrm{E}^1+\sigma_\mathrm{M}\sigma_\theta\rho)}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\ & + \frac{\theta}{r-\alpha}\left[-\frac{\sigma_\mathrm{M}^2}{r+2\delta-\alpha} - \frac{2(\beta\mathrm{E}^1+\sigma_\theta\sigma_\mathrm{M}\rho)^2}{(r+2\delta-\alpha)(r+\delta-\alpha)}\right]. \end{aligned}$$

2.4 The general solution of $\mathrm{W}^\mathrm{N}(\theta, \mathrm{M})$

The general solution $\varphi(\theta, \mathrm{M})$ in Equation (7) represents the additional option value for waiting. It is the solution to the homogeneous part of the HJB equation (4) of $\mathrm{W}^\mathrm{N}(\theta, \mathrm{M})$ excluding the running cost $-\theta\mathrm{M}^2$:

$$\begin{aligned} r\varphi = (\alpha\theta)\frac{\partial\varphi}{\partial\theta} + (\beta\mathrm{E}^1 - \delta\mathrm{M})\frac{\partial\varphi}{\partial\mathrm{M}} + \frac{1}{2}(\theta^2\sigma_\theta^2)\frac{\partial^2\varphi}{\partial\theta^2} \\ + \frac{1}{2}(\sigma_\mathrm{M}^2)\frac{\partial^2\varphi}{\partial\mathrm{M}^2} + (\theta\sigma_\theta\sigma_\mathrm{M}\rho)\frac{\partial^2\varphi}{\partial\theta\partial\mathrm{M}}. \end{aligned} \tag{12}$$

Inspired by the first term of Equation (10) in Pindyck (2000), we assume and later verify that the general solution $\varphi(\theta, \mathrm{M})$ takes the following form:

$$\varphi(\theta,\mathrm{M}) = \mathrm{A}f(\mathrm{M})\theta^\gamma. \tag{13}$$

We can prove that Equation (12) reduces to an ordinary differential equation (ODE) for the function $f(\mathrm{M})$:

$$\frac{1}{2}(\sigma_\mathrm{M}^2)f''(\mathrm{M}) + \left[(\beta\mathrm{E}^1 - \delta\mathrm{M}) + (\rho\sigma_\mathrm{M}\sigma_\theta\gamma)\right]f'(\mathrm{M}) = 0, \tag{14}$$

where

$$\gamma = \gamma_1 = \frac{1}{2} - \frac{\alpha}{\sigma_\theta^2} + \sqrt{(\frac{\alpha}{\sigma_\theta^2} - \frac{1}{2})^2 + \frac{2r}{\sigma_\theta^2}} > 1. \tag{15}$$

One can then obtain the general solution $\varphi(\theta, \mathrm{M})$ as:

$$\varphi(\theta,\mathrm{M}) = \theta^\gamma\left[\mathrm{A}_1\,\mathrm{erfi}\left[\frac{\sqrt{\delta}}{\sigma_\mathrm{M}}\left(\mathrm{M} - \frac{\eta}{\delta}\right)\right] + \mathrm{A}_2\right], \tag{16}$$

where

$$\begin{cases} \mathrm{A}_1 = \mathrm{A} \cdot \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right) \frac{\sqrt{\pi\sigma_{\mathrm{M}}^2}}{2\sqrt{\delta}} \\ \mathrm{A}_2 = \mathrm{A}\left[\mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right)\sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}}\mathrm{C}_3 + \mathrm{C}_2\right] \\ \eta = \beta \mathrm{E}^1 + \rho\sigma_{\mathrm{M}}\sigma_\theta\gamma. \end{cases}$$

Here, A, C_1, C_2, and C_3 are undetermined coefficients (or integration constants), determined by the boundary conditions at the optimal free boundary. The explicit derivation of these parameters is provided in Section 2.6. All detailed derivations for the general solution can be found in Appendix B.

2.5 Limiting behavior of the value function

In this section, we examine the economic significance of extreme initial values of the state variables θ_t and M_t to see whether these results align with intuition when θ and M reach their limits. To this end, we will analyse the limit behaviour of the particular solution $\mathrm{P}^{\mathrm{N}}(\theta, \mathrm{M})$ associated with policy adoption, and the general solution component $\varphi(\theta, \mathrm{M})$, which captures the option value of waiting. Since the particular solution $\mathrm{P}^{\mathrm{A}}(\theta, \mathrm{M})$ in the non-adoption region has an analogous structure, we omit it for brevity.

We begin with the economic cost factor. The process θ_t follows a geometric Brownian motion. When the initial value θ is close to zero, θ_t remains small with high probability. Consequently, the flow of social cost $\mathrm{B}(\theta_t, \mathrm{M}_t) = -\theta_t \mathrm{M}_t^2$ vanishes for all values of M_t. Economically, this corresponds to a situation in which pollution generates no social damage. In this case, the particular solution $\mathrm{P}^{\mathrm{N}}(\theta, \mathrm{M})$ converges to zero. In addition, this also implies that the social benefit of reducing emissions from E^1 to E^2 becomes negligible. However, the sunk cost $\mathrm{K}(\mathrm{E}^2)$ for adopting the policy remains positive, so immediate adoption is never optimal. Consistently, the option value of waiting $\varphi(\theta, \mathrm{M}) = \mathrm{A}f(\mathrm{M})\theta^\gamma$ also approaches zero as $\theta \to 0$.

Conversely, when the initial value θ is large ($\theta \to \infty$), the geometric Brownian motion implies that θ_t is likely to remain high. In this case, the social cost $\mathrm{B}(\theta_t, \mathrm{M}_t)$ becomes enormous, even for small values of M_t. This indicates that the impact of the pollutant could be catastrophic. As a result, paying the cost $\mathrm{K}(\mathrm{E}^2)$ to reduce emissions becomes extremely attractive. Thus, the option value $\varphi(\theta, \mathrm{M})$ approaches infinity.

We next consider the role of the initial pollution stock M. As $\mathrm{M} \to 0$, current social damages vanish. However, since emissions $\mathrm{E}^1 > 0$, the pollution stock process M_t is mean-reverting and tends to its long-run mean $\beta\mathrm{E}^1/\delta$. Thus, even starting from a clean state, future pollution accumulation is inevitable, generating positive expected social costs. This feature is reflected in the particular solution $\mathrm{P}^{\mathrm{N}}(\theta, \mathrm{M})$, which remains strictly positive. Consequently, the general solution retains a non-zero component as $\mathrm{M} \to 0$, as evidenced by the presence of the term $\mathrm{A}_2\theta^\gamma$ in Equation (16). Economically, this confirms that the option to adopt the policy remains valuable even when current pollution levels are negligible, because it mitigates future environmental damages.

Finally, an important difference between our framework and that of Pindyck (2000) appears when $\mathrm{M} \to \infty$. In deriving Equation (10) of his paper, Pindyck (2000) assumes that M_t is deterministic and the social cost $\mathrm{B}(\theta_t, \mathrm{M}_t)$ is a linear function of M_t. Under these simplified assumptions, the general solution $\varphi(\theta, \mathrm{M})$ becomes independent of M, which in turn leads to the conclusion that the coefficient A_1 of the general solution must be zero (cf. Pindyck (2000, Section 3.2)). In contrast, we adopt a more general and economically realistic specification in Equation (13). We explicitly model M_t as a stochastic, mean-reverting process. This specification implies that the future evolution of pollution stocks M_t depends on their current level M. As a result, the option value of waiting, $\varphi(\theta, M)$, depends not only on the economic cost factor θ but also on the current pollution stock M in a nonlinear manner, yielding $A_1 \neq 0$. Economically, this reflects the fact that when pollution stocks are high, the marginal value of delaying policy adoption is affected by both the persistence and the stochastic evolution of environmental damages. Consequently, assuming $A_1 = 0$ may substantially understate the value of waiting when economic costs and environmental dynamics interact, highlighting a key limitation of models that treat pollution stocks deterministically.

2.6 Value matching and smooth pasting conditions

In this section, we use the value matching and smooth pasting conditions to simultaneously solve for the optimal free boundary and the two unknown constants A_1 and A_2 in (16). Specifically, we characterise this optimal boundary not as a single adoption point, but as a continuous curve $(\theta^\star(M^\star), M^\star)$ within the two-dimensional state space. For notation simplicity, we let $c_1 = c_{A,1} = c_{N,1}$ (as shown in Equations (9) and (11)), and define $Z(M^\star) = \frac{\sqrt{\delta}}{\sigma_M}(M^\star - \frac{\eta}{\delta})$. First, the value matching condition for any point $(\theta^\star, M^\star)$ with $\theta^\star := \theta^\star(M^\star)$ on the free boundary is given by:

$$W^N(\theta^\star, M^\star) = W^A(\theta^\star, M^\star) - K(E^2).$$

Substituting the structures from Equations (6) and (7), we get

$$P^N(\theta^\star, M^\star) + \varphi(\theta^\star, M^\star) = P^A(\theta^\star, M^\star) - K(E^2).$$

By expanding this using the particular and general solutions, we obtain:

$$\begin{aligned}&\left(c_1\theta^\star(M^\star)^2 + c_{N,2}\theta^\star M^\star + c_{N,3}\theta^\star\right) + \left[(\theta^\star)^\gamma(A_1 \operatorname{erfi}(Z(M^\star)) + A_2)\right]\\ &= \left(c_1\theta^\star(M^\star)^2 + c_{A,2}\theta^\star M^\star + c_{A,3}\theta^\star\right) - K(E^2).\end{aligned}$$

Rearranging the terms yields:

$$(\theta^\star)^\gamma\left[A_1 \operatorname{erfi}(Z(M^\star)) + A_2\right] = (c_{A,2} - c_{N,2})\theta^\star M^\star + (c_{A,3} - c_{N,3})\theta^\star - K(E^2). \quad (17)$$

Next, we apply the smooth pasting condition with respect to $M^\star$

$$\frac{\partial W^N}{\partial M}(\theta^\star, M^\star) = \frac{\partial W^A}{\partial M}(\theta^\star, M^\star).$$

Using the general and particular solutions, this becomes:

$$\frac{\partial P^N}{\partial M}(\theta^\star, M^\star) + \frac{\partial \varphi}{\partial M}(\theta^\star, M^\star) = \frac{\partial P^A}{\partial M}(\theta^\star, M^\star). \quad (18)$$

To facilitate the calculation, we list and evaluate the required partial derivatives for any point $(\theta^\star, M^\star)$

on the free boundary below:

$$\begin{cases} \frac{\partial \mathrm{P^N}}{\partial \mathrm{M}} = 2c_1\theta\mathrm{M} + c_{\mathrm{N},2}\theta \\ \frac{\partial \mathrm{P^N}}{\partial \theta} = c_1\mathrm{M}^2 + c_{\mathrm{N},2}\mathrm{M} + c_{\mathrm{N},3} \\ \frac{\partial \mathrm{P^A}}{\partial \mathrm{M}} = 2c_1\theta\mathrm{M} + c_{\mathrm{A},2}\theta \\ \frac{\partial \mathrm{P^A}}{\partial \theta} = c_1\mathrm{M}^2 + c_{\mathrm{A},2}\mathrm{M} + c_{\mathrm{A},3} \\ \frac{\partial \varphi}{\partial \mathrm{M}} = \mathrm{A}_1\frac{2\sqrt{\delta}}{\sqrt{\pi}\sigma_{\mathrm{M}}}\theta^{\gamma}\exp\big(\mathrm{Z(M)}^2\big) \\ \frac{\partial \varphi}{\partial \theta} = \gamma\theta^{\gamma-1}[\mathrm{A}_1\,\mathrm{erfi}(\mathrm{Z(M)}) + \mathrm{A}_2]. \end{cases}$$

Substituting these derivatives into Equation (18) gives:

$$(2c_1\theta^\star\mathrm{M}^\star + c_{\mathrm{N},2}\theta^\star) + \left[\mathrm{A}_1\frac{2\sqrt{\delta}}{\sqrt{\pi}\sigma_{\mathrm{M}}}(\theta^\star)^{\gamma}\exp\Big(\mathrm{Z}(\mathrm{M}^\star)^2\Big)\right] = (2c_1\theta^\star\mathrm{M}^\star + c_{\mathrm{A},2}\theta^\star).$$

Simplifying this equation yields:

$$(c_{\mathrm{A},2} - c_{\mathrm{N},2})\theta^\star = \mathrm{A}_1\frac{2\sqrt{\delta}}{\sqrt{\pi}\sigma_{\mathrm{M}}}(\theta^\star)^{\gamma}\exp\Big(\mathrm{Z}(\mathrm{M}^\star)^2\Big). \tag{19}$$

Similarly, the smooth pasting condition with respect to $\theta^\star(\mathrm{M}^\star)$ yields:

$$\frac{\partial \mathrm{W^N}}{\partial \theta}(\theta^\star, \mathrm{M}^\star) = \frac{\partial \mathrm{W^A}}{\partial \theta}(\theta^\star, \mathrm{M}^\star).$$

This expands to:

$$\frac{\partial \mathrm{P^N}}{\partial \theta}(\theta^\star, \mathrm{M}^\star) + \frac{\partial \varphi}{\partial \theta}(\theta^\star, \mathrm{M}^\star) = \frac{\partial \mathrm{P^A}}{\partial \theta}(\theta^\star, \mathrm{M}^\star).$$

Substituting the relevant derivatives and rearranging gives:

$$(c_1(\mathrm{M}^\star)^2 + c_{\mathrm{N},2}\mathrm{M}^\star + c_{\mathrm{N},3}) + \gamma(\theta^\star)^{\gamma-1}[\mathrm{A}_1\,\mathrm{erfi}(\mathrm{Z}(\mathrm{M}^\star)) + \mathrm{A}_2] = (c_1(\mathrm{M}^\star)^2 + c_{\mathrm{A},2}\mathrm{M}^\star + c_{\mathrm{A},3}). \tag{20}$$

Assuming $\gamma \neq 0$ and $\theta^\star(\mathrm{M}^\star) \neq 0$, we solve the system of equations formed by Equations (17), (19), and (20). This yields the following relationship:

$$\underbrace{\mathrm{K}(\mathrm{E}^2)\left(\frac{\gamma}{\gamma-1}\right)}_{\text{Total cost for policy adoption}} = \underbrace{\theta^\star[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})]}_{\text{Total benefit for policy adoption}}, \tag{21}$$

where

$$\begin{cases} \mathrm{A}_1 = \frac{(c_{\mathrm{A},2}-c_{\mathrm{N},2})\sqrt{\pi}\sigma_{\mathrm{M}}}{2\sqrt{\delta}}(\theta^\star)^{1-\gamma}\exp\big(-\mathrm{Z}(\mathrm{M}^\star)^2\big) \\ \mathrm{A}_2 = (\theta^\star)^{1-\gamma}\left[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3}) - \frac{\mathrm{K}(\mathrm{E}^2)}{\theta^\star}\right] - \mathrm{A}_1\,\mathrm{erfi}\left(\mathrm{Z}(\mathrm{M}^\star)\right). \end{cases}$$

The optimal free boundary can then be expressed implicitly as:

$$(1 - \frac{1}{\gamma})\theta^\star\left[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})\right] = \mathrm{K}(\mathrm{E}^2), \tag{22}$$

or by deriving $\theta^\star(\mathrm{M}^\star)$ as an explicit function of $\mathrm{M}^\star$:

$$\theta^\star(\mathrm{M}^\star) = \frac{\mathrm{K}(\mathrm{E}^2)^{\frac{\gamma}{\gamma-1}}}{(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})}. \tag{23}$$

Detailed steps for solving this system are provided in Appendix C. Unlike Pindyck (2000), which derives a single optimal point, our framework results in a system of three simultaneous equations for the optimal free boundary $(\theta^\star(\mathrm{M}^\star), \mathrm{M}^\star)$ and the constants A_1 and A_2. Since this system involves transcendental equations with special functions and exponential terms, a fully closed-form analytical solution is not possible. Therefore, numerical methods are required to solve for the optimal free boundary $(\theta^\star(\mathrm{M}^\star), \mathrm{M}^\star)$, A_1, and A_2.

3 Further analysis

3.1 Comparison with the single stochastic state variable models in Pindyck (2000)

Pindyck (2000) considers a framework in which only one state variable is stochastic at a time. Specifically, either the economic cost factor θ_t is modelled as stochastic, while the pollution stock M_t is treated as deterministic, corresponding to $\sigma_\mathrm{M} = 0$, or the pollution stock M_t is stochastic, while the economic cost factor θ_t is deterministic, corresponding to $\sigma_\theta = 0$. In both cases, the model effectively reduces to a single source of uncertainty.

In this section, we compare our two-state variables model with the deterministic benchmark cases of Pindyck (2000) by setting the relevant diffusion coefficients (σ_M or σ_θ) to zero. We first show that our particular solutions reduce to those obtained in Pindyck (2000) under these restrictions. We then compare the resulting expressions for the optimal stopping boundary to confirm the robustness of our results under the two-dimensional setting.

First, setting $\sigma_\mathrm{M} = 0$ in Equation (10) for $\mathrm{P}^\mathrm{A}(\theta, \mathrm{M})$, we have

$$\mathrm{P}^\mathrm{A}(\theta, \mathrm{M}) = -\frac{\theta \mathrm{M}^2}{r + 2\delta - \alpha} - \frac{2\theta \mathrm{M}(\beta \mathrm{E}^2)}{(r + 2\delta - \alpha)(r + \delta - \alpha)} - \frac{2\theta(\beta \mathrm{E}^2)^2}{(r - \alpha)(r + 2\delta - \alpha)(r + \delta - \alpha)}.$$

Applying the same assumption as in Pindyck (2000) as $\mathrm{B}(\theta_t, \mathrm{M}_t) = -\theta_t \mathrm{M}_t^2$ and $\beta \mathrm{E}^2 = 0$ yields

$$\mathrm{P}^\mathrm{A}(\theta, \mathrm{M}) = -\frac{\theta \mathrm{M}^2}{r + 2\delta - \alpha},$$

which is identical to Equation (29) of Pindyck (2000). Secondly, setting $\sigma_\theta = 0$ in Equation (10) gives:

$$\mathrm{P}^\mathrm{A}(\theta, \mathrm{M}) = -\frac{\theta \mathrm{M}^2}{r + 2\delta - \alpha} - \frac{2\theta \mathrm{M}(\beta \mathrm{E}^2)}{(r + 2\delta - \alpha)(r + \delta - \alpha)} + \frac{\theta}{r - \alpha}\left[-\frac{\sigma_\mathrm{M}^2}{r + 2\delta - \alpha} - \frac{2(\beta \mathrm{E}^2)^2}{(r + 2\delta - \alpha)(r + \delta - \alpha)}\right].$$

Applying the assumption in Section 5.1 of Pindyck (2000) including $\mathrm{B}(\theta_t, \mathrm{M}_t) = -\theta_t \mathrm{M}_t^2$, $\beta \mathrm{E}^2 = 0$, and $\alpha = 0$ derived from remaining θ fixed, we have

$$\mathrm{P}^\mathrm{A}(\theta, \mathrm{M}) = -\frac{\theta \mathrm{M}^2}{r + 2\delta} - \frac{\theta \sigma_\mathrm{M}^2}{r(r + 2\delta)},$$

which is identical to Equation (48) of Pindyck (2000). These results confirm the robustness of our particular solution derivations.

Having established this consistency at the level of value functions, we then turn to the implied optimal stopping boundary. Under the deterministic assumption used in Section 3.3 of Pindyck (2000), we have

$$\begin{cases} \sigma_{\mathrm{M}} = 0 \Longrightarrow \rho = 0 \\ \mathrm{E}^2 = 0 \\ \mathrm{K} = k(\mathrm{E}^1 - \mathrm{E}^2) = k\mathrm{E}^1. \end{cases}$$

The coefficients in Equations (9) and (11) become:

$$\begin{cases} c_{\mathrm{A},2} = 0 \\ c_{\mathrm{A},3} = 0 \\ c_{\mathrm{N},2} = -\frac{2\beta\mathrm{E}^1}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\ c_{\mathrm{N},3} = \frac{1}{r-\alpha}\left[\frac{-2(\beta\mathrm{E}^1)^2}{(r+2\delta-\alpha)(r+\delta-\alpha)}\right]. \end{cases}$$

Substituting the simplified coefficients into the optimal free boundary (22) gives

$$(1-\frac{1}{\gamma})\theta^\star\left[\frac{2\beta\mathrm{E}^1}{(r+2\delta-\alpha)(r+\delta-\alpha)}\mathrm{M}^\star + \frac{2(\beta\mathrm{E}^1)^2}{(r-\alpha)(r+2\delta-\alpha)(r+\delta-\alpha)}\right] = k\mathrm{E}^1.$$

Solving for $\theta^\star$ yields:

$$\begin{aligned} \theta^\star(\mathrm{M}^\star) &= \frac{(k\mathrm{E}^1)\frac{\gamma}{\gamma-1}}{\left[\frac{2\beta\mathrm{E}^1}{(r+2\delta-\alpha)(r+\delta-\alpha)}\mathrm{M}^\star + \frac{2(\beta\mathrm{E}^1)^2}{(r-\alpha)(r+2\delta-\alpha)(r+\delta-\alpha)}\right]} \\ &= \frac{(k\mathrm{E}^1)\frac{\gamma}{\gamma-1}}{\frac{2\beta\mathrm{E}^1}{(r+2\delta-\alpha)(r+\delta-\alpha)}\left[\mathrm{M}^\star + \frac{\beta\mathrm{E}^1}{r-\alpha}\right]} \\ &= \frac{\gamma}{\gamma-1}k\frac{(r+2\delta-\alpha)(r+\delta-\alpha)}{2\beta(\mathrm{M}^\star + \frac{\beta\mathrm{E}^1}{r-\alpha})} \\ &= \frac{(r-\alpha)(r+2\delta-\alpha)(r+\delta-\alpha)k\gamma}{2\beta(\gamma-1)[\beta\mathrm{E}^1 + (r-\alpha)\mathrm{M}^\star]}. \end{aligned}$$

This result is consistent with Equation (31) in Pindyck (2000).

3.2 Comparison with the two-dimensional framework of Pindyck (2002)

As discussed in Section 2.1, while Pindyck (2000) considers only one source of uncertainty at a time, Pindyck (2002) extends the analysis to a two-state-variable framework in which both the cost factor and the pollution stock evolve stochastically. However, a key restriction in his model is that the two underlying Brownian motions are assumed to be uncorrelated, i.e. $\rho = 0$. In this section, we therefore compare the optimal stopping boundary implied by our model with that obtained in Pindyck (2002) by setting the correlation parameter to zero and show that relaxing the strict condition of $\rho = 0$ has important consequences on the qualitative behavior of both the value function and the exercise boundary.

Applying the assumption $\rho = 0$ and decay rate $\delta = 0$, and simplifying Equations (9) and (11), we can obtain the differences of coefficient as:

$$\begin{cases} c_{\mathrm{A},2} - c_{\mathrm{N},2} = \frac{2\beta \mathrm{E}^1}{(r-\alpha)^2} \\ c_{\mathrm{A},3} - c_{\mathrm{N},3} = \frac{2(\beta \mathrm{E}^1)^2}{(r-\alpha)^3}. \end{cases}$$

Substituting these terms into Equation (23) gives:

$$\begin{aligned} \theta^\star(\mathrm{M}^\star) &= \frac{\mathrm{K}(\mathrm{E}^2)\frac{\gamma}{\gamma-1}}{\frac{2\beta\mathrm{E}^1}{(r-\alpha)^2}\mathrm{M}^\star + \frac{2(\beta\mathrm{E}^1)^2}{(r-\alpha)^3}} \\ &= \frac{\mathrm{K}(\mathrm{E}^2)\gamma(r-\alpha)^3}{(\gamma-1)\left[2\beta\mathrm{E}^1(r-\alpha)\mathrm{M}^\star + 2\beta^2(\mathrm{E}^1)^2\right]} \\ &= \frac{\mathrm{K}(\mathrm{E}^2)\gamma(r-\alpha)^3}{2\beta(\gamma-1)\mathrm{E}^1\left[(r-\alpha)\mathrm{M}^\star + \beta\mathrm{E}^1\right]}. \end{aligned}$$

This result aligns with Pindyck (2002)'s Equation (46) when we let $\Omega(\mathrm{M}^\star) \equiv (r-\alpha)\mathrm{M}^\star + \beta\mathrm{E}^1$. These comparisons confirm that our model is a generalised form of the previous simplified models.

3.3 Sensitivity analysis

In this section, we analyse how changes in the key parameters (σ_θ, σ_M, and ρ) affect the optimal free boundary $(\theta^\star(\mathrm{M}^\star), \mathrm{M}^\star)$ in Equation (21). Note that these parameters primarily influence the boundary through γ, $(c_{\mathrm{A},2} - c_{\mathrm{N},2})$, and $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$. The coefficients γ, $c_{\mathrm{A},2}$, $c_{\mathrm{N},2}$, $c_{\mathrm{A},3}$, and $c_{\mathrm{N},3}$ are defined by the systems in Equations (9), (11), and (15). Using these definitions, the relevant expressions can be written explicitly as

$$\begin{cases} c_{\mathrm{A},2} - c_{\mathrm{N},2} = \frac{2\beta(\mathrm{E}^1 - \mathrm{E}^2)}{(r+2\delta-\alpha)(r+\delta-\alpha)} \\ c_{\mathrm{A},3} - c_{\mathrm{N},3} = \frac{2[(\beta\mathrm{E}^1 + \sigma_\theta\sigma_\mathrm{M}\rho)^2 - (\beta\mathrm{E}^2 + \sigma_\theta\sigma_\mathrm{M}\rho)^2]}{(r-\alpha)(r+2\delta-\alpha)(r+\delta-\alpha)} \\ \gamma = \gamma_1 = \frac{1}{2} - \frac{\alpha}{\sigma_\theta^2} + \sqrt{(\frac{\alpha}{\sigma_\theta^2} - \frac{1}{2})^2 + \frac{2r}{\sigma_\theta^2}}. \end{cases}$$

It can be shown that both $(c_{\mathrm{A},2} - c_{\mathrm{N},2})$, $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$, and $\mathrm{K}(\mathrm{E}^2)\frac{\gamma}{\gamma-1}$ are positive since we have $\mathrm{K}(\mathrm{E}^2) > 0$ and $\gamma > 1$. Therefore, the first and second order derivatives of $\theta^\star$ with respect to $\mathrm{M}^\star$ become

$$\begin{cases} \frac{\mathrm{d}\theta^\star(\mathrm{M}^\star)}{\mathrm{dM}^\star} = \frac{-\left[\mathrm{K}(\mathrm{E}^2)\frac{\gamma}{\gamma-1}\right](c_{\mathrm{A},2} - c_{\mathrm{N},2})}{\left[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})\right]^2} < 0 \\ \frac{d^2\theta^\star(\mathrm{M}^\star)}{(\mathrm{dM}^\star)^2} = \frac{2\left[\mathrm{K}(\mathrm{E}^2)\frac{\gamma}{\gamma-1}\right](c_{\mathrm{A},2} - c_{\mathrm{N},2})^2}{\left[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})\right]^3} > 0. \end{cases}$$

We consequently conclude that the optimal free boundary is a convex function that exhibits a negative slope, regardless of the specific parameter levels, which determine the location of the optimal free boundary. This means that a higher threshold of ecological pollution stock $\mathrm{M}^\star$ lowers the required economic cost threshold $\theta^\star$ for policy execution, reflecting the urgency caused by environmental deterioration.

3.3.1 Sensitivity to economic cost volatility σ_θ

The changes of σ_θ influence the decision through two channels: the total cost (via γ) and the total benefit (via the term $c_{A,3} - c_{N,3}$). Firstly, we analyse γ. When σ_θ increases, γ decreases and approaches 1 from above (denoted here as 1^+). Consequently, the denominator $(\gamma - 1)$ approaches zero from the positive side (0^+), causing the cost multiplier $\frac{\gamma}{\gamma-1} \to \infty$. From this perspective, high volatility of the economic cost factor increases the threshold cost for policy adoption. Therefore, execution should be delayed. Next, we analyze the term $(c_{A,3} - c_{N,3})$. Define the function:

$$f(\sigma_\theta) = (\beta \mathrm{E}^1 + \sigma_\theta \sigma_{\mathrm{M}} \rho)^2 - (\beta \mathrm{E}^2 + \sigma_\theta \sigma_{\mathrm{M}} \rho)^2.$$

Taking the derivative with respect to σ_θ, we obtain:

$$\begin{aligned} f'(\sigma_\theta) &= 2(\beta \mathrm{E}^1 + \sigma_\theta \sigma_{\mathrm{M}} \rho)(\sigma_{\mathrm{M}} \rho) - 2(\beta \mathrm{E}^2 + \sigma_\theta \sigma_{\mathrm{M}} \rho)(\sigma_{\mathrm{M}} \rho) \\ &= 2(\sigma_{\mathrm{M}} \rho)\beta(\mathrm{E}^1 - \mathrm{E}^2). \end{aligned}$$

Since σ_{M}, β, and $(\mathrm{E}^1 - \mathrm{E}^2)$ are all positive, the sign of this derivative depends entirely on the sign of the correlation coefficient ρ. The resulting shifts in the optimal free boundary are illustrated in the first row of Figure 1.

We analyse three specific cases. Let us recall the definitions of the total cost and total benefit for policy adoption introduced in Equation (21). If $\rho = 0$, then $f'(\sigma_\theta) = 0$. The total benefit remains unchanged. However, since the total cost side increases, the net effect is that policy adoption should be delayed. The optimal free boundary shifts to the upper right, as shown in Case A of Figure 1. If $\rho > 0$, which corresponds to Case B in Figure 1, the derivative is positive. When σ_θ increases, both $(c_{A,3} - c_{N,3})$ and the total benefit increase. In this scenario, the final decision depends on the competition between the increasing cost (from $\frac{\gamma}{\gamma-1}$) and the increasing benefit. If $\rho < 0$, the term $(c_{A,3} - c_{N,3})$ and the total benefit decrease when σ_θ increases. Since the cost side is increasing and the benefit side is decreasing, this leads to a strong signal to delay adoption. As shown in Case C, the optimal free boundary moves significantly further to the upper right.

3.3.2 Sensitivity to environmental pollutants volatility σ_{M}

By examining the optimal free boundary (21), we observe that both the total cost for policy adoption and the term $(c_{A,2} - c_{N,2})$ are independent of σ_{M}. Only the term $(c_{A,3} - c_{N,3})$ is affacted by the changes of σ_{M}. To analyze this, let $f(\sigma_{\mathrm{M}}) = (\beta \mathrm{E}^1 + \sigma_\theta \sigma_{\mathrm{M}} \rho)^2 - (\beta \mathrm{E}^2 + \sigma_\theta \sigma_{\mathrm{M}} \rho)^2$. Taking the derivative with respect to σ_{M}, we get:

$$\begin{aligned} f'(\sigma_{\mathrm{M}}) &= 2(\beta \mathrm{E}^1 + \sigma_\theta \sigma_{\mathrm{M}} \rho)(\sigma_\theta \rho) - 2(\beta \mathrm{E}^2 + \sigma_\theta \sigma_{\mathrm{M}} \rho)(\sigma_\theta \rho) \\ &= 2(\sigma_\theta \rho)\beta(\mathrm{E}^1 - \mathrm{E}^2). \end{aligned}$$

Since σ_θ, β, and $(\mathrm{E}^1 - \mathrm{E}^2)$ are all positive, the sign of this derivative depends entirely on the sign of the correlation coefficient ρ. The resulting shifts in the curves are shown in the second row of Figure 1.

In the first scenario, if the correlation coefficient ρ equals zero, then $f'(\sigma_{\mathrm{M}}) = 0$. This means the total benefit remains unchanged as σ_{M} varies. Consequently, changes in σ_{M} do not influence the optimal free boundary $(\theta^\star(\mathrm{M}^\star), \mathrm{M}^\star)$. As shown in Case D of Figure 1, the optimal free boundaries for different values of σ_{M} overlap completely. Noteably, this conclusion differs from that in Section 5.1 of Pindyck (2000),

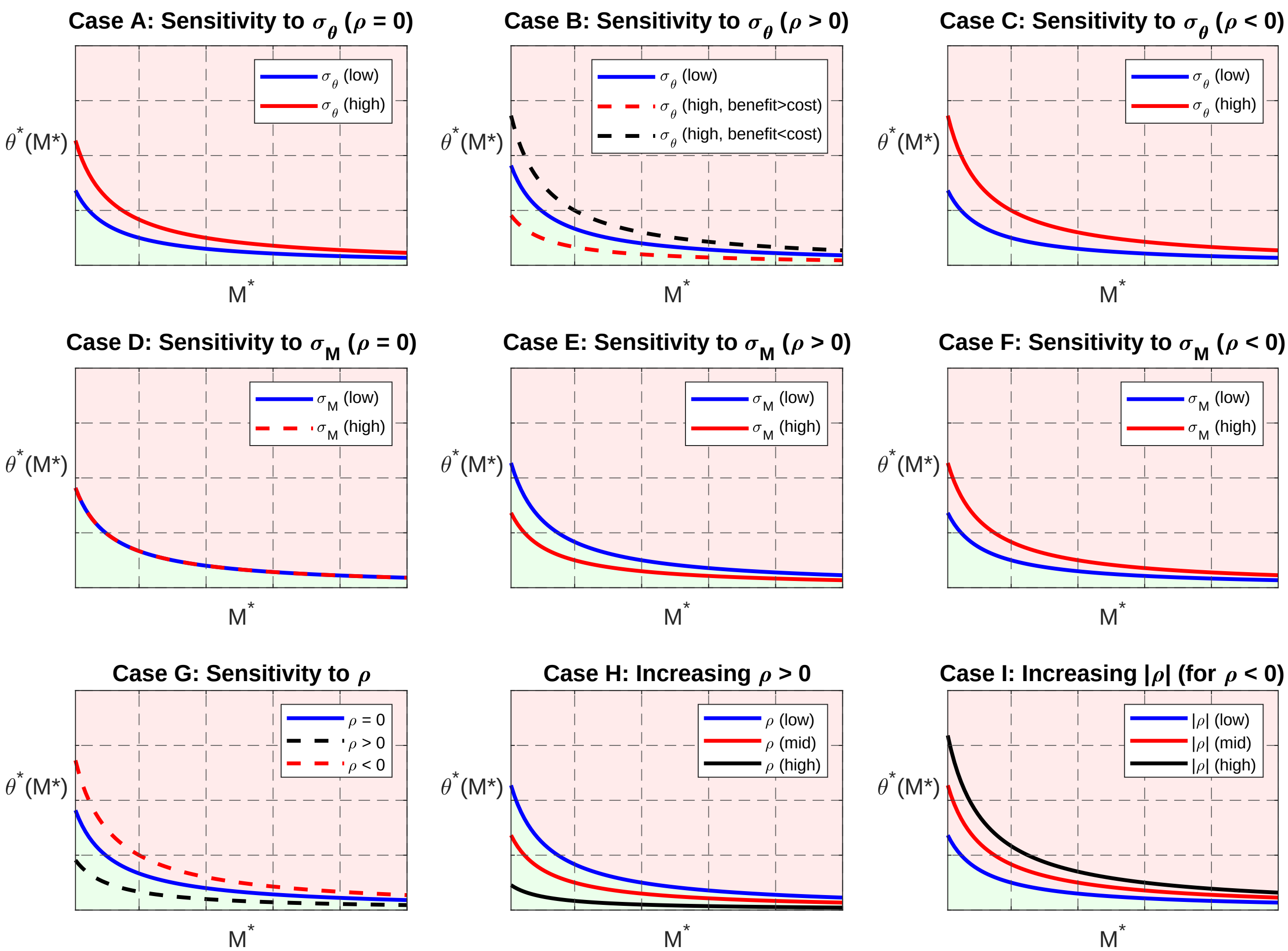

Figure 1: Changes of optimal free boundary

Notes: This figure illustrates how the optimal free boundary $(\theta^\star(\mathrm{M}^\star), \mathrm{M}^\star)$ shifts in response to changes in σ_θ, σ_{M}, and ρ from low to high. The first row shows the sensitivity to the volatility of economic cost (σ_θ). The second row shows the sensitivity to the volatility of stocks of environmental pollutants (σ_{M}). The final row shows the sensitivity to the correlation coefficient (ρ). The columns separate the effects based on the sign of ρ. The green and red areas represent the continuation region and the policy adoption region, respectively.

which suggests that increased ecological volatility delays climate policy. We attribute this difference to the structure of the models: In our model, the effects of σ_{M} on the total benefit are completely cancelled out because the terms $c_{\mathrm{A},3}$ and $c_{\mathrm{N},3}$ contain identical components regarding ecological volatility. The additional costs caused by σ_{M} changes in the non-adoption and adoption regions are the same. However, the economic cost factor θ is treated as fixed, and the square of the ecological volatility σ_{M}^2 appears directly in the value function $\mathrm{W}^{\mathrm{N}}(\mathrm{M})$, which is shown in Equations (50) and (51) of Pindyck (2000). This directly affects the curvature of the boundary, leading to a delay in execution.

In the second case when $\rho > 0$, the derivative $f'(\sigma_{\mathrm{M}})$ is positive. As σ_{M} increases, the term $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$ and the total benefit both increase. This accelerates policy adoption. Graphically, the curve in Case E of Figure 1 shifts downward and to the left (meaning adoption happens at lower thresholds). When $\rho < 0$, the derivative $f'(\sigma_{\mathrm{M}}) < 0$ holds. As σ_{M} increases, the term $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$ and the total benefit decrease. This delays policy adoption. Graphically, the optimal free boundary shifts upward and to the right, as depicted in Case F. When volatility increases under positive correlation ρ, our results are consistent with a "wait-and-see" mechanism in the sense discussed by Basei et al. (2024), insofar as uncertainty affects the timing option through its interaction with irreversibility. However, their analysis also highlights that higher uncertainty does not generically imply delayed action. In particular, when uncertainty operates through discrete changes in the future drift—captured via Bayesian learning—an increase in uncertainty may instead lower the real option exercise threshold and induce earlier intervention. This mechanism is distinct from ours, but it underscores that the effect of uncertainty on optimal timing is model-dependent and need not conform to the standard uncertainty-delay paradigm.

3.3.3 Sensitivity to correlation coefficient ρ

We now examine the influence of the correlation coefficient ρ. We observe that ρ only affects the term $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$. Both the total cost of policy adoption and the term $(c_{\mathrm{A},2} - c_{\mathrm{N},2})$ are independent of ρ. Let $f(\rho) = (\beta\mathrm{E}^1 + \sigma_\theta\sigma_{\mathrm{M}}\rho)^2 - (\beta\mathrm{E}^2 + \sigma_\theta\sigma_{\mathrm{M}}\rho)^2$, Differentiating with respect to ρ, we demonstrate that:

$$\begin{aligned} f'(\rho) &= 2(\beta\mathrm{E}^1 + \sigma_\theta\sigma_{\mathrm{M}}\rho)(\sigma_\theta\sigma_{\mathrm{M}}) - 2(\beta\mathrm{E}^2 + \sigma_\theta\sigma_{\mathrm{M}}\rho)(\sigma_\theta\sigma_{\mathrm{M}}) \\ &= 2(\sigma_\theta\sigma_{\mathrm{M}})\beta(\mathrm{E}^1 - \mathrm{E}^2) > 0. \end{aligned}$$

This derivative is strictly positive under all conditions. The corresponding shifts in the optimal free boundary are shown in the third row of Figure 1.

First, we consider Case G in Figure 1 where $\rho = 0$. Here, the economic cost factor $\theta^\star(\mathrm{M}^\star)$ and stocks of environmental pollutants $\mathrm{M}^\star$ are independent. By examining the equation for $(c_{\mathrm{A},2} - c_{\mathrm{N},2})$ and $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$, we conclude that the total benefit of policy adoption is determined solely by the difference between E^1 and E^2 (which remains positive). Economically, the two sources of state variables contribute to the option value separately; they neither reinforce nor offset each other.

Second, when the state variables are positively correlated, both $(c_{\mathrm{A},3} - c_{\mathrm{N},3})$ and the total benefit increase since $f'(\rho) > 0$. Consequently, policy adoption should be accelerated. As shown in Case H of Figure 1, the optimal free boundary moves downward and to the left. In this scenario, the state variables reinforce each other. A higher pollution stock $\mathrm{M}^\star$ tends to occur simultaneously with a higher economic loss $\theta^\star(\mathrm{M}^\star)$. This reflects a scenario of physical risk: an increased probability of natural disasters is accompanied by greater future economic losses. Faced with this compound risk, the social planner must adopt emission reduction policies as early as possible. The conclusion here is consistent with Ewald and

Taub (2022), which also demonstrates that a higher positive correlation will have the effect of a lower threshold from the perspective of the convenience yield.

Finally, if $\rho < 0$ then $(c_{A,3} - c_{N,3})$ and the total benefit decrease as the correlation coefficient becomes more negative (i.e., as ρ decreases). This raises the threshold for action, meaning policy adoption should be delayed. The optimal free boundary moves upward and to the right, as in Case I. In this case, the state variables hedge against each other. High stocks of environmental pollutants $M^\star$ tend to coincide with lower economic cost $\theta^\star(M^\star)$. We interpret this as transition risk: the social planner takes necessary measures (such as carbon taxes or pollution penalties) to prevent natural disasters. These interventions help decouple the economic loss from the ecological damage, reducing the urgency of the specific policy adoption modelled here.

4 Conclusions

This paper develops a real options framework for the optimal timing of climate policy under two correlated sources of uncertainty: economic damages per unit of pollution and the stock of atmospheric pollutants. By allowing both state variables to follow stochastic dynamics and explicitly modelling their correlation, we move beyond the assumptions in Pindyck (2000), where one state variable is deterministic, and Pindyck (2002), which sets correlation to zero and eliminates pollutant decay. Our approach yields a richer and more realistic characterisation of the incentives driving delayed climate action.

Our results highlight three core contributions. First, we show that the option value of waiting necessarily depends on the current pollution stock, as reflected in the presence of a strictly positive coefficient on ecological conditions in the continuation value. This structurally contradicts the simplifications in earlier studies that imply independence from environmental risk. Second, we derive the optimal policy not as a single trigger point but as a two-dimensional exercise boundary. This boundary is monotonically decreasing and convex: a more damaging ecological state lowers the required economic trigger for action. Third, and most importantly, we demonstrate that the relationship between uncertainty and the timing of policy is governed by correlation. Under positive correlation, greater volatility accelerates climate action, directly overturning Pindyck's classical "uncertainty-delay" conclusion. Negative correlation restores the standard real options intuition, while the uncorrelated case implies ecological volatility is irrelevant for timing.

These findings have clear implications. Policymakers should not assume that waiting is always valuable: when climate and economic risks rise together, deferral becomes costly and irreversible damages accumulate. Likewise, firms exposed to transition and physical risks should expect earlier and more decisive policy responses when risks become strongly connected.

Looking ahead, several natural extensions of our framework remain. These include incorporating jump risks associated with extreme climate events, modelling endogenous and stochastic abatement costs, and expanding the set of uncertainties relevant for climate governance, possibly with the inclusion of ambiguity and parameter uncertainty as introduced in Agarwal et al. (2025). Ultimately, recognising the interaction between multiple sources of uncertainty is essential: climate policy should be designed for a world where risks rarely evolve in isolation.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the authors used ChatGPT and Grammarly in order to improve grammar and wording. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Declaration of competing interest

There are no conflicts of interest to declare.

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A Details for deriving the Hamilton-Jacobi-Bellman equations

In the non-adoption region, $\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M})$ represents the expected discounted present value of the future flow of social costs:

$$\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) = \mathrm{E}\left[\int_0^{\infty} \mathrm{B}(\theta_s, \mathrm{M}_s) e^{-rs} \mathrm{d}s\right].$$

We split this integral into two parts: the contribution over a small time interval $[0, \mathrm{d}t]$, and the remaining value from time $\mathrm{d}t$ onwards:

$$\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) = \mathrm{E}\left[\int_0^{\mathrm{d}t} \mathrm{B}(\theta_s, \mathrm{M}_s) e^{-rs} \mathrm{d}s + \int_{\mathrm{d}t}^{\infty} \mathrm{B}(\theta_s, \mathrm{M}_s) e^{-rs} \mathrm{d}s\right].$$

We approximate the first term as $\mathrm{B}(\theta, \mathrm{M})\mathrm{d}t$ in a small time interval $\mathrm{d}t$. The second term represents the value function $\mathrm{W}^{\mathrm{N}}(\theta + \mathrm{d}\theta, \mathrm{M} + \mathrm{dM})$ at the new state $(\theta + \mathrm{d}\theta, \mathrm{M} + \mathrm{dM})$, discounted back to time 0:

$$\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) = \mathrm{B}(\theta, \mathrm{M})\mathrm{d}t + e^{-r\mathrm{d}t} \mathrm{E}\left[\mathrm{W}^{\mathrm{N}}(\theta + \mathrm{d}\theta, \mathrm{M} + \mathrm{dM})\right]. \tag{24}$$

Then we introduce the total differential $\mathrm{E}[\mathrm{dW}^{\mathrm{N}}] = \mathrm{E}[\mathrm{W}^{\mathrm{N}}(\theta + \mathrm{d}\theta, \mathrm{M} + \mathrm{dM})] - \mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M})$. Using the first-order Taylor expansion $e^{-r\mathrm{d}t} = 1 - r\mathrm{d}t + o(\mathrm{d}t)$, we substitute these terms into Equation (24):

$$\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) = \mathrm{B}(\theta, \mathrm{M})\mathrm{d}t + (1 - r\mathrm{d}t + o(\mathrm{d}t))\left(\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) + \mathrm{E}\left[\mathrm{dW}^{\mathrm{N}}\right]\right).$$

Ignoring higher-order terms in $\mathrm{d}t$ and rearranging yields the infinitesimal form implied by the DPP for $\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M})$:

$$r\mathrm{W}^{\mathrm{N}}\mathrm{d}t = \mathrm{B}(\theta, \mathrm{M})\mathrm{d}t + \mathrm{E}\left[\mathrm{dW}^{\mathrm{N}}\right].$$

We then apply the two-dimensional Itô's Lemma to $\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M})$. Note that the term involving $\frac{\partial \mathrm{W}^{\mathrm{N}}}{\partial t}$ is zero because the value function is time-invariant in this infinite-horizon problem. This gives:

$$\mathrm{dW}^{\mathrm{N}} = \frac{\partial \mathrm{W}^{\mathrm{N}}}{\partial \theta}\mathrm{d}\theta_t + \frac{\partial \mathrm{W}^{\mathrm{N}}}{\partial \mathrm{M}}\mathrm{dM}_t^1 + \frac{1}{2}\frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \theta^2}(\mathrm{d}\theta_t)^2 + \frac{1}{2}\frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \mathrm{M}^2}(\mathrm{dM}_t^1)^2 + \frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \theta \partial \mathrm{M}}\mathrm{d}\theta_t \mathrm{dM}_t^1.$$

The stochastic differentials satisfy the following rules:

$$\begin{cases} (\mathrm{d}\theta_t)^2 = (\sigma_\theta \theta_t \mathrm{dW}_t^1)^2 = \theta_t^2 \sigma_\theta^2 \mathrm{d}t \\ (\mathrm{dM}_t^1)^2 = (\sigma_{\mathrm{M}} \mathrm{dW}_t^2)^2 = \sigma_{\mathrm{M}}^2 \mathrm{d}t \\ \mathrm{d}\theta_t \mathrm{dM}_t^1 = (\sigma_\theta \theta_t \mathrm{dW}_t^1)(\sigma_{\mathrm{M}} \mathrm{dW}^2) = \theta_t \sigma_\theta \sigma_{\mathrm{M}} \rho \mathrm{d}t. \end{cases}$$

Therefore, the expansion of dW^{N} is:

$$\begin{aligned} \mathrm{dW}^{\mathrm{N}} = &(\alpha \theta_t \mathrm{d}t + \sigma_\theta \theta_t \mathrm{dW}_t^1)\frac{\partial \mathrm{W}^{\mathrm{N}}}{\partial \theta} + \left[(\beta \mathrm{E}^1 - \delta \mathrm{M}_t^1)\mathrm{d}t + \sigma_{\mathrm{M}} \mathrm{dW}^2\right]\frac{\partial \mathrm{W}^{\mathrm{N}}}{\partial \mathrm{M}} \\ &+ \frac{1}{2}(\theta_t^2 \sigma_\theta^2)\frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \theta^2}\mathrm{d}t + \frac{1}{2}(\sigma_{\mathrm{M}}^2)\frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \mathrm{M}^2}\mathrm{d}t + (\theta_t \sigma_\theta \sigma_{\mathrm{M}} \rho)\frac{\partial^2 \mathrm{W}^{\mathrm{N}}}{\partial \theta \partial \mathrm{M}}\mathrm{d}t. \end{aligned}$$

Grouping the deterministic terms (dt) and the stochastic terms (dW^1_t, dW^2_t) gives:

$$dW^N = \left[(\alpha\theta_t)\frac{\partial W^N}{\partial\theta} + (\beta E^1 - \delta M^1_t)\frac{\partial W^N}{\partial M} + \frac{1}{2}(\theta_t^2\sigma_\theta^2)\frac{\partial^2 W^N}{\partial\theta^2} + \frac{1}{2}(\sigma_M^2)\frac{\partial^2 W^N}{\partial M^2} + (\theta_t\sigma_\theta\sigma_M\rho)\frac{\partial^2 W^N}{\partial\theta\partial M}\right]dt$$
$$+ (\sigma_\theta\theta_t)\frac{\partial W^N}{\partial\theta}dW^1_t + (\sigma_M)\frac{\partial W^N}{\partial M}dW^2_t.$$

We substitute this expression back into the DPP of $W^N(\theta, M)$. Since the expectations of the Brownian motions are zero ($E[dW^1_t] = 0$ and $E[dW^2_t] = 0$), those terms vanish. Dividing both sides by dt, we obtain Equation (4). Similarly, for the policy adoption region, the DPP for $W^A(\theta, M)$ is:

$$rW^A dt = B(\theta, M)dt + E\left[dW^A\right].$$

Applying Itô's Lemma again:

$$dW^A = \frac{\partial W^A}{\partial\theta}d\theta_t + \frac{\partial W^A}{\partial M}dM^2_t + \frac{1}{2}\frac{\partial^2 W^A}{\partial\theta^2}(d\theta_t)^2 + \frac{1}{2}\frac{\partial^2 W^A}{\partial M^2}(dM^2_t)^2 + \frac{\partial^2 W^A}{\partial\theta\partial M}d\theta_t dM^2_t.$$

Expanding and rearranging dW^A yields:

$$dW^A = \left[(\alpha\theta_t)\frac{\partial W^A}{\partial\theta} + (\beta E^2 - \delta M^2_t)\frac{\partial W^A}{\partial M} + \frac{1}{2}(\theta_t^2\sigma_\theta^2)\frac{\partial^2 W^A}{\partial\theta^2} + \frac{1}{2}(\sigma_M^2)\frac{\partial^2 W^A}{\partial M^2} + (\theta_t\sigma_\theta\sigma_M\rho)\frac{\partial^2 W^A}{\partial\theta\partial M}\right]dt$$
$$+ (\sigma_\theta\theta_t)\frac{\partial W^A}{\partial\theta}dW^1_t + (\sigma_M)\frac{\partial W^A}{\partial M}dW^2_t.$$

Substituting this into the DPP for $W^A(\theta, M)$ yields Equation (5).

B Derivation of the general solution

We assume the general solution $\varphi(\theta, M)$ takes the form given in Equation (13). The partial derivatives are:

$$\begin{cases} \frac{\partial\varphi}{\partial\theta} = A\gamma f(M)\theta^{\gamma-1} \\ \frac{\partial^2\varphi}{\partial\theta^2} = A\gamma(\gamma-1)f(M)\theta^{\gamma-2} \\ \frac{\partial\varphi}{\partial M} = Af'(M)\theta^\gamma \\ \frac{\partial^2\varphi}{\partial M^2} = Af''(M)\theta^\gamma \\ \frac{\partial^2\varphi}{\partial\theta\partial M} = A\gamma f'(M)\theta^{\gamma-1}. \end{cases}$$

Substituting these into the PDE (12), we get:

$$r\left[Af(M)\theta^\gamma\right] = (\alpha\theta)\left[A\gamma f(M)\theta^{\gamma-1}\right] + (\beta E^1 - \delta M)\left[Af'(M)\theta^\gamma\right] + \frac{1}{2}(\sigma_\theta^2\theta^2)\left[A\gamma(\gamma-1)f(M)\theta^{\gamma-2}\right]$$
$$+ \frac{1}{2}(\sigma_M^2)\left[Af''(M)\theta^\gamma\right] + (\rho\sigma_M\sigma_\theta\theta)\left[A\gamma f'(M)\theta^{\gamma-1}\right].$$

Assuming $A \neq 0$ and $\theta \neq 0$, we factor out the common term $A\theta^\gamma$:

$$A\theta^\gamma\left[\alpha\gamma f(M) + (\beta E^1 - \delta M)f'(M) + \frac{1}{2}\sigma_\theta^2\gamma(\gamma-1)f(M) + \frac{1}{2}(\sigma_M^2)f''(M) + (\rho\sigma_M\sigma_\theta\gamma)f'(M) - rf(M)\right] = 0.$$

For this equation to hold, the expression inside the brackets must be zero. We regroup the terms based on the derivatives of $f(\mathrm{M})$:

$$\underbrace{\frac{1}{2}(\sigma_{\mathrm{M}}^2)f''(\mathrm{M}) + \Big[(\beta \mathrm{E}^1 - \delta \mathrm{M}) + (\rho\sigma_{\mathrm{M}}\sigma_\theta\gamma)\Big]f'(\mathrm{M})}_{\text{Part depending only on M}} + \underbrace{\Big[\alpha\gamma + \frac{1}{2}\sigma_\theta^2\gamma(\gamma-1) - r\Big]}_{\text{Part depending only on constant and } \gamma (\text{or } \theta)} f(\mathrm{M}) = 0.$$

Since this equation must hold for all values of M, the coefficient of $f(\mathrm{M})$ must be zero:

$$\alpha\gamma + \frac{1}{2}\sigma_\theta^2\gamma(\gamma-1) - r = 0.$$

This yields the characteristic equation for γ, which matches Pindyck (2000). The roots are:

$$\begin{cases} \gamma_1 = \frac{1}{2} - \frac{\alpha}{\sigma_\theta^2} + \sqrt{(\frac{\alpha}{\sigma_\theta^2} - \frac{1}{2})^2 + \frac{2r}{\sigma_\theta^2}} > 1 \\ \gamma_2 = \frac{1}{2} - \frac{\alpha}{\sigma_\theta^2} - \sqrt{(\frac{\alpha}{\sigma_\theta^2} - \frac{1}{2})^2 + \frac{2r}{\sigma_\theta^2}} < 0. \end{cases}$$

To satisfy the boundary condition $\mathrm{W}^{\mathrm{N}}(\theta, \mathrm{M}) = 0$, γ must be positive. Therefore, we select the root $\gamma = \gamma_1$. To solve for the general solution $\varphi(\theta, \mathrm{M})$, we reduce the order of the ODE by defining $g(\mathrm{M}) = f'(\mathrm{M})$ and $g'(\mathrm{M}) = f''(\mathrm{M})$. For simplicity, let $\eta = \beta\mathrm{E}^1 + \rho\sigma_{\mathrm{M}}\sigma_\theta\gamma$. The ODE (14) becomes:

$$\frac{1}{2}(\sigma_{\mathrm{M}}^2)g'(\mathrm{M}) + (\eta - \delta\mathrm{M})g(\mathrm{M}) = 0.$$

Since $g'(\mathrm{M}) = \frac{\mathrm{d}g(\mathrm{M})}{\mathrm{dM}}$, rearranging and separating variables gives:

$$\frac{\mathrm{d}g(\mathrm{M})}{g(\mathrm{M})} = -\frac{2}{\sigma_{\mathrm{M}}^2}(\eta - \delta\mathrm{M})\mathrm{dM}.$$

Integrating both sides yields:

$$\int \frac{1}{g(\mathrm{M})}\mathrm{d}g = \int \left[\frac{-2(\eta - \delta\mathrm{M})}{\sigma_{\mathrm{M}}^2}\right]\mathrm{dM}.$$

Performing the integration results in:

$$\ln|g(\mathrm{M})| = \frac{\delta}{\sigma_{\mathrm{M}}^2}\mathrm{M}^2 + \frac{-2\eta}{\sigma_{\mathrm{M}}^2}\mathrm{M} + \mathrm{C}.$$

Exponenting both sides, we get:

$$|g(\mathrm{M})| = e^{\mathrm{C}}\exp\left(\frac{\delta}{\sigma_{\mathrm{M}}^2}\mathrm{M}^2 + \frac{-2\eta}{\sigma_{\mathrm{M}}^2}\mathrm{M}\right).$$

Let $\mathrm{C}_1 = \pm e^{\mathrm{C}}$ be an arbitrary non-zero constant, so we can derive that

$$g(\mathrm{M}) = \mathrm{C}_1\exp\left(\frac{\delta}{\sigma_{\mathrm{M}}^2}\mathrm{M}^2 + \frac{-2\eta}{\sigma_{\mathrm{M}}^2}\mathrm{M}\right) \tag{25}$$

is the general solution of g(M). Since $f'(\mathrm{M}) = g(\mathrm{M})$, we integrate Equation (25) to find $f(\mathrm{M})$:

$$\begin{aligned} f(\mathrm{M}) &= \int g(\mathrm{M})\mathrm{dM} + \mathrm{C}_2 \qquad (26) \\ &= \int \mathrm{C}_1 \exp\left(\frac{\delta}{\sigma_{\mathrm{M}}^2}\mathrm{M}^2 + \frac{-2\eta}{\sigma_{\mathrm{M}}^2}\mathrm{M}\right)\mathrm{dM} + \mathrm{C}_2. \end{aligned}$$

Assuming $\delta \neq 0$, we complete the square for the exponent: $\frac{\delta}{\sigma_{\mathrm{M}}^2}\mathrm{M}^2 + \frac{-2\eta}{\sigma_{\mathrm{M}}^2}\mathrm{M} = \frac{\delta}{\sigma_{\mathrm{M}}^2}(\mathrm{M} - \frac{\eta}{\delta})^2 - \frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}$ in Equation (26), Substituting this back into the integral gives:

$$\begin{aligned} f(\mathrm{M}) &= \int \mathrm{C}_1 \exp\left(\frac{\delta}{\sigma_{\mathrm{M}}^2}(\mathrm{M} - \frac{\eta}{\delta})^2 - \frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right)\mathrm{dM} + \mathrm{C}_2 \\ &= \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right) \int \exp\left(\frac{\delta}{\sigma_{\mathrm{M}}^2}(\mathrm{M} - \frac{\eta}{\delta})^2\right)\mathrm{dM} + \mathrm{C}_2. \end{aligned}$$

We define $u = \sqrt{\frac{\delta}{\sigma_{\mathrm{M}}^2}}(\mathrm{M} - \frac{\eta}{\delta})$, differentiating with respect to M gives $du = \sqrt{\frac{\delta}{\sigma_{\mathrm{M}}^2}}\mathrm{dM}$, and then $\mathrm{dM} = \sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}}du$, Substituting this change of variable:

$$\begin{aligned} f(\mathrm{M}) &= \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right) \int_0^u \exp\left(v^2\right)\sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}}dv + \mathrm{C}_2 \qquad (27) \\ &= \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right)\sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}} \int_0^u e^{v^2}dv + \mathrm{C}_2. \end{aligned}$$

We introduce the imaginary error function, erfi to represent the integral $\int_0^u e^{v^2}dv$. We define this function as

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}\mathrm{d}t.$$

Rearranging and letting $z = u$ gives

$$\int e^{u^2}du = \frac{\sqrt{\pi}}{2}\mathrm{erfi}(u) + \mathrm{C}_3.$$

Substituting this back into Equation (27):

$$f(\mathrm{M}) = \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right)\sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}}\left[\frac{\sqrt{\pi}}{2}\mathrm{erfi}(u) + \mathrm{C}_3\right] + \mathrm{C}_2.$$

Finally, substituting $u = \sqrt{\frac{\delta}{\sigma_{\mathrm{M}}^2}}(\mathrm{M} - \frac{\eta}{\delta})$ back into the expression and grouping constants yields:

$$f(\mathrm{M}) = \mathrm{C}_1 \exp\left(-\frac{\eta^2}{\delta\sigma_{\mathrm{M}}^2}\right)\left[\frac{\sqrt{\pi\sigma_{\mathrm{M}}^2}}{2\sqrt{\delta}}\mathrm{erfi}\left[\frac{\sqrt{\delta}}{\sigma_{\mathrm{M}}}(\mathrm{M} - \frac{\eta}{\delta})\right] + \sqrt{\frac{\sigma_{\mathrm{M}}^2}{\delta}}\mathrm{C}_3\right] + \mathrm{C}_2.$$

Thus, the general solution $\varphi(\theta, \mathrm{M})$ in the form of (13) can be written in the semi-analytical form presented in Equation (16).

C Solving the optimal free boundary

We start with the smooth pasting condition for $\theta^\star$ (20). Assuming $\gamma \neq 0$ and $\theta^\star \neq 0$, we isolate the term with A_1 and A_2:

$$[\mathrm{A}_1 \operatorname{erfi}(\mathrm{Z}(\mathrm{M}^\star)) + \mathrm{A}_2] = \frac{1}{\gamma(\theta^\star)^{\gamma-1}}[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})].$$

Substituting this expression into the value matching Equation (17) gives

$$(\theta^\star)^\gamma \left\{ \frac{1}{\gamma(\theta^\star)^{\gamma-1}}[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})] \right\} = (c_{\mathrm{A},2} - c_{\mathrm{N},2})\theta^\star \mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})\theta^\star - \mathrm{K}(\mathrm{E}^2).$$

Rearranging the terms, we find that the optimal free boundary $(\theta^\star, \mathrm{M}^\star)$ satisfies:

$$(1 - \frac{1}{\gamma})\theta^\star[(c_{\mathrm{A},2} - c_{\mathrm{N},2})\mathrm{M}^\star + (c_{\mathrm{A},3} - c_{\mathrm{N},3})] = \mathrm{K}(\mathrm{E}^2),$$

or the form presented in Equation (21). Finally, we derive the constants A_1 and A_2 using Equations (19) and (17), respectively.