Model Combination Strategies for Covariance Forecasting

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Motivation

- Accurate multivariate volatility forecasts are essential for asset allocation strategies such as *minimum-variance* or *risk parity*
- Empirical evidence shows that combining forecasts from different models or information sets **consistently improves out-of-sample accuracy** compared to individual models (Patton and Sheppard, 2009)
- Existing literature is scarce and limited:
  - Focuses exclusively on **daily forecasting horizon**
  - **Neglects practical feasibility** for real-world portfolio management
  - Shrinks forecasts towards a **single static target**
  - Considers only a **narrow set of candidate models**

★ *How to design a combination strategy that combines accuracy and economic relevance, while remaining feasible for portfolio implementation with monthly rebalancing?*

Contribution

- We propose a multi-target combination strategy with advanced weighting and dynamic model selection, featuring
  - **Monthly** forecasting horizon for practical portfolio applications
  - Robust in **high-dimensional** environments (large N, small T)
  - Integration of various models ranging from **multivariate GARCH** models, **realized covariance** estimators, and **dynamic factor** structures
  - **Stable, adaptive weights** that respond to recent forecasting performance and portfolio utility
- We characterize the set of candidate models and analyze estimation settings using a **simulation study**

Related Literature

Literature is rather heterogeneous

- Typical linear shrinkage approaches with one target (Ledoit and Wolf, 2003; 2004b;a)
- Model averaging approaches for daily data (Amendola and Storti, 2015; Caldeira et al., 2017; Clements and Doolan, 2020; Amendola et al., 2020)
- Decomposition approaches for the covariance matrix (Ledoit and Wolf, 2012; Halbleib and Voev, 2016)
- Some theoretical work on multi-target shrinkage approaches (Halbe et al., 2013; Lancewicki and Aladjem, 2014; Oriol, 2024)

Gap in the Literature

No **model averaging/combination** work for monthly horizons suitable for practical institutional portfolio management

The Challenge: High-Dimensional Covariance Estimation

The problem: Too many parameters, not enough data

What happens:

- Sample covariance is very noisy
- Small changes in data → large changes in estimates
- Estimation error dominates

Portfolio impact:

- Extreme, unstable weights
- High turnover
- Poor out-of-sample performance

Solution: Shrinkage

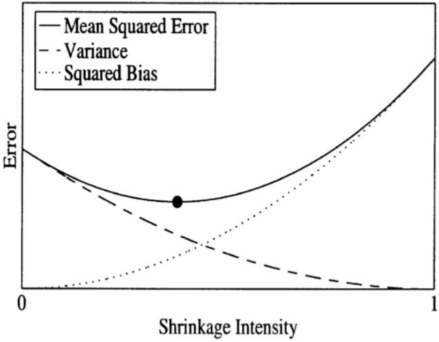
Combine noisy sample S with stable structured target F:

$$\widehat{\Sigma} = (1 - \rho)S + \rho F$$

Pull extreme estimates toward reasonable structure

The Shrinkage Idea: Ledoit-Wolf Intuition

Key insight: Balance bias and variance by shrinking toward a structured target



The tradeoff:

- $\rho = 0$: Use only sample S
  Unbiased but high variance
- $\rho = 1$: Use only target F
  Stable but biased
- ρ^*: Optimal blend
  Minimizes total error

When N/T is large:
Optimal ρ^* shifts right
(shrink more toward F)

Common Shrinkage Targets

Different targets embed different structural assumptions:

1. **Identity Matrix:** $F = \sigma^2 I$
   - Assumes equal variances, zero correlations
2. **Constant Correlation:** $F_{ij} = \sigma_i \sigma_j \bar{\rho}$ for $i \neq j$
   - Most popular in practice (Ledoit-Wolf 2004)
3. **Factor Models:** $F = B \Sigma_f B^\top + D$
   - Multi-factor: Fama-French, macro factors, industry factors, PCA

The Challenge

Which target is correct? Market conditions change, no single structure dominates

Multi-Target Shrinkage

- Use multiple targets $T_1, \ldots, T_K$ simultaneously
- Multi-target shrinkage (difference form):

$$\Sigma^*(\alpha) = S + \sum_{k=1}^{K} \alpha_k (T_k - S)$$

- Equivalent mixture representation:

$$\Sigma^*(\rho) = \rho_0 S + \sum_{k=1}^{K} \rho_k T_k, \qquad \rho_0 = 1 - \sum_{k=1}^{K} \rho_k$$

- Advantages:
  - Flexibility: combine identity, constant-corr, factor models, PCA
  - Diversifies across structural assumptions
  - Optimal weights ρ^* minimize forecast error

Risk Decomposition and Notation

We study the Frobenius risk $R(\rho) := E\left[\|\Sigma^*(\rho) - \Sigma\|_F^2\right]$

Define the estimation error and target directions

$$\Delta := S - \Sigma, \qquad U_k := T_k - S, \quad k = 1, \ldots, K.$$

Using the Frobenius inner product $\langle A, B \rangle := \mathrm{tr}(A^\top B)$, the risk expands to:

$$R(\rho) = \beta^2 - 2\rho^\top b + \rho^\top C \rho$$

where $\beta^2 = E\|\Delta\|_F^2,\ C_{kl} = E\langle U_k, U_l \rangle,\ b_k = E\langle \Sigma - S,\ U_k \rangle$

Optimal weights: $\boxed{\rho^* = C^{-1} b}$ (if we knew true Σ)

In practice: Estimate from data, use cross-validation, or equal weighting

Key Theoretical Result

Proposition: Adding Non-Redundant Models Reduces Risk

Adding a new target T_{m+1} that captures information orthogonal to existing targets $T_1, \ldots, T_m$ **strictly reduces** forecast error:

$$R(\rho^*_{m+1}) < R(\rho^*_m)$$

Practical implications:

- No downside to including more models (if they're truly different)
- Diversification benefit: combine uncorrelated "model assets"
- Use 3-5 models with different structural assumptions

Optimal weights: $\rho^* = C^{-1}b$ where

- b_k measures how well target k aligns with true structure
- C_{kl} captures redundancy between targets

From Deterministic Weights to Probabilistic Weights

- Classical shrinkage chooses ρ_k via:
  - analytic formulas (Frobenius risk)
  - cross-validation (GMV risk)
  - heuristic rules
- Problem 1: **model uncertainty**. Which target is correct?
- Problem 2: How to avoid very wrong and *bad* targets?
- Solution: treat each target F_k as a model M_k
- Use the data to assess model plausibility
- This leads naturally to **Bayesian model averaging (BMA)**

Simulation Setup

True monthly covariance: Factor model

$$\Sigma_{\text{true}} = BFB^\top + D,$$

where

- B: $N \times 3$ random factor loadings,
- F: diagonal factor covariance,
- D: idiosyncratic variances.

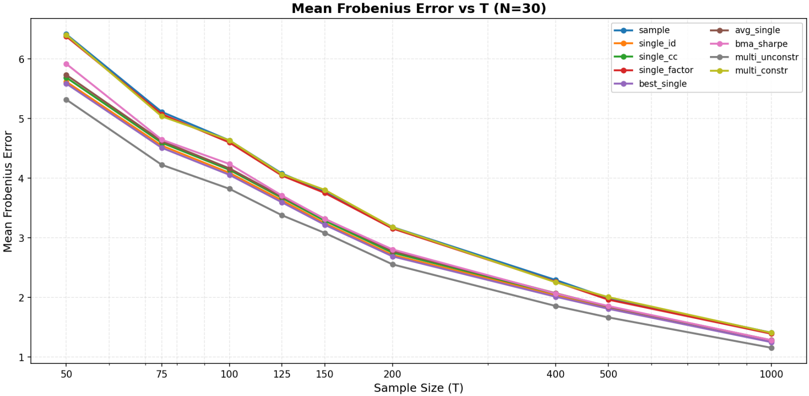
Monthly returns:

$$R_t \sim \mathcal{N}(0, \Sigma_{\text{true}}), \quad t = 1, \ldots, T$$

Dimensions tested:

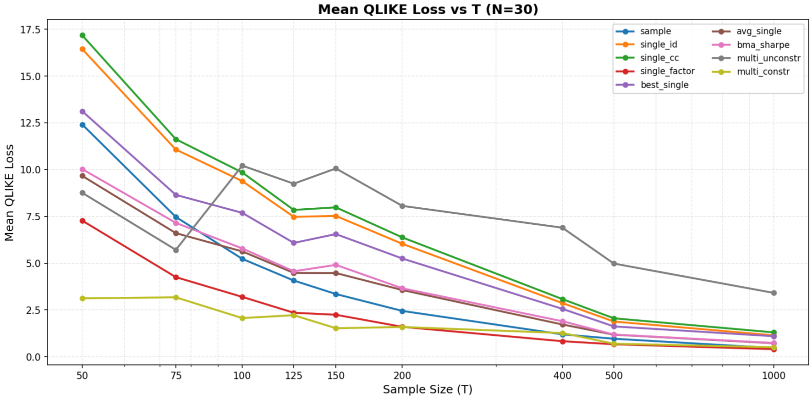
$$N \in \{10, 20, 50, 100\}, \qquad T \in \{50, 100, 200, 500, 1000\}$$

Frobenius Risk



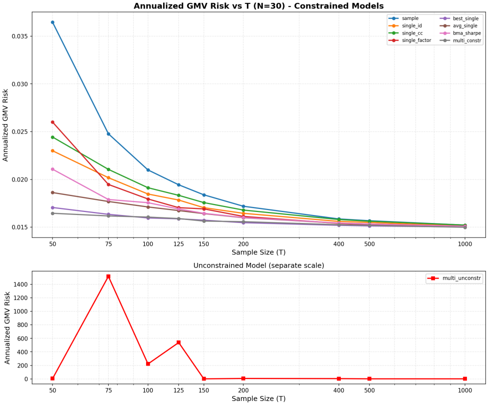
- Sample covariance (top line, yellow dotted) performs worst
- Single models with structure (constant-corr, factor) perform well
- Note: Multi-target combination shows *higher* Frobenius error but delivers economic gains (see GMV portfolio risk)

Qlike Risk



- QLIKE measures quality for portfolio optimization (likelihood-based)
- Multi-target (yellow/bottom) shows even stronger dominance
- Gap narrows as T increases but combination still wins

GMV Portfolio Risk



- **Realized annualized volatility** of minimum-variance portfolio
- At $T = 100$: multi-target $\approx 1.6\%$, sample $\approx 2.1\%$ (**24% reduction**)
- Better covariance forecasts $\rightarrow$ lower realized portfolio volatility

Portfolio Application

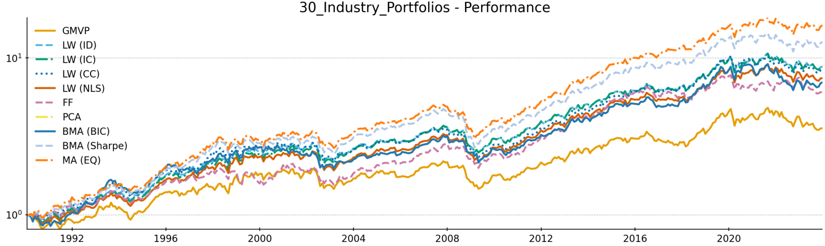
- Test combination idea on a simple portfolio application, here: GMVP weights
- Use various models and combine them to test the impact
- Considered models: Various LW shrinkage estimators, PCA factor model, Fama-French 5-factor model
- Pseudo-out-of-sample application (rolling window) on monthly data
- Dataset: Fama French portfolio data (30 industry portfolios, 1990-2023)

Performance for 30 Industry Portfolios I/II

	MeanReturn	Volatility	MDD	Sharpe	Certainty Equivalent	MAD
GMVP	4.59	13.05	33.18	0.35	-1.93	3.86
LW (ID)	7.12	11.92	32.46	0.60	1.16	2.54
LW (IC)	7.12	11.86	32.68	0.60	1.19	2.49
LW (CC)	7.00	12.12	34.39	0.58	0.94	2.60
LW (NLS)	6.63	11.98	33.37	0.55	0.64	2.69
FF	6.09	12.62	29.75	0.48	-0.22	2.66
PCA	6.57	12.94	32.89	0.51	0.10	3.04
BMA (BIC)	6.57	12.94	32.89	0.51	0.10	3.04
BMA (Sharpe)	8.21	11.95	43.13	0.69	2.24	1.63
MA (EQ)	8.89	11.64	39.45	0.76	3.07	1.49

- MA(EQ) and BMA (Sharpe) dominate with highest Sharpe ratios (0.76, 0.69) and certainty equivalents (3.07, 2.24)
- Linear weighting methods deliver moderate performance (Sharpe 0.55-0.60)
- Sample GMVP and FF show poor risk-adjusted returns with negative certainty equivalents

Performance for 30 Industry Portfolios II/II



- MA(EQ) and BMA (Sharpe) show superior cumulative growth, reaching 20x by 2023
- GMVP (yellow) suffers severe 2008 crash and stagnates around 3x, never recovering
- Model averaging approaches demonstrate resilience with smoother trajectories and lower volatility

Conclusion

- Risk management for different strategies can be costly
- Aggregating over different covariance estimators can be more beneficial than aggregating over portfolio weights
- Multi-target shrinkage is theoretically optimal and empirically beneficial

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Research Agenda:

- Dynamic weighting schemes (regime-dependent, performance-based)
- Include DCC and realized covariance measures
- Connection to non-linear shrinkage approaches

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