

# A Mapping Technique for Selectivity Theory

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This version: February 24, 2026.  
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## Abstract

Selectivity theory predicts that enhanced index portfolio managers should hold between 50% and 80% of benchmark stocks in their portfolio. The theory was based on some very strict assumptions, including the form of the distribution of returns. In this paper, we relax these assumptions and derive analytical formulas for the portfolio information ratio. We then present the mathematics to map from the information ratio of a portfolio manager to his omega as required in selectivity theory. This technique may allow researchers to map empirical data on portfolio managers to their respective selectivity parameters and underlying skill. We also suggest a potential application of this work, which, in certain circumstances, improves on estimating a portfolio manager's information ratio.

*JEL Classification: G0, G13*

*Key Words: enhanced indexing, selectivity theory, information ratio, portfolio management, active management, simulations*

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\*We would like to thank Pierangelo DePace for working with us on the mixed distribution simulations described in the appendix, Joseph Losonczy, Daniel Zanger, Steven Evans, Jason Abrayav, Phil Dybvig, and Ron Kahn for comments. We also would like to thank Marco Riani, Aldo Corbellini, Domenico Perrotta, and the FSDA team for helping us add the Wallenius and Fischer package to MATLAB to make our work easier. Contact: Ludwig Chincarini, CFA, Ph.D., is an Professor of Finance at the University of San Francisco School of Management, Director of Quantitative Strategies at United States Commodity Funds, and Advisor to Wedge Capital, Office MH 117, University of San Francisco, School of Management, 101 Howard Street, Suite 500, San Francisco, CA 94105. Email: [chincarini@hotmail.com](mailto:chincarini@hotmail.com) or [lbchincarini@usfca.edu](mailto:lbchincarini@usfca.edu) Phone: 415-422-6992. Andrei Bolshakov, CFA, is a general partner of Wedge Capital Management, 301 South College Street, Suite 3800, Charlotte, NC 28202. [abolshakov@wedgcapital.com](mailto:abolshakov@wedgcapital.com) Phone: 704-334-6475. Daniel Jerison is an Assistant Professor of mathematics at the University of San Francisco School, University of San Francisco, San Francisco, CA 94105. Email: [dcjerison@usfca.edu](mailto:dcjerison@usfca.edu).

## 1 Introduction

One can think of the portfolio manager's process as choosing one's securities (e.g. stocks), choosing the appropriate weights for those securities, and achieving a desired level of diversification. Modern portfolio theory was a large step forward in the process of choosing securities by understanding the relationship between a portfolio of securities in terms of its average return and risk (Markowitz (1952), Lintner (1965), Mossin (1966), Sharpe (1964), and Treynor (1961), (Merton (1973), Ross (1976)). Bolshakov and Chincarini (2020), as well as others (Hallerback (2014), Van Loon (2018), Constable and Armitage (2006)), considered a different approach that relies on skill in active portfolio management. These papers looked at investors that have the ability to select winner stocks more frequently than loser stocks. Winners and losers can be defined in a variety of ways, but the central idea is that winner stocks will have a higher return than the average stock. Bolshakov and Chincarini (2020) considered a portfolio manager that has an ability to pick winner stocks from a benchmark. That is, a manager has a probability greater than 50% to pick a stock in the upper half of the distribution. This naturally leads to a stock pricing process without replacement, but is slightly more complicated and is represented by the Wallenius or Fischer hypergeometric distribution. In this framework, the optimal amount of stocks that the portfolio manager should choose is somewhere between 50% and 80% of the benchmark stocks. One is that the manager picks a level of stocks to own, say  $N_p$ , and selects them all at once with a selectivity level of  $\frac{N_p}{N}$  and the other is that the manager picks them sequentially until he reaches  $\frac{N_p}{N}$ . The method of selection affects the optimal selectivity level. It is 50% if we believe that the portfolio manager selection process involves grabbing an entire group of stocks at once, for example, by screening for a particular factor and grabbing a certain decile. It is 80% if it is believed that portfolio manager

selects stocks one-by-one from the benchmark universe.<sup>1</sup>

Although a provocative and interesting paper, it would be difficult to apply the framework to stock return data since the distribution of stock returns is certainly not binary, i.e. every stock has an  $x\%$  return or a  $-x\%$  return. In order to apply these concepts to real world data, we must allow for stock distributions that are more realistic. The first step in this direction is to assume that stock returns follow a normal distribution. Using this more realistic assumption, the paper derives the mathematical link from an information ratio of a portfolio to the parameter in selectivity theory,  $\omega$ . That is, using the actual distribution of stock returns, this paper proposes a technique to map from the portfolio manager's observable performance to his selectivity parameters. This is an important first step in using selectivity theory in real world situations.

This derivation is non-trivial, since the Wallenius' and Fisher's Noncentral Hypergeometric Distributions are very complicated. Using a collection of statistical techniques, we have been able to link the Wallenius and Fischer distribution to the underlying information ratio of the portfolio. The strength of our derivation is that it can be used when the portfolio is chosen using a Wallenius or Fischer process, or any other stock picking process. To our knowledge, our paper is the first to make this apparent mapping between stock selection skill and the corresponding information ratio when a portfolio manager has stock picking ability. We then use this to provide an application for selecting money managers using the parameters of selectivity theory rather than the actual measured information ratio. One of the most provocative applications of this mapping is to show that an investor will obtain a more accurate information ratio, in many circumstances, using our mapping theorem along with sample statistics rather than using the actual measured information

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<sup>1</sup>This is a surprising number, since many active enhanced portfolio managers hold much less than this, while at the other extreme, passive managers hold 100%. For a more detailed discussion see Bolshakov and Chincarini (2020).

ratio measured from historical data. Although we have not developed a proof for this application, some realistic examples illustrate the concept.

The rest of the paper is as follows: Section 2 develops a general framework for analyzing the information ratio of a portfolio manager with stock picking skill. We model the portfolio, manager's alphas as i.i.d. from a normal distribution of stock returns. We are able to then map a manager's IR to his selectivity parameters, whatever his stock selection process is. Section 3 extends the analysis using a single factor model of stock returns. That is, we integrate this alpha model into a single index model and find that the basic results do not change much other than an adjustment factor related to the portfolio and benchmark beta. Thus, our mapping from the manager's information ratio to the selectivity parameters are extended to an accepted model of stock returns. Section 4 examines our formulas in the context of a Wallenius and Fischer stock picking manager comparing simulations to our theoretical model. Although our derivation is general for many types of stock picking processes, we find that when using the Wallenius and Fischer, regardless of the distribution of stock returns, the optimal selectivity ratio is still 50% and 80% for the bulk and sequential selection methods respectively, which is qualitatively similar to the simpler binary distribution of returns in Bolshakov and Chincarini (2020). We run simulations to verify that this theoretical model is accurate. Section 5 discusses a particular application of this work to selecting a portfolio manager from observable performance data and proposes an alternative to the traditional approach of measuring the portfolio manager's information ratio. Section 6 concludes the paper.<sup>2</sup>

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<sup>2</sup>We have an internet appendix with many other results that may be of interest to the reader.

## 2 General Framework

### 2.1 Basic Definitions and Framework

We consider a portfolio manager who is choosing a portfolio of stocks.<sup>3</sup> Suppose that there are  $N$  stocks numbered 1 through  $N$  available for selection from a benchmark of  $N$  stocks. The manager chooses a subset  $P \subseteq \{1, \dots, N\}$  of these stocks to be in the portfolio. Let  $Q = \{1, \dots, N\} - P$  be the set of remaining stocks that are not in the portfolio.

Let  $r_1, \dots, r_N$  be the returns of the stocks. We will measure the return of the portfolio relative to a benchmark in which the stocks have weights  $w_1^B, \dots, w_N^B$ , with

$$\sum_{i=1}^N w_i^B = 1.$$

The benchmark return is

$$r_B = \sum_{i=1}^N w_i^B r_i. \quad (1)$$

For each  $i \in P$ , let  $w_i^P$  be the weight of stock  $i$  in the portfolio, with

$$\sum_{i \in P} w_i^P = 1.$$

(By assumption, if  $i \in Q$  then  $w_i^P = 0$ .) The portfolio return is

$$r_P = \sum_{i \in P} w_i^P r_i. \quad (2)$$

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<sup>3</sup>In addition to the online appendix for this paper, there is also a working paper with many useful results related to this area of portfolio management (See Bolshakov, Chincarini, and Jerison (2025)).

We will derive formulas for the expected active return  $\mathbf{E}(r_P - r_B)$ , the tracking error  $\sqrt{\text{Var}(r_P - r_B)}$ , and the information ratio

$$\text{IR} = \frac{\mathbf{E}(r_P - r_B)}{\sqrt{\text{Var}(r_P - r_B)}}$$

under several different model specifications. Underlying all our formulas is the following:

We make a fundamental assumption that the portfolio manager has some ability to choose stocks that outperform the benchmark. We assume that the manager's skill is limited to deciding which stocks to include in the portfolio and which stocks to exclude. That is, the manager can choose sets  $P$  and  $Q$  so that the stocks in  $P$  are likely to have higher returns than the stocks in  $Q$ . But, under our assumption, the manager has no special insight to distinguish between stocks within  $P$  or stocks within  $Q$ . Mathematically, we assume that the joint distribution of returns  $\{r_i\}_{1 \leq i \leq N}$  is invariant under any permutation of the indices that preserves the sets  $P$  and  $Q$ . In particular, conditioned on  $P$  and  $Q$ , the returns  $\{r_i\}_{i \in P}$  are an *exchangeable* family of random variables, and the returns  $\{r_i\}_{i \in Q}$  are another exchangeable family.<sup>4</sup>

We assume that the  $N$  stocks are chosen from stock distribution representing the distribution of stock returns. These  $N$  stock returns are drawn independently from a  $N(\mu, \sigma)$  distribution. Split these into two equal groups, the  $N/2$  above the median and the  $N/2$  below the median. Let  $N_P \leq N$  be the total number of stocks that will be in the portfolio. Of these,  $x$  will be randomly drawn

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<sup>4</sup>Our formulation also allows for a portfolio manager who has no ability to select stocks that outperform the benchmark. However, in the case that they do, it is reasonable to question whether this fundamental assumption holds in practice. On one level, the answer is clearly not: there is no reason to suppose that the portfolio manager is knowledgeable enough to split the stocks into two categories but completely unable to draw finer-grained distinctions. Nevertheless, we propose the fundamental assumption as a starting point for an empirically grounded theory of portfolio selection. This is because it is a testable hypothesis. As we demonstrate below, under the fundamental assumption we can find formulas for the expected active return, tracking error, and information ratio of a portfolio. These formulas constitute predictions for how portfolio returns would look in a world where the fundamental assumption holds true. In future work, we plan to test these predictions against real data. The results will show how much of the "story" of market outperformance by portfolio managers is explained by the fundamental assumption and how much is due to other factors such as heavy bets on individual stocks or fine-grained knowledge of covariances.

without replacement from the “good” group (above the median) and  $N_P - x$  will be randomly drawn without replacement from the “bad” group (below the median), where the number  $x$  is chosen randomly from some distribution with mean  $\mu_x$  and variance  $\sigma_x^2$ .

## 2.2 An Equal-Weight Portfolio and Benchmark

The average return of the portfolio is<sup>5</sup>

$$r_P = \frac{1}{N_P} \left( \sum_{i=1}^x r_i^{(g)} + \sum_{j=1}^{N_P-x} r_j^{(b)} \right)$$

where  $r_i^{(g)}$  are the returns from the good group and  $r_j^{(b)}$  are the returns from the bad group.

The figure  $r_P$  should be compared against the average return of all  $N$  stocks:

$$r_B = \frac{1}{N} \left( \sum_{i=1}^{N/2} r_i^{(g)} + \sum_{j=1}^{N/2} r_j^{(b)} \right)$$

Note that  $r_B$  is an average of i.i.d. Normals, so its distribution is Normal with mean  $\mu$  and standard deviation  $\sigma/\sqrt{N}$ . It is positively correlated with  $r_P$ .

We are interested in the mean and variance of  $r_P$ , and in the information ratio

$$\text{IR} = \frac{\mathbf{E}(r_P - r_B)}{\text{SD}(r_P - r_B)}.$$

Below, we derive approximate formulas for  $\mathbf{E}(r_P)$ ,  $\text{Var}(r_P)$ , and the information ratio in terms of

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<sup>5</sup>The derivations in this paper are for both an equal-weighted benchmark and equal-weighted portfolio. We believe the results will hold with a set of fixed weights for the benchmark and portfolio that are not equal, but it is for future work to show this.

$N, \mu, \sigma, N_P, \mu_x, \sigma_x^2$ .<sup>6</sup> Here are the main results:<sup>7</sup>

$$\mathbf{E}(r_P) \approx \mu + \sigma \left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96+40\pi-3\pi^2}{16\sqrt{2\pi}} \right). \quad (3)$$

$$\begin{aligned} \text{Var}(r_P) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &+ \frac{\sigma^2}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi} \right] \\ &+ \frac{\sigma^2}{N^2} \left[ -\frac{16(\pi-3)-(4-\pi)^2}{4\pi} - \frac{1}{N_P} \cdot \frac{120-60\pi+7\pi^2}{4\pi} \right. \\ &\quad \left. - \frac{\mu_x}{N_P} \cdot \frac{(4-\pi)^2}{\pi} + \frac{\mu_x^2}{N_P^2} \cdot \frac{(4-\pi)^2}{\pi} \right. \\ &\quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4-\pi)^2+(96-40\pi+3\pi^2)}{2\pi} \right] \end{aligned} \quad (4)$$

$$\mathbf{E}(r_P - r_B) \approx \sigma \left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96+40\pi-3\pi^2}{16\sqrt{2\pi}} \right). \quad (5)$$

$$\begin{aligned} \text{Var}(r_P - r_B) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &+ \frac{\sigma^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi} \right] \\ &+ \frac{\sigma^2}{N^2} \left[ -\frac{2(\pi-3)}{\pi} - \frac{1}{N_P} \cdot \frac{120-60\pi+7\pi^2}{4\pi} \right. \\ &\quad \left. + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \frac{(4-\pi)^2}{\pi} \right. \\ &\quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4-\pi)^2+(96-40\pi+3\pi^2)}{2\pi} \right]. \end{aligned} \quad (6)$$

$$\text{IR} \approx \frac{\left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96+40\pi-3\pi^2}{16\sqrt{2\pi}} \right)}{\sqrt{\blacksquare}} \quad (7)$$

<sup>6</sup>The formulas for  $\mathbf{E}(r_P)$  and  $\text{Var}(r_P)$ , along with both the numerator and denominator of the IR formula, are written as a “constant” term (with no dependence on  $N$ ) plus a correction term with a  $1/N$  factor. These are in fact the beginnings of infinite series in powers of  $1/N$ . In online Appendix A, we will compute these formulas up to the  $1/N^2$  term. From our numerical simulations, it makes very little difference whether the  $1/N^2$  term is included or not. The error from other approximations made during the derivation outweighs any increased accuracy from including the extra term.

<sup>7</sup>See the online Appendix A for the proof and derivation. We present the results to the third order, but our simulations show that the formula works quite well even for a 2nd order approximation.

where

$$\blacksquare = \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] + \frac{1}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\ + \frac{1}{N^2} \left[ -\frac{2(\pi - 3)}{\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \frac{(4 - \pi)^2}{\pi} \right. \\ \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right].$$

The technique or strategy we used to find the expressions above was as follows.<sup>8</sup> Let  $M$  be the median of the  $N$  returns. Suppose that we condition on the value of  $M$  and consider the upper and lower parts of the  $N(\mu, \sigma)$  distribution truncated at  $M$ . The “good” and “bad” returns will be mutually conditionally independent, or nearly so, with the good returns drawn from the upper truncated distribution and the bad returns drawn from the lower truncated distribution. We obtain the formulas for  $\mathbf{E}(r_P)$ ,  $\text{Var}(r_P)$ , and IR by first sampling  $M$  and then sampling the good and bad returns independently from the truncated distributions.<sup>9</sup>

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<sup>8</sup>The entire proof and approach is contained in the online appendix to this paper.

<sup>9</sup>It is well-known [Chu–Hotelling 1955] that the distribution of  $M$  is approximately Normal when  $N$  is large. We use this simplifying approximation in our derivation in online Appendix A. A more delicate issue is the relationship between the parity of  $N$  and the conditional independence statement in the paragraph above. So far, we have implicitly assumed that  $N$  is even by speaking of two equal groups of size  $N/2$ . When  $N$  is even, it is not quite true that the good and bad returns are mutually conditionally independent given  $M$ . On the other hand, when  $N$  is odd, it is true that the upper  $(N - 1)/2$  returns and the lower  $(N - 1)/2$  returns are conditionally independent given the median  $M$ . To make the computation tractable, we assume that the good and bad returns are conditionally independent given  $M$  even when  $N$  is even. The effect of this assumption can be observed in simulation data: when  $N$  is even, the given formulas for  $\mathbf{E}(r_P)$  and IR may be too large by a factor of about  $1 + \frac{1}{N}$ .

## 3 A Single Factor World

### 3.1 Motivation

Up until this point, we have assumed that the stock picks are i.i.d. from a normal distribution. This might be realistic for managers selecting the alpha of stocks, however, it is not realistic for picking stock returns, since stocks tend to be correlated amongst one another.<sup>10</sup> In order to set selectivity theory into a more acceptable world, we use the dynamics described earlier as the alpha generation model of stock returns and place this model within an accepted model of stock returns; the single factor model.

Suppose we live in a single-factor world like the CAPM or the single-index model of APT. That is, every stock has a  $\beta$  with respect to the market. That is,

$$\tilde{r}_i = \alpha_i + \beta_i \tilde{r}_M + \epsilon_i \quad (8)$$

where  $\tilde{r}_i$  indicates the respective variable minus the risk-free rate and for simplicity we are going to also assume (as does the single index model) that  $\mathbb{E}[\alpha_i] = 0$ .<sup>11</sup>

Then, we know that the expected return and variance of any given stock is given by:

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<sup>10</sup>If stock returns were truly uncorrelated, many perturbing attributes would occur to portfolios. First, as the number of stocks in benchmark became large ( $N$ ), the variance of the benchmark would converge to zero. This is natural as the picks are uncorrelated, but it is a departure from the real world. Second, if number of stocks in the portfolio and the benchmark are both large, even the tracking error will converge to zero. Although, there are realistic limits to the  $N_P$  and  $N$ , this is still an unrealistic result of the uncorrelated stock picks.

<sup>11</sup>In simulations, this is perfectly acceptable, however, if we bring the analysis to real data and estimate the beta of each stock, we will need to include a constant to allow for OLS assumptions to be valid, like error independent of independent variable, etc. If we move the analysis to real data, we will deal with those issues at that time.

$$E[\tilde{r}_i] = \beta_i E[\tilde{r}_B] \quad (9)$$

$$\text{Var}(\tilde{r}_i) = \beta_i^2 \text{Var}(\tilde{r}_B) + \text{Var}(\epsilon_i) \quad (10)$$

At this point, we can combine our portfolio manager selection with an accepted model of the stock market. We will assume that the skilled portfolio manager picks only the  $\epsilon_i$  of each stock from an i.i.d. normal distribution of stock  $\epsilon$ s. That is, stock returns are given by:  $\tilde{r}_i = \beta_i \tilde{r}_M + \epsilon_i$  and the only “real difference” between the benchmark portfolio and the selected portfolio is the selectivity ratio and the probability of picking a good  $\epsilon$ . That is, the portfolio manager with skill has the ability to pick which stocks will have positive shocks not explained by their beta and the market return. To make things even more realistic, we can center the alpha distribution around a mean of 0, that is  $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ . Thus, on average, an index manager will have no excess returns above that predicted by the single index model. However, the skilled portfolio manager is able to pick more stocks from the positive side of the error distribution.<sup>12</sup>

The corresponding formulas for the mapping theorem of the single-index model are:<sup>13</sup>

$$\mathbf{E}(\tilde{r}_P - \tilde{r}_B) \approx (\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + (2p^* - 1) \cdot \sigma_\epsilon \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} \right). \quad (11)$$

$$\begin{aligned} \text{Var}(\tilde{r}_P - \tilde{r}_B) &\approx (\bar{\beta}_P - \bar{\beta}_B)^2 \cdot \sigma_M^2 + \sigma_\epsilon^2 \left[ \frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_\epsilon^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &\quad + \frac{\sigma_\epsilon^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_\epsilon^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi} \right]. \end{aligned} \quad (12)$$

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<sup>12</sup>Although in this context this seems like market timing, in some sense, identifying good stocks before others is always market timing of some sort.

<sup>13</sup>The proofs are in the online Appendix B.

Finally, the information ratio is given by

$$\begin{aligned} \text{IR} &= \frac{\mathbf{E}(\tilde{r}_P - \tilde{r}_B)}{\text{SD}(\tilde{r}_P - \tilde{r}_B)} \\ &\approx \frac{(\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + (2p^* - 1) \cdot \sigma_\epsilon \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} \right)}{\sqrt{(\bar{\beta}_P - \bar{\beta}_B)^2 \sigma_M^2 + \sigma_\epsilon^2 \left[ \frac{1}{N_P} \frac{\pi-2}{\pi} + \frac{\sigma_\epsilon^2}{N_P^2} \frac{8}{\pi} \right] + \frac{\sigma_\epsilon^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \frac{2(\pi-3)}{\pi} + \frac{\sigma_\epsilon^2}{N_P^2} \frac{4(4-\pi)}{\pi} \right]}}. \end{aligned} \quad (13)$$

It is instructive to compare these formulas with the ones from Section 2. The difference here is that the individual stock returns are correlated with each other due to the  $\tilde{r}_M$  term. If we suppose that  $\bar{\beta}_P = \bar{\beta}_B$ , that is, the average beta of the portfolio matches the average beta of the overall market, then the formulas for  $\mathbf{E}(\tilde{r}_P - \tilde{r}_B)$  and the information ratio are identical to the case in the previous section with uncorrelated returns. If we suppose that  $\bar{\beta}_P \neq \bar{\beta}_B$  and  $\mu_M = 0$ , then  $\mathbf{E}(\tilde{r}_P - \tilde{r}_B)$  is the same as in the uncorrelated case, but  $\text{SD}(\tilde{r}_P - \tilde{r}_B)$  is larger, so the information ratio is smaller.

A natural interpretation of this is that in the single index model, there are two different ways to beat the market. The first way is to choose individual stocks with high  $\epsilon_i$  values. The second way is to predict accurately whether  $\tilde{r}_M$  will be positive or negative, and make a portfolio of high beta stocks if  $\tilde{r}_M$  is likely positive or low beta stocks if  $\tilde{r}_M$  is likely negative. In our scenario, the portfolio manager follows the first strategy only, which is just as successful as it would have been in the uncorrelated case. But, the existence of the second strategy inflates the denominator in the information ratio.

## 4 Wallenius and Fischer Selectivity and the Mapping Between IR and $\omega$

Now that we have a statistical relationship between the stock picking distribution (e.g. Wallenius, Fischer, Binomial, or other) and the expected excess return and tracking error of a portfolio with respect to the benchmark, we can map the information ratio (IR) of a portfolio to the omega ( $\omega$ ) of the portfolio manager, given a process for stock returns.<sup>14</sup>

Thus, suppose we know the distribution of stock returns, we also know the selectivity ratio of the portfolio manager (i.e.  $\phi = \frac{N_P}{N}$ ), then we should be able to link the portfolio manager's information ratio with their  $\omega$ . That is, even if the portfolio manager does not believe that he or she is picking according to a Wallenius or Fisher process, we can still describe that particular manager in terms of the  $\omega$  of the Wallenius or Fischer distribution.<sup>15</sup> With exact knowledge of the underlying parameters, this mapping will be exact. In the practical world, we may have sample statistics about the distribution of stock returns, which will make the mapping less precise, but even in this case, one can place bounds on the mapping.<sup>16</sup>

For example, take a world of independent stock returns (Section 2.2), then guided by Equation 7), if we know the distribution of stock returns from which the benchmark stocks are selected (i.e.  $N(\mu, \sigma)$ ), the number of stocks chosen for the portfolio ( $N_P$ ), the number of stocks chosen for the benchmark ( $N$ ), and the information ratio of the portfolio (IR), then we have a unique determination of the  $\omega$  (which determines  $\mu_x$  and  $\sigma_x$ ) for the portfolio manager.<sup>17</sup>

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<sup>14</sup>For more information on the Wallenius and Fischer distribution, see Wallenius (1963), Fog (2013), or Bolshakov and Chincarini (2020).

<sup>15</sup>The reader should note that since our formulas are independent of the actual stock picking process, this would work with other selection processes that are not based on the Wallenius or Fischer selection method.

<sup>16</sup>Understanding this and applying to real world data will be part of our future research.

<sup>17</sup>This is for a one period horizon. In a dynamic horizon, the results are more complicated, which we will discuss

In our simulations, we first show how are formulas compare to actual simulations and then we use this to create an example of a mapping table from IR to  $\omega$ . We will do this for the uncorrelated alphas, as well as within a single factor model world.<sup>18</sup>

#### 4.1 Simulations of Alpha Uncorrelated Returns

In the case of uncorrelated returns, we use the set of equations in Section 2 of the paper. Our simulation methodology is as follows. First, we select random returns from a normal  $N(\mu, \sigma)$  for all the stocks in the benchmark.<sup>19</sup> For approximate method with draws from the benchmark, we use  $\mu_\epsilon = 0$  and  $\sigma_\epsilon = 0.50$  to coincide with the single index parameters in Section 4.2. Second, the manager uses his selection technique (i.e. Wallenius or Fischer) to pick the  $x$  good stocks and the  $N_P - x$  bad stocks from the benchmark. Third, we compute the return of the portfolio and the return of the benchmark. Fourth, we compute the excess return, the tracking error and the information ratio of the portfolio over many 100,000 investment periods. In each investment period, we repeat steps (1) to (3).

[INSERT TABLE 1 ABOUT HERE]

Table 1 contains the results from the 100,000 simulations. As one can see, the approximate method is very precise at estimating the information ratio of the portfolios. For example, for an  $\omega$  of 1.20, a

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in further work.

<sup>18</sup>In the tables presented in the paper, we use only the Wallenius selection method. In other work, we have created similar tables to show the accuracy of our formula for the Fisher selection method. Of course, the reader should remember that our derivations are very general and would apply to a variety of selection methods.

<sup>19</sup>In the online Appendix C, we derive a proof for a version of indexing, whereby returns for the benchmark and portfolio are chosen independently of each other. This special case is not really in the spirit of enhanced indexing, because the returns of the portfolio are generated independently of the benchmark. Nevertheless, it is an interesting version of our work, which provides exact solutions rather than approximate solutions and allows for the weights to vary. For the exact method simulations, we use the parameters  $\mu_\epsilon = 0.12$  and  $\sigma_\epsilon = 0.20$  and report these results in online Appendix C.

manager managing with respect to a 500 stock benchmark (e.g. the S&P 500), should be expected to have an information ratio of 1.1271 and 1.3092 for a 50 percent and 80 percent selectivity ratio respectively. In fact, in our 100,000 simulations, the actual information ratios are 1.1280 and 1.3077 respectively. Thus, in the case of uncorrelated stock returns, this table provides a mapping from information ratio to omega.<sup>20</sup>

## 4.2 Simulations of Single Factor World

In the case of a single factor world, we use the set of equations in Section 3 of the paper. Our simulation methodology is as follows. First, we randomly choose a set of betas for the benchmark that remain fixed for the every investment horizon. Second, we choose a set of those betas to be the betas of the portfolio of  $N_P$  stocks chosen by the portfolio manager.<sup>21</sup> Third, we simulate a random draw from the market portfolio from a normal distribution, i.e.  $\tilde{r}_M \sim N(\mu_M, \sigma_M)$ . Fourth, we simulate the error terms of the individual stocks in the benchmark. That is,  $\epsilon_i \sim N(0, \sigma_\epsilon)$ .<sup>22</sup> Fifth, the manager uses his selection technique (i.e. Wallenius or Fischer) to pick the  $x$  good stocks and the  $N_P - x$  bad stocks from the benchmark. Sixth, we compute the return of the portfolio and the return of the benchmark (i.e.  $r_{P,t} = \bar{\beta}_P \tilde{r}_{M,t} + \frac{1}{N_P} \sum_{i=1}^{N_P} \tilde{\epsilon}_{it}^*$  and  $r_{B,t} = \bar{\beta}_B \tilde{r}_{M,t} + \frac{1}{N_B} \sum_{i=1}^{N_B} \tilde{\epsilon}_{it}$ ). Seventh, we compute the excess return, the tracking error and the information ratio of the portfolio over many investment periods.<sup>23</sup>

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<sup>20</sup>Of course, for implementation, the actual distribution of stock returns over the sample must be estimated.

<sup>21</sup>As we mentioned earlier, we could attach betas to each stock instead and as the manager predicts the alpha of each stock, his or her portfolio beta changes, but that also seems unrealistic, since a manager could control this overall beta.

<sup>22</sup>Our formulas are valid for any value of these parameters. However, for the simulations, we followed the parameters estimated by Goyal and Santa Clara (2003). In particular,  $\mu_M = 0.12$ ,  $\sigma_M = 0.20$ , and  $\sigma_\epsilon = 0.50$ .

<sup>23</sup>Due to the computer time required to run these simulations, we present our results for 100,000 simulations. With our particular code, this was run on a computer with 11 workers and for both Tables 1 and 2 still ran for about 1 day.

[INSERT TABLE 2 ABOUT HERE]

Table 2 contains the results from the 100,000 simulations for the single index model. In each case, the beta of the portfolio and the benchmark may differ from each other. For example, for an  $\omega$  of 1.20, a manager managing with respect to a 500 stock benchmark (e.g. the S&P 500), should be expected to have an information ratio of 1.1281 and 1.2788 for a 50 percent and 80 percent selectivity ratio respectively. In fact, in our 100,000 simulations, the actual information ratios are 1.1290 and 1.2771 respectively. For these portfolios, the portfolio betas are 0.9969 and 0.9961, while the benchmark betas are 0.9968 and 0.9988. These are the average betas over the entire simulations. One again, given a set of parameters and average betas, one can use the formula to closely approximate the information ratios of the portfolios.

## 5 A Potential Application

The main purpose of this mapping technique is to make a connection between selectivity theory and empirical data on portfolio performance. However, one of the benefits of the mathematics we have derived is that an evaluator that has a sense of the generation process for stock picking, and knowledge about the empirical stock distribution can infer the theoretical information ratio of a particular portfolio manager. The only necessary information is the mean and the standard deviation of good versus bad stock picks by the manager out of a population of stocks. For example, a binomial picking manager that is likely to pick a winner 60% of the time, would likely pick 18 good stocks ( $np$ ) out of 30 selected with a standard deviation of 2.683 ( $\sqrt{p(1-p)n}$ ). This process can be used with our formulas to predict his likely information ratio.

Many investors must evaluate past portfolio performance to make future allocation decisions. One very used criteria is the ex-post information ratio of the manager. However, the ex-post information ratio may be heavily influenced by luck rather than skill. In our context, luck means that some of the good stocks had an enormous return or a bad stock had a severe negative return. Implicit in our work on selectivity theory is that we assume managers can pick good stocks, but not necessarily the magnitude of those stock returns. This, we assume is random. There is evidence that this assumption is reasonable. Several studies indicate that mutual fund managers have some stock picking ability, but no evidence that they can predict the magnitude of those returns (Baker et al. (2010), Wermers (2000)), as well as studies that show that even high hit rates do not translate into superior portfolio performance (Shannon (2024)). One of the reasons for this could be that although managers can pick the right stocks, they cannot pick the magnitude of the returns.

Given our mapping technique, an investor might have an additional tool to evaluate historical portfolio performance. The current practice is that an investor would measure the portfolio manager's sample or historical information ratio, call this,  $IR^h$ . The investor can also use our mapping technique as follows. Measure the number of winners that the portfolio picks in every investment period and measure the standard deviation of those winners over the investment periods. Then enter these estimated parameters along with the moments of the stock distribution into our formula to obtain the theoretical information ratio for that manager. Call this,  $IR^{M.T.}$  (M.T. is for mapping technique). In fact, in the examples below we show that using our mapping theorem produces more reliable information ratios than the standard approach.<sup>24</sup> The examples below shows that an investor might wish to use our technique as a complement to the typical method used.<sup>25</sup>

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<sup>24</sup>If the investor also knows the stock picking process of the portfolio manager (or maybe that portfolio manager describes the process to the investor), this would allow the investor to verify which method produces a more reliable information ratio.

<sup>25</sup>Although our technique assumed a normal distribution, the problems we discuss most likely become more amplified if returns are skewed as documented by many papers including Bessembinder (2023).

### 5.1 Example 1

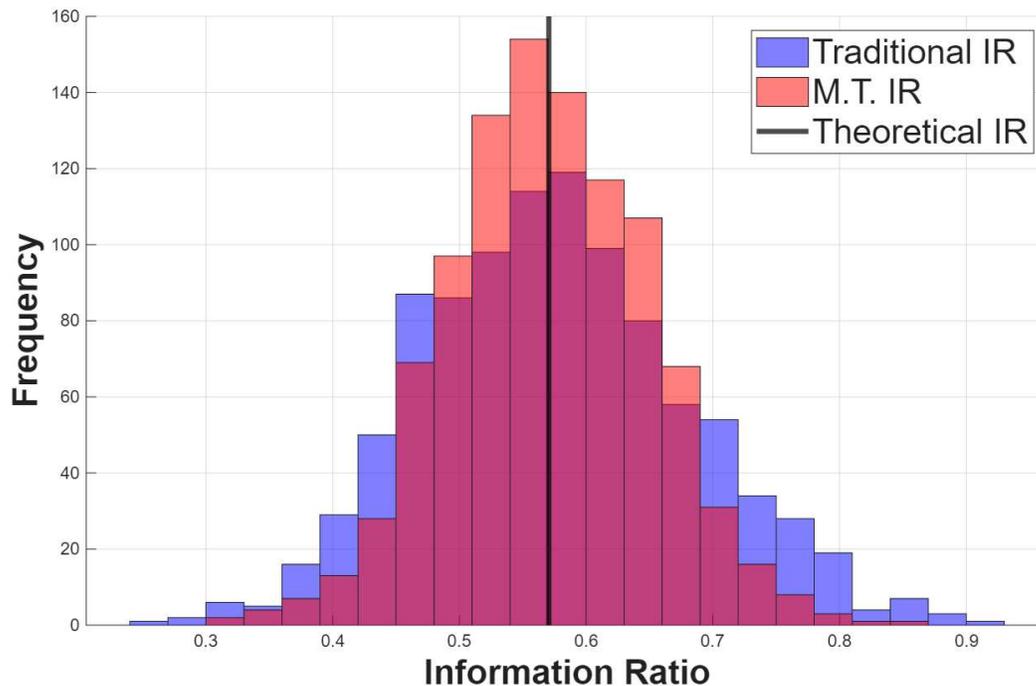
We start with a Wallenius portfolio manager with an  $\omega = 1.3$ , a benchmark of 100 stocks, and a selectivity ratio of 30%. Using our mapping formula, we can show that the theoretical information ratio of such a manager is 0.5705. Figure 1 below shows the measured information ratios from a 1000 simulations of  $IR^h$  and  $IR^{M.T.}$ . It is clear from this figure that  $IR^{M.T.}$  is a more accurate measure of the portfolio manager's true information ratio.

In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.08504 for the traditional method and 0.062945 for the mapping technique method (or 14.905% error versus 11.0324% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.011451 for the traditional method and 0.0062332 for the mapping technique method. The black line is the theoretical information ratio of the manager at 0.57055.

### 5.2 Example 2

Suppose we take a similar situation to the previous example, but consider a manager selecting in bulk (i.e. the Fisher distribution). We consider a Fisher portfolio manager with an  $\omega = 1.3$ , a benchmark of 100 stocks, and a selectivity ratio of 30%. Using our mapping formula, we can show that the theoretical information ratio of such a manager is 0.48306. Figure 2 below shows the measured information ratios from a 1000 simulations of  $IR^h$  and  $IR^{M.T.}$ , and it is clear that the mapping technique ( $IR^{M.T.}$ ) produces a more accurate view of the manager's information ratio than using a historical measured information ratio ( $IR^h$ ).

Figure 1: **The Distribution of Information Ratios for Wallenius Process.** This figure shows the empirically measured information ratios of Wallenius portfolio managers with 100 stocks in the benchmark, a selectivity ratio of 30%, an  $\omega = 1.3$ , the cross-sectional mean of stock returns of 12% with a standard deviation of 20%. The traditional estimates are obtained by taking the historical returns of the portfolio manager and benchmark and computing their sample information ratio. The mapping technique estimate takes the empirical number of winner stocks and the standard deviation of those winner stocks in each period for the portfolio manager and enters these values into the formula for the information ratio derived in this paper and uses that as the calculation of the information ratio. In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.08504 for the traditional method and 0.062945 for the mapping technique method method (or 14.905% error versus 11.0324% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.011451 for the traditional method and 0.0062332 for the mapping technique method method. The black line is the theoretical information ratio of the manager at 0.57055.



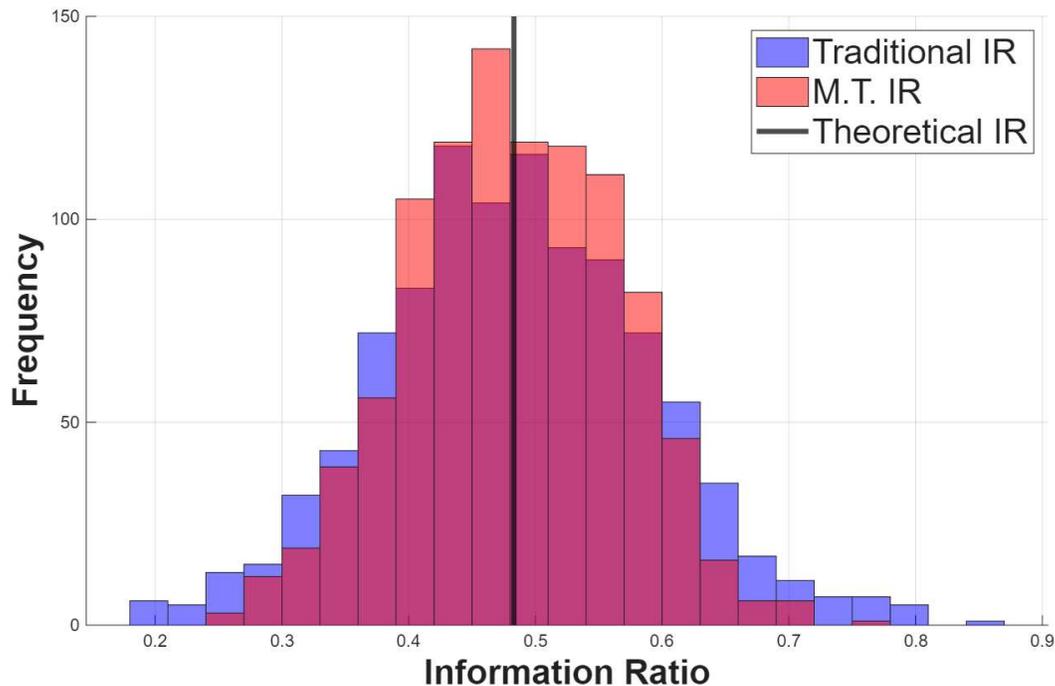
In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.085101 for the traditional method and 0.067448 for the mapping technique method method (or 17.6171% error versus 13.9628% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.011461 for the traditional method and 0.0068684 for the mapping technique method method. The black line is the theoretical information ratio of the manager at 0.48306.

### 5.3 Example 3

In this example, we consider a manager with a binomial-like process to pick stocks. That is, he has a 60% chance to pick any given winner stock regardless of what he has already picked. This is a bit unrealistic, but a more familiar process to many people. We also modify two other items, we allow only for 36 months of data, something which might be available in the practical world, and we also estimate the cross-sectional mean and standard deviation of the actual stock returns of the benchmark, rather than assuming we know this process.

We continue to use a benchmark of 100 stocks, and a selectivity ratio of 30%. Using our mapping formula, we can show that the theoretical information ratio of such a manager is 0.9397. Figure 3 below shows the measured information ratios from a 1000 simulations of  $IR^h$  and  $IR^{M.T.}$ , and it is clear that the mapping technique ( $IR^{M.T.}$ ) produces a more accurate view of the manager's information ratio than using a historical measured information ratio ( $IR^h$ ). In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.16601 for the traditional method and 0.13445 for the mapping technique method method (or 17.6661% error versus 14.307% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.04346 for the traditional method and 0.028385

Figure 2: **The Distribution of Information Ratios for Fisher Process.** This figure shows the empirically measured information ratios of Fisher portfolio managers with 100 stocks in the benchmark, a selectivity ratio of 30%, an  $\omega = 1.3$ , the cross-sectional mean of stock returns of 12% with a standard deviation of 20%. The traditional estimates are obtained by taking the historical returns of the portfolio manager and benchmark and computing their sample information ratio. The mapping technique estimate takes the empirical number of winner stocks and the standard deviation of those winner stocks in each period for the portfolio manager and enters these values into the formula for the information ratio derived in this paper and uses that as the calculation of the information ratio. In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.085101 for the traditional method and 0.067448 for the mapping technique method method (or 17.6171% error versus 13.9628% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.011461 for the traditional method and 0.0068684 for the mapping technique method method. The black line is the theoretical information ratio of the manager at 0.48306.



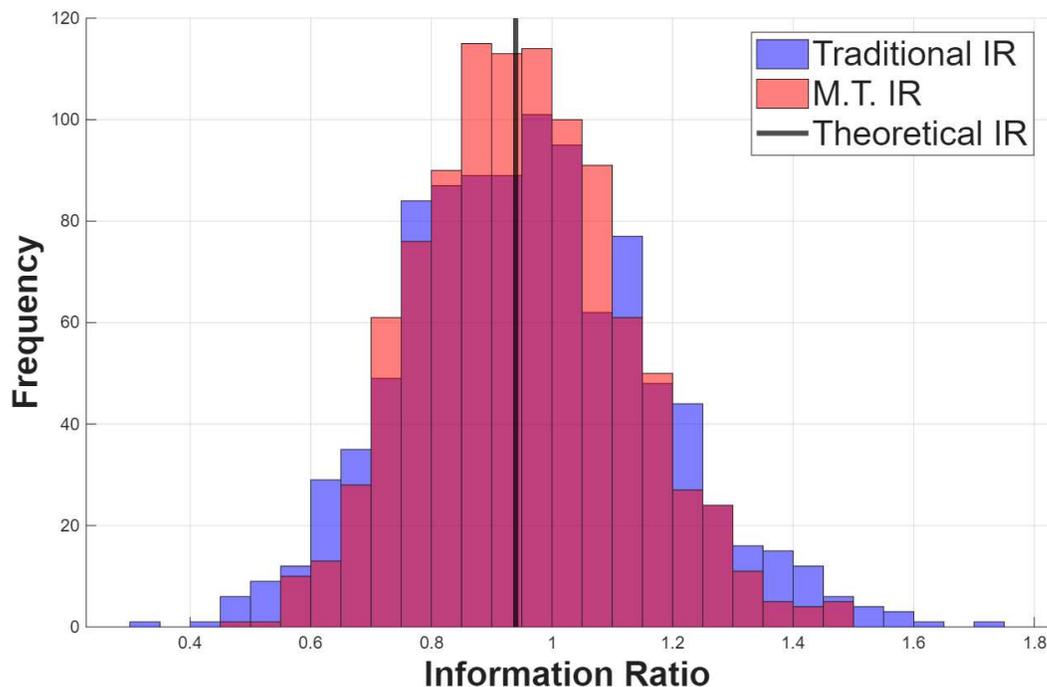
for the mapping technique method method. The black line is the theoretical information ratio of the manager at 0.93973.

## 5.4 Discussion

Although our initial evidence provides a promising application of our mapping theorem that might be used as a complement to the more traditional method of measuring a portfolio manager's historical information ratio, some comments are in order. First, we do not have a proof that the mapping technique will be a better estimate of the information ratio in all circumstances. Second, our mapping technique only applies to equally weighted portfolios and benchmarks. Conceptually, a similar formula for a benchmark with fixed weights and a portfolio with fixed weights seems very plausible, since it would just add an extra variance term for the cross-sectional variance of the weights. This is the scope of further research. Third, we use a normal distribution process, and have not specifically focused on a skewed stock return process. One might believe that the application of our mapping technique should be stronger under skewed distributions, since the "luck" part would likely be larger than in the normal case, but we currently do not have a mapping formula for other distributions.

We hope that future research by the investment community can extend the work in these important directions. Despite this, these early results on the use of the mapping theorem are very inspiring and promising.

**Figure 3: The Distribution of Information Ratios for Binomial Process.** This figure shows the empirically measured information ratios of Binomial portfolio managers with 100 stocks in the benchmark, a selectivity ratio of 30%, and a binomial probability of 60% to pick a winning stock, the cross-sectional mean of stock returns of 12% with a standard deviation of 20%, but parameters were measured from the actual sample of data with 0.1209 and 0.2010. The traditional estimates are obtained by taking the historical returns of the portfolio manager and benchmark and computing their sample information ratio. The mapping technique estimate takes the empirical number of winner stocks and the standard deviation of those winner stocks in each period for the portfolio manager and enters these values into the formula for the information ratio derived in this paper and uses that as the calculation of the information ratio. In this particular simulation, the mean absolute error (MAE) of the information ratio compared to the theoretical information is 0.16601 for the traditional method and 0.13445 for the mapping technique method (or 17.6661% error versus 14.307% error). The mean squared error (MSE) information ratio compared to the theoretical information is 0.04346 for the traditional method and 0.028385 for the mapping technique method. The black line is the theoretical information ratio of the manager at 0.93973.



## 6 Conclusion

In this paper, we have derived the mathematics to map selectivity theory portfolio selection to the more traditional world of information ratios based on the distribution of the underlying stock returns. Selectivity theory (Bolshakov and Chincarini (2020)) states that if a portfolio manager is selecting stocks from a benchmark whose performance is measured against that benchmark, then the optimal amount of stocks he or she should choose is somewhere between 50% to 80% of the benchmark stocks. The original study on selectivity theory focused on an unrealistic distribution of stock returns for simplicity. However, in order to understand the behavior of portfolio managers with respect to stock picking skill, one must have a mapping from the picking of stocks to the traditional information ratio. This paper has made a first large step in that direction. In particular, if one knows the percentage of stocks from a benchmark that a manager chooses and their information ratio, one can use our technique to map these known values to their selectivity parameters in a given stock picking framework, like Wallenius or Fischer. This mapping might ultimately prove useful in describing actual behavior of enhanced portfolio managers. In addition, our mapping formula is valid regardless of the stock picking process provided it can be represented by a mean and standard deviation of selected winner stocks.

In order to generate our mapping, we select stock returns from a normal distribution with mean,  $\mu$ , and standard deviation,  $\sigma$ , such that stock returns are uncorrelated with one another. Although quite unrealistic, this builds the mathematics that can be used for modelling the alpha ( $\alpha$ ) of stocks. We then transport this alpha generation process to a single-factor index world, where stock returns are correlated as would be predicted by a single-factor model. Although the mathematics is more complicated, we find that even within a single factor world, we can map from selectivity theory to

a portfolio manager's information ratio.

In particular, we show that if we know a portfolio manager's selectivity ratio (i.e. the percentage of stocks of the benchmark that he holds in his portfolio), the number of stocks in the benchmark, the empirical distribution of stock returns, and the portfolio's information ratio, we can infer the portfolio manager's omega ( $\omega$ ) if he was selecting stocks according to a Wallenius or Fischer distribution as used in selectivity theory.

For example, if a portfolio manager is sampling stocks from a 1000 stock benchmark in which the historical or estimated characteristics of the stock market are such that the expected market return is 12%, with an annualized standard deviation of 20%, with stock specific errors of expected return 0% and standard deviation of 50%, and we observe their information ratio of 2.2821 for an 80% selectivity ratio (see Table 2), then we can infer that this portfolio manager behaves "as if" they are a Wallenius stock picker with an omega close to 1.25.<sup>26</sup>

While this mapping from a Wallenius or Fisher stock picking process to the information ratio is valuable in itself, we find a potentially even more important application of the mapping technique. Given our assumptions, if one has empirical data on a portfolio's returns, one can get a more reliable estimate of the true information ratio of the portfolio manager by using our technique rather than measuring the portfolio's historical information ratio even if the underlying picking process is not known, but has a mean and standard deviation of winner picks. The intuition for this result is that the randomness of stock returns can make a manager luck really good in a lucky period or really bad in an unlucky period unrelated to his true skill set. This seems to be a very promising result that we hope can be extended to more general circumstances.

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<sup>26</sup>Of course, the real world of stock returns is messier, since we will not have exact distributions, but rather estimated distributions and the sample error associated with this.

The work on the mapping theorem is a very strong step forward. In fact, in our online appendices, we have derived many useful mathematical applications for portfolio theory and beyond. Future research should expand these results to consider non equal-weighted portfolios, whereby the benchmark and the portfolio have a fixed vector of portfolio weights. Further research should attempt to determine in what broad categories of situations, the mapping technique will provide a better measure of the portfolio manager's information ratio. Finally, although we did examine some of the empirical estimation issues in our application section, future research should examine in much greater detail the empirical returns with the underlying parameter estimation errors and its effect on the mapping theorem.

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## 7 Appendices

### A Tables and Figures

Table 1: Mapping from Error Returns and Omega to Information Ratio (Portfolio Drawn from Benchmark - Approximate Method)

		N= 100.00			N= 500.00			N= 1000.00		
	$\omega$	10%	50%	80%	10%	50%	80%	10%	50%	80%
Theory	1.05	0.0616	0.1349	0.1558	0.1376	0.3017	0.3499	0.1946	0.4266	0.4951
Sims.	1.05	0.0612	0.1318	0.1630	0.1356	0.3063	0.3453	0.1970	0.4242	0.5002
Theory	1.10	0.1204	0.2635	0.3045	0.2688	0.5893	0.6837	0.3801	0.8334	0.9674
Sims.	1.10	0.1180	0.2606	0.3045	0.2669	0.5866	0.6808	0.3800	0.8302	0.9617
Theory	1.15	0.1766	0.3864	0.4467	0.3942	0.8641	1.0030	0.5573	1.2220	1.4193
Sims.	1.15	0.1717	0.3858	0.4443	0.4016	0.8638	1.0001	0.5564	1.2273	1.4170
Theory	1.20	0.2303	0.5040	0.5831	0.5142	1.1271	1.3092	0.7270	1.5940	1.8525
Sims.	1.20	0.2374	0.5051	0.5751	0.5138	1.1280	1.3077	0.7283	1.5903	1.8522
Theory	1.25	0.2818	0.6168	0.7141	0.6292	1.3793	1.6035	0.8897	1.9507	2.2689
Sims.	1.25	0.2838	0.6171	0.7116	0.6346	1.3854	1.6042	0.8873	1.9463	2.2717
Theory	1.30	0.3313	0.7252	0.8403	0.7397	1.6216	1.8868	1.0459	2.2933	2.6698
Sims.	1.30	0.3301	0.7169	0.8379	0.7368	1.6234	1.8928	1.0371	2.2870	2.6612
Theory	1.35	0.3789	0.8294	0.9620	0.8460	1.8546	2.1602	1.1962	2.6228	3.0566
Sims.	1.35	0.3744	0.8303	0.9577	0.8480	1.8525	2.1588	1.1935	2.6286	3.0529
Theory	1.40	0.4248	0.9298	1.0797	0.9484	2.0791	2.4244	1.3409	2.9403	3.4304
Sims.	1.40	0.4242	0.9254	1.0770	0.9452	2.0798	2.4127	1.3391	2.9197	3.4375
Theory	1.80	0.7409	1.6219	1.9059	1.6540	3.6269	4.2805	2.3387	5.1293	6.0568
Sims.	1.80	0.7366	1.6174	1.8945	1.6534	3.6080	4.2692	2.3430	5.1316	6.0451
Theory	2.00	0.8729	1.9111	2.2623	1.9486	4.2736	5.0815	2.7552	6.0438	7.1904
Sims.	2.00	0.8707	1.9001	2.2582	1.9384	4.2632	5.0765	2.7630	6.0335	7.2172
Theory	3.00	1.3761	3.0159	3.7247	3.0712	6.7444	8.3807	4.3424	9.5382	11.8615
Sims.	3.00	1.3692	2.9986	3.6937	3.0672	6.7403	8.3970	4.3587	9.5495	11.8414

*Note:* This table shows the simulated and theoretical results of information ratio based on a Wallenius picking procedure by a manager with a given  $\omega$  for 100,000 simulations. The returns of the underlying errors are  $N(0, 0.50)$ . Selectivity levels are listed for 10, 50, and 80%.

Table 2: Mapping from Single Index Returns and Omega to Information Ratio for the Single Index Model (Portfolio Drawn Independently - Approximate Method)

		N= 100.00			N= 500.00			N= 1000.00		
	$\omega$	10%	50%	80%	10%	50%	80%	10%	50%	80%
Theory	1.05	0.0550	0.1470	0.1275	0.0967	0.3027	0.3207	0.2346	0.3788	0.5058
Sims.	1.05	0.0546	0.1438	0.1344	0.0949	0.3073	0.3164	0.2367	0.3762	0.5109
Theory	1.10	0.1138	0.2756	0.2760	0.2276	0.5903	0.6541	0.4197	0.7843	0.9781
Sims.	1.10	0.1113	0.2727	0.2758	0.2257	0.5875	0.6514	0.4194	0.7808	0.9723
Theory	1.15	0.1699	0.3984	0.4180	0.3526	0.8651	0.9730	0.5965	1.1717	1.4298
Sims.	1.15	0.1650	0.3978	0.4155	0.3598	0.8647	0.9699	0.5950	1.1763	1.4274
Theory	1.20	0.2237	0.5161	0.5541	0.4723	1.1281	1.2788	0.7658	1.5425	1.8630
Sims.	1.20	0.2307	0.5171	0.5460	0.4716	1.1290	1.2771	0.7670	1.5391	1.8628
Theory	1.25	0.2752	0.6288	0.6850	0.5870	1.3803	1.5726	0.9282	1.8981	2.2793
Sims.	1.25	0.2771	0.6292	0.6823	0.5922	1.3864	1.5733	0.9257	1.8939	2.2821
Theory	1.30	0.3247	0.7372	0.8109	0.6972	1.6226	1.8555	1.0841	2.2397	2.6802
Sims.	1.30	0.3233	0.7288	0.8087	0.6944	1.6244	1.8611	1.0761	2.2328	2.6717
Theory	1.35	0.3723	0.8414	0.9324	0.8032	1.8556	2.1285	1.2341	2.5682	3.0670
Sims.	1.35	0.3677	0.8424	0.9281	0.8057	1.8535	2.1272	1.2313	2.5738	3.0632
Theory	1.40	0.4181	0.9417	1.0498	0.9053	2.0801	2.3923	1.3785	2.8846	3.4408
Sims.	1.40	0.4175	0.9374	1.0471	0.9021	2.0807	2.3803	1.3773	2.8647	3.4483
Theory	1.80	0.7341	1.6338	1.8740	1.6086	3.6279	4.2447	2.3744	5.0664	6.0671
Sims.	1.80	0.7297	1.6293	1.8623	1.6081	3.6090	4.2328	2.3784	5.0686	6.0555
Theory	2.00	0.8661	1.9230	2.2292	1.9021	4.2746	5.0439	2.7902	5.9778	7.2007
Sims.	2.00	0.8637	1.9120	2.2247	1.8917	4.2642	5.0393	2.7976	5.9680	7.2272
Theory	3.00	1.3689	3.0278	3.6845	3.0195	6.7454	8.3317	4.3744	9.4590	11.8717
Sims.	3.00	1.3619	3.0105	3.6532	3.0147	6.7413	8.3494	4.3910	9.4707	11.8523

*Note:* This table shows the simulated and theoretical results of information ratio based on a Wallenius picking procedure by a manager with a given  $\omega$  for 100,000 simulations. The returns of the underlying errors are  $N(0, 0.50)$  and for the market of  $N(0.12, 0.20)$ . The results are generated for the single index portfolio, where returns of all stocks are given by  $\tilde{r}_i = \beta_i \tilde{r}_M + \epsilon_i$ . The betas for the benchmark are chosen once for each benchmark and for each portfolio. That is, we assume the betas are fixed. The average beta for the portfolios are 0.9785, 1.0106, 0.9944, 0.9753, 0.9969, 0.9961, 1.0211, 0.9851, and 1.0004 and for the benchmark are 0.9868, 1.0056, 1.0003, 0.9981, 0.9968, 0.9988, 1.0050, 0.9912, and 0.9997. Selectivity levels are listed for 10, 50, and 80%. This took 35968.59 seconds to run on a supercomputer with 11 processors.

# Internet Appendix for A Mapping Technique for Selectivity Theory

This version: February 23, 2026.  
Comments Welcome.

## Abstract

Selectivity theory predicts that enhanced index portfolio managers should hold between 50% and 80% of benchmark stocks in their portfolio. The theory was based on some very strict assumptions, including the form of the distribution of returns. In this paper, we relax these assumptions and derive analytical formulas for the portfolio information ratio. We then present the mathematics to map from the information ratio of a portfolio manager to his omega as required in selectivity theory. This technique may allow researchers to map empirical data on portfolio managers to their respective selectivity parameters and underlying skill. We also suggest a potential application of this work, which, in certain circumstances, improves on estimating a portfolio manager's information ratio.

*JEL Classification: G0, G13*

*Key Words: enhanced indexing, selectivity theory, information ratio, portfolio management, active management, simulations*

## A Proof for Equations (3)-(7)

$$\mathbb{E}[r_P] \approx \left(\frac{2\mu_x}{N_P} - 1\right) \left(\sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}}\right) \quad (1)$$

$$\text{Var}(r_P) \approx \sigma^2 \left[\frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi}\right] \quad (2)$$

$$+ \frac{\sigma^2}{N} \left[\frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi}\right] \quad (3)$$

$$IR \approx \frac{\left(\frac{2\mu_x}{N_P} - 1\right) \left(\sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}}\right)}{\sqrt{\left[\frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi}\right] + \frac{1}{N} \left[-1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi}\right]}} \quad (4)$$

. This represents the mean, variance, and information ratio of a portfolio that is chosen from a non-negative integer valued function of “good” and “bad” stocks, where good stocks are chosen from the upper half of a sample from a normal distribution,  $N(\mu, \text{sigma})$ , and the bad stocks are chosen from the lower half of the same sample. Portfolio and benchmark weights are equal-weighted.

### A.1 Basic Approach

Temporarily set  $\mu = 0$ ,  $\sigma = 1$ . Assume that we will draw  $x$  stocks independently from the upper part of the standard Normal distribution truncated at  $M$ , and  $N_P - x$  stocks independently from the lower part of the standard Normal distribution truncated at  $M$ . (The section heading “exact formulas” means that the formulas which follow are exactly correct subject to this assumption.) Note that  $M$  and  $x$  are independent.

Let  $U$  be a random variable drawn from the upper part of the standard Normal distribution truncated at  $M$ , and let  $L$  be a random variable drawn from the lower part of the standard Normal distribution truncated at  $M$ . Given the sampling scheme described above, we have

$$\mathbf{E}[r_P] = \mathbf{E}[\mathbf{E}[r_P | M, x]]$$

where

$$\mathbf{E}(r_P | M, x) = \frac{1}{N_P}(x\mathbf{E}(U | M) + (N_P - x)\mathbf{E}(L | M)).$$

Since  $M$  and  $x$  are independent,

$$\mathbf{E}(r_P) = \frac{1}{N_P}(\mu_x\mathbf{E}(U) + (N_P - \mu_x)\mathbf{E}(L)).$$

As the distribution of  $M$  is symmetric about zero, we have  $\mathbf{E}(L) = -\mathbf{E}(U)$ , so

$$\mathbf{E}(r_P) = \frac{1}{N_P}(2\mu_x - N_P)\mathbf{E}(U) = \left(\frac{2\mu_x}{N_P} - 1\right)\mathbf{E}(U).$$

Moving on to consider the variance of  $r_P$ , we have

$$\text{Var}(r_P) = \mathbf{E}(\text{Var}(r_P | M, x)) + \text{Var}(\mathbf{E}(r_P | M, x)).$$

From the formula for  $r_P$  and the conditional independence of the  $r_i^{(g)}$  and  $r_j^{(b)}$  given  $M$  and

$x$ , it follows that

$$\text{Var}(r_P | M, x) = \frac{1}{N_P^2}(x\text{Var}(U | M) + (N_P - x)\text{Var}(L | M))$$

and

$$\mathbf{E}(\text{Var}(r_P | M, x)) = \frac{1}{N_P^2}(\mu_x \mathbf{E}(\text{Var}(U | M)) + (N_P - \mu_x) \mathbf{E}(\text{Var}(L | M))).$$

By symmetry of the distribution of  $M$ , we have

$$\mathbf{E}(\text{Var}(U | M)) = \mathbf{E}(\text{Var}(L | M))$$

which means that

$$\mathbf{E}(\text{Var}(r_P | M, x)) = \frac{1}{N_P} \mathbf{E}(\text{Var}(U | M)).$$

As well, we need to take the variance of

$$\begin{aligned} \mathbf{E}(r_P | M, x) &= \frac{1}{N_P}(x\mathbf{E}(U | M) + (N_P - x)\mathbf{E}(L | M)) \\ &= \mathbf{E}(L | M) + \frac{x}{N_P}\mathbf{E}(U - L | M). \end{aligned}$$

We have

$$\begin{aligned} \text{Var}\left(\mathbf{E}(L | M) + \frac{x}{N_P}\mathbf{E}(U - L | M)\right) &= \text{Var}(\mathbf{E}(L | M)) \\ &+ \frac{2}{N_P}\text{Cov}(\mathbf{E}(L | M), x\mathbf{E}(U - L | M)) + \frac{1}{N_P^2}\text{Var}(x\mathbf{E}(U - L | M)). \end{aligned}$$

Note that  $x$  is independent of  $M$ . In general, for random variables  $A, B, C$  with  $B$  independent of both  $A$  and  $C$ , it holds that

$$\begin{aligned} \text{Cov}(A, BC) &= \mathbf{E}(ABC) - \mathbf{E}(A)\mathbf{E}(BC) = \mathbf{E}(B)\mathbf{E}(AC) - \mathbf{E}(A)\mathbf{E}(B)\mathbf{E}(C) \\ &= \mathbf{E}(B)\text{Cov}(A, C) \end{aligned}$$

and

$$\begin{aligned} \text{Var}(BC) &= \mathbf{E}(B^2C^2) - E(BC)^2 = \mathbf{E}(B^2)\mathbf{E}(C^2) - \mathbf{E}(B)^2\mathbf{E}(C)^2 \\ &= [\mathbf{E}(B^2) - \mathbf{E}(B)^2]\mathbf{E}(C^2) + \mathbf{E}(B)^2[\mathbf{E}(C^2) - \mathbf{E}(C)^2] \\ &= \text{Var}(B)\mathbf{E}(C^2) + \mathbf{E}(B)^2\text{Var}(C). \end{aligned}$$

Using these identities, we obtain

$$\text{Cov}(\mathbf{E}(L | M), x\mathbf{E}(U - L | M)) = \mathbf{E}(x)\text{Cov}(\mathbf{E}(L | M), \mathbf{E}(U - L | M))$$

and

$$\text{Var}(x\mathbf{E}(U - L | M)) = \text{Var}(x)\mathbf{E}(\mathbf{E}(U - L | M)^2) + \mathbf{E}(x)^2\text{Var}(\mathbf{E}(U - L | M)).$$

It follows that

$$\begin{aligned}\text{Var}(\mathbf{E}(r_P | M, x)) &= \text{Var}(\mathbf{E}(L | M)) + \frac{2\mu_x}{N_P} \text{Cov}(\mathbf{E}(L | M), \mathbf{E}(U - L | M)) \\ &\quad + \frac{\sigma_x^2}{N_P^2} \mathbf{E}(\mathbf{E}(U - L | M)^2) + \frac{\mu_x^2}{N_P^2} \text{Var}(\mathbf{E}(U - L | M)).\end{aligned}$$

We put everything together to get the formula

$$\begin{aligned}\text{Var}(r_P) &= \frac{1}{N_P} \mathbf{E}(\text{Var}(U | M)) + \text{Var}(\mathbf{E}(L | M)) \\ &\quad + \frac{2\mu_x}{N_P} \text{Cov}(\mathbf{E}(L | M), \mathbf{E}(U - L | M)) \\ &\quad + \frac{\sigma_x^2}{N_P^2} \mathbf{E}(\mathbf{E}(U - L | M)^2) + \frac{\mu_x^2}{N_P^2} \text{Var}(\mathbf{E}(U - L | M)).\end{aligned}$$

The simple average of all  $N$  returns may be written as

$$r_B = \frac{1}{N} \left( \sum_{i=1}^{N/2} r_i^{(g)} + \sum_{j=1}^{N/2} r_j^{(b)} \right).$$

As each individual return has a  $N(\mu, \sigma)$  distribution, we have

$$r_B \sim N \left( \mu, \frac{\sigma}{\sqrt{N}} \right).$$

We are interested in the information ratio

$$\text{IR} = \frac{\mathbf{E}(r_P - r_B)}{\text{SD}(r_P - r_B)}.$$

To compute the denominator, it is useful to set

$$r_Q = \frac{1}{N - N_P} \left( \sum_{i=x+1}^{N/2} r_i^{(g)} + \sum_{j=N_P-x+1}^{N/2} r_j^{(b)} \right)$$

to be the average of the remaining returns that were not included in the portfolio. We have

$$r_B = \frac{N_P}{N} r_P + \frac{N - N_P}{N} r_Q$$

and in addition, the random variables  $r_P$  and  $r_Q$  are conditionally independent given  $M$  and  $x$ . We are concerned with the variance of

$$r_P - r_B = \frac{N - N_P}{N} (r_P - r_Q).$$

We have

$$\text{Var}(r_P - r_Q) = \mathbf{E}(\text{Var}(r_P - r_Q \mid M, x)) + \text{Var}(\mathbf{E}(r_P - r_Q \mid M, x)).$$

By conditional independence,

$$\text{Var}(r_P - r_Q \mid M, x) = \text{Var}(r_P \mid M, x) + \text{Var}(r_Q \mid M, x).$$

We already saw that

$$\mathbf{E}(\text{Var}(r_P \mid M, x)) = \frac{1}{N_P} \mathbf{E}(\text{Var}(U \mid M))$$

and by similar reasoning it is also true that

$$\mathbf{E}(\text{Var}(r_Q | M, x)) = \frac{1}{N - N_P} \mathbf{E}(\text{Var}(U | M)).$$

Thus,

$$\begin{aligned} \mathbf{E}(\text{Var}(r_P - r_Q | M, x)) &= \left( \frac{1}{N_P} + \frac{1}{N - N_P} \right) \mathbf{E}(\text{Var}(U | M)) \\ &= \frac{N}{N_P(N - N_P)} \mathbf{E}(\text{Var}(U | M)). \end{aligned}$$

Next,

$$\begin{aligned} \mathbf{E}(r_P - r_Q | M, x) &= \frac{1}{N_P} (x \mathbf{E}(U | M) + (N_P - x) \mathbf{E}(L | M)) \\ &\quad - \frac{1}{N - N_P} \left[ \left( \frac{N}{2} - x \right) \mathbf{E}(U | M) + \left( \frac{N}{2} - (N_P - x) \right) \mathbf{E}(L | M) \right] \\ &= \left( \frac{x}{N_P} - \frac{\frac{N}{2} - x}{N - N_P} \right) \mathbf{E}(U - L | M) \\ &= \left( \frac{N}{N_P(N - N_P)} x - \frac{N/2}{N - N_P} \right) \mathbf{E}(U - L | M). \end{aligned}$$

If we take

$$B = \frac{N}{N_P(N - N_P)} x - \frac{N/2}{N - N_P}, \quad C = \mathbf{E}(U - L | M)$$

then we have  $\text{Var}(BC) = \text{Var}(B) \mathbf{E}(C^2) + \mathbf{E}(B)^2 \text{Var}(C)$ , since  $B$  and  $C$  are independent.

This yields

$$\begin{aligned}
\text{Var}(\mathbf{E}(r_P - r_Q \mid M, x)) &= \frac{N^2}{N_P^2(N - N_P)^2} \text{Var}(x) \mathbf{E}(\mathbf{E}(U - L \mid M)^2) \\
&\quad + \left( \frac{N}{N_P(N - N_P)} \mathbf{E}(x) - \frac{N/2}{N - N_P} \right)^2 \text{Var}(\mathbf{E}(U - L \mid M)) \\
&= \frac{N^2 \sigma_x^2}{N_P^2(N - N_P)^2} \mathbf{E}(\mathbf{E}(U - L \mid M)^2) + \frac{(N\mu_x - NN_P/2)^2}{N_P^2(N - N_P)^2} \text{Var}(\mathbf{E}(U - L \mid M)).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\text{Var}(r_P - r_B) &= \frac{(N - N_P)^2}{N^2} \text{Var}(r_P - r_Q) \\
&= \frac{(N - N_P)^2}{N^2} [\mathbf{E}(\text{Var}(r_P - r_Q \mid M, x)) + \text{Var}(\mathbf{E}(r_P - r_Q \mid M, x))] \\
&= \left( \frac{1}{N_P} - \frac{1}{N} \right) \mathbf{E}(\text{Var}(U \mid M)) + \frac{\sigma_x^2}{N_P^2} \mathbf{E}(\mathbf{E}(U - L \mid M)^2) \\
&\quad + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \text{Var}(\mathbf{E}(U - L \mid M)).
\end{aligned}$$

## A.2 Truncating the Normal Sample

The conditional density function for  $U$  given  $M$  is

$$f_U(x \mid M) = \frac{\varphi(x)}{1 - \Phi(M)} \quad (x > M)$$

where  $\varphi(x) = \Phi'(x)$  is the standard Normal density function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

We compute

$$\begin{aligned}\mathbf{E}(U | M) &= \int_M^\infty x f_U(x | M) dx = \frac{1}{1 - \Phi(M)} \int_M^\infty x \varphi(x) dx \\ &= \frac{1}{1 - \Phi(M)} (-\varphi(x)) \Big|_{x=M}^{x=\infty} = \frac{\varphi(M)}{1 - \Phi(M)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}(U^2 | M) &= \int_M^\infty x^2 f_U(x | M) dx = \frac{1}{1 - \Phi(M)} \int_M^\infty x^2 \varphi(x) dx \\ &= \frac{1}{1 - \Phi(M)} (\Phi(x) - x\varphi(x)) \Big|_{x=M}^{x=\infty} = \frac{1 - \Phi(M) + M\varphi(M)}{1 - \Phi(M)} \\ &= 1 + \frac{M\varphi(M)}{1 - \Phi(M)}.\end{aligned}$$

Then,

$$\text{Var}(U | M) = \mathbf{E}(U^2 | M) - \mathbf{E}(U | M)^2 = 1 + \frac{M\varphi(M)}{1 - \Phi(M)} - \frac{\varphi(M)^2}{(1 - \Phi(M))^2}.$$

Similarly,

$$\mathbf{E}(L | M) = -\frac{\varphi(M)}{\Phi(M)}, \quad \mathbf{E}(U - L | M) = \frac{\varphi(M)}{1 - \Phi(M)} + \frac{\varphi(M)}{\Phi(M)}.$$

### A.3 Taylor Approximation Approach

We provide Taylor approximations in  $M$  up to the  $M^4$  term for  $\text{Var}(U | M)$ ,  $\mathbf{E}(L | M)$ , and  $\mathbf{E}(U - L | M)$ . These were computed by a computer algebra system using the formulas

above.

$$\begin{aligned}
\text{Var}(U | M) &= \frac{\pi - 2}{\pi} - \frac{\sqrt{2}(4 - \pi)}{\pi^{3/2}}M + \frac{4(\pi - 3)}{\pi^2}M^2 + \frac{-96 + 40\pi - 3\pi^2}{3\sqrt{2} \cdot \pi^{5/2}}M^3 \\
&\quad - \frac{120 - 60\pi + 7\pi^2}{3\pi^3}M^4 + O(M^5), \\
\mathbf{E}(L | M) &= -\sqrt{\frac{2}{\pi}} + \frac{2}{\pi}M - \frac{4 - \pi}{\sqrt{2} \cdot \pi^{3/2}}M^2 - \frac{4(\pi - 3)}{3\pi^2}M^3 \\
&\quad + \frac{-96 + 40\pi - 3\pi^2}{12\sqrt{2} \cdot \pi^{5/2}}M^4 + O(M^5), \\
\mathbf{E}(U - L | M) &= \sqrt{\frac{8}{\pi}} + \frac{\sqrt{2}(4 - \pi)}{\pi^{3/2}}M^2 - \frac{-96 + 40\pi - 3\pi^2}{6\sqrt{2} \cdot \pi^{5/2}}M^4 + O(M^6).
\end{aligned}$$

## A.4 The Moments of the Median

According to [Chu–Hotelling 1955], the median  $M$  is approximately Normal with mean zero and variance

$$\frac{1}{4N\varphi(0)^2} = \frac{\pi}{2N}.$$

We have therefore

$$\mathbf{E}(M^2) \approx \frac{\pi}{2N}, \quad \mathbf{E}(M^4) \approx \frac{3\pi^2}{4N^2}.$$

## A.5 Final Approximation

We can now write approximate formulas in  $N$  for the quantities needed to compute  $\mathbf{E}(r_P)$ ,  $\text{Var}(r_P)$ , and  $\text{Var}(r_P - r_B)$ , namely  $\mathbf{E}(U)$ ,  $\mathbf{E}(\text{Var}(U | M))$ ,  $\text{Var}(\mathbf{E}(L | M))$ ,  $\text{Cov}(\mathbf{E}(L |$

$M$ ),  $\mathbf{E}(U - L | M)$ ),  $\mathbf{E}(\mathbf{E}(U - L | M)^2)$ , and  $\text{Var}(\mathbf{E}(U - L | M))$ . To begin,

$$\begin{aligned}\mathbf{E}(L) &= \mathbf{E}(\mathbf{E}(L | M)) = -\sqrt{\frac{2}{\pi}} - \frac{4 - \pi}{\sqrt{2} \cdot \pi^{3/2}} \mathbf{E}(M^2) + \frac{-96 + 40\pi - 3\pi^2}{12\sqrt{2} \cdot \pi^{5/2}} \mathbf{E}(M^4) \\ &\approx -\sqrt{\frac{2}{\pi}} - \frac{4 - \pi}{2\sqrt{2\pi}} \cdot \frac{1}{N} + \frac{-96 + 40\pi - 3\pi^2}{16\sqrt{2\pi}} \cdot \frac{1}{N^2}\end{aligned}$$

which means that

$$\mathbf{E}(U) = -\mathbf{E}(L) \approx \sqrt{\frac{2}{\pi}} + \frac{4 - \pi}{2\sqrt{2\pi}} \cdot \frac{1}{N} - \frac{-96 + 40\pi - 3\pi^2}{16\sqrt{2\pi}} \cdot \frac{1}{N^2}.$$

As well,

$$\begin{aligned}\mathbf{E}(\text{Var}(U | M)) &\approx \frac{\pi - 2}{\pi} + \frac{4(\pi - 3)}{\pi^2} \mathbf{E}(M^2) - \frac{120 - 60\pi + 7\pi^2}{3\pi^3} \mathbf{E}(M^4) \\ &\approx \frac{\pi - 2}{\pi} + \frac{2(\pi - 3)}{\pi N} - \frac{120 - 60\pi + 7\pi^2}{4\pi N^2}.\end{aligned}$$

For the next terms, we first see that

$$\begin{aligned}\text{Cov}\left(\sum_i c_i M^i, \sum_j d_j M^j\right) &= \sum_{i,j} c_i d_j \text{Cov}(M^i, M^j) \\ &\approx c_1 d_1 \text{Cov}(M, M) + c_1 d_3 \text{Cov}(M, M^3) \\ &\quad + c_2 d_2 \text{Cov}(M^2, M^2) + c_3 d_1 \text{Cov}(M^3, M) \\ &= c_1 d_1 \mathbf{E}(M^2) + (c_1 d_3 + c_3 d_1) \mathbf{E}(M^4) \\ &\quad + c_2 d_2 [\mathbf{E}(M^4) - \mathbf{E}(M^2)^2]\end{aligned}$$

where we eliminated terms with odd moments of  $M$  that equal zero, and where the approximate equality excludes higher-order terms with  $M$  raised to the sixth power or more.

Now,

$$\begin{aligned}
\text{Var}(\mathbf{E}(L | M)) &\approx \frac{4}{\pi^2} \mathbf{E}(M^2) - \frac{16(\pi - 3)}{3\pi^3} \mathbf{E}(M^4) + \frac{(4 - \pi)^2}{2\pi^3} [\mathbf{E}(M^4) - \mathbf{E}(M^2)^2] \\
&\approx \frac{2}{\pi N} - \frac{4(\pi - 3)}{\pi N^2} + \frac{(4 - \pi)^2}{4\pi N^2} \\
&= \frac{2}{\pi N} - \frac{16(\pi - 3) - (4 - \pi)^2}{4\pi N^2}
\end{aligned}$$

and

$$\begin{aligned}
\text{Cov}(\mathbf{E}(L | M), \mathbf{E}(U - L | M)) &\approx -\frac{(4 - \pi)^2}{\pi^3} [\mathbf{E}(M^4) - \mathbf{E}(M^2)^2] \\
&\approx -\frac{(4 - \pi)^2}{2\pi N^2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathbf{E}(\mathbf{E}(U - L | M)^2) &\approx \frac{8}{\pi} + \frac{8(4 - \pi)}{\pi^2} \mathbf{E}(M^2) + \frac{2(4 - \pi)^2}{\pi^3} \mathbf{E}(M^4) \\
&\quad - \frac{2(-96 + 40\pi - 3\pi^2)}{3\pi^3} \mathbf{E}(M^4) \\
&\approx \frac{8}{\pi} + \frac{4(4 - \pi)}{\pi N} \\
&\quad + \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi N^2}
\end{aligned}$$

and

$$\text{Var}(\mathbf{E}(U - L | M)) \approx \frac{2(4 - \pi)^2}{\pi^3} [\mathbf{E}(M^4) - \mathbf{E}(M^2)^2] \approx \frac{(4 - \pi)^2}{\pi N^2}.$$

We use the formulas above to obtain the approximations

$$\mathbf{E}(r_P) \approx \left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96 + 40\pi - 3\pi^2}{16\sqrt{2\pi}} \right)$$

as well as

$$\begin{aligned}
\text{Var}(r_P) &\approx \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \\
&+ \frac{1}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
&+ \frac{1}{N^2} \left[ -\frac{16(\pi - 3) - (4 - \pi)^2}{4\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} \right. \\
&\quad \left. - \frac{\mu_x}{N_P} \cdot \frac{(4 - \pi)^2}{\pi} + \frac{\mu_x^2}{N_P^2} \cdot \frac{(4 - \pi)^2}{\pi} \right. \\
&\quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right]
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(r_P - r_B) &\approx \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \\
&+ \frac{1}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
&+ \frac{1}{N^2} \left[ -\frac{2(\pi - 3)}{\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} \right. \\
&\quad \left. + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \frac{(4 - \pi)^2}{\pi} \right. \\
&\quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right].
\end{aligned}$$

## A.6 Adjustment for Actual Mean and Variance of Normal and Varying Accuracy

For general  $\mu, \sigma$  (instead of  $\mu = 0$  and  $\sigma = 1$ ), we get

$$\mathbf{E}(r_P) \approx \mu + \sigma \left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96 + 40\pi - 3\pi^2}{16\sqrt{2\pi}} \right).$$

Based on the equation above, we can present three formulas for variance based on increasing accuracy for  $\text{Var}(r_P)$ :<sup>1</sup>

$$\begin{aligned}
(1) \quad \text{Var}(r_P) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
(2) \quad \text{Var}(r_P) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
(3) \quad \text{Var}(r_P) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N^2} \left[ -\frac{16(\pi - 3) - (4 - \pi)^2}{4\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} \right. \\
&\quad \quad - \frac{\mu_x}{N_P} \cdot \frac{(4 - \pi)^2}{\pi} + \frac{\mu_x^2}{N_P^2} \cdot \frac{(4 - \pi)^2}{\pi} \\
&\quad \quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right].
\end{aligned}$$

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<sup>1</sup>In the main body of the paper, we actually use the second formula in all case for our calculations of variance and IR.

And three formulas of increasing accuracy for  $\text{Var}(r_P - r_B)$ :

$$\begin{aligned}
(4) \quad \text{Var}(r_P - r_B) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
(5) \quad \text{Var}(r_P - r_B) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
(6) \quad \text{Var}(r_P - r_B) &\approx \sigma^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\
&\quad + \frac{\sigma^2}{N^2} \left[ -\frac{2(\pi - 3)}{\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} \right. \\
&\quad \quad \left. + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \frac{(4 - \pi)^2}{\pi} \right. \\
&\quad \quad \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right].
\end{aligned}$$

For the information ratio

$$\text{IR} = \frac{\mathbf{E}(r_P - r_B)}{\text{SD}(r_P - r_B)},$$

since  $\mathbf{E}(r_B) = \mu$  we have formulas

$$\begin{aligned}
(7) \quad \text{IR} &\approx \frac{\left( \frac{2\mu_x}{N_P} - 1 \right) \sqrt{2/\pi}}{\sqrt{\frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi}}} \\
(8) \quad \text{IR} &\approx \frac{\left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} \right)}{\sqrt{\left[ \frac{1}{N_P} \cdot \frac{\pi-2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] + \frac{1}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi-3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4-\pi)}{\pi} \right]}} \\
(9) \quad \text{IR} &\approx \frac{\left( \frac{2\mu_x}{N_P} - 1 \right) \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4-\pi}{2\sqrt{2\pi}} - \frac{1}{N^2} \cdot \frac{-96+40\pi-3\pi^2}{16\sqrt{2\pi}} \right)}{\sqrt{\blacksquare}}
\end{aligned}$$

where

$$\blacksquare = \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] + \frac{1}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right] \\ + \frac{1}{N^2} \left[ -\frac{2(\pi - 3)}{\pi} - \frac{1}{N_P} \cdot \frac{120 - 60\pi + 7\pi^2}{4\pi} + \frac{1}{4} \left( \frac{2\mu_x}{N_P} - 1 \right)^2 \frac{(4 - \pi)^2}{\pi} \right. \\ \left. + \frac{\sigma_x^2}{N_P^2} \cdot \frac{3(4 - \pi)^2 + (96 - 40\pi + 3\pi^2)}{2\pi} \right].$$

## B Derivations of Single Index Model for Equal Weight Portfolio and Benchmark

Given  $N$  stocks numbered  $1, \dots, N$ . We model the (adjusted) return of the  $i$ -th stock as

$$\tilde{r}_i = \beta_i \tilde{r}_M + \epsilon_i$$

where:

- the  $\beta_i$  are non-random constants;<sup>2</sup>
- $\tilde{r}_M$  is a random variable representing overall market conditions, with mean  $\mu_M$  and variance  $\sigma_M^2$ ;
- the  $\epsilon_i$  are random variables representing the idiosyncratic risk associated with the  $i$ -th

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<sup>2</sup>In our work, we assign betas to the stocks in the benchmark, and then the portfolio picks stocks randomly, but keeps the beta of the portfolio the same for each investment period. Hence, the beta of the portfolio and the benchmark is constant. Lemma 10 from Chincarini, Bolshakov, and Jerison (2025) shows how to adjust for randomly drawn betas. That is, if we force the alphas of stocks to be linked to whatever beta the stock has, then the portfolio's beta will change in every period based on those selections. This will result in a cross-sectional variance of the beta.

stock, which we assume to be i.i.d.  $N(0, \sigma_\epsilon^2)$ ;

- the market return  $\tilde{r}_M$  is independent of the idiosyncratic risks  $\epsilon_i$ .

The average return of all the stocks is

$$\tilde{r}_B = \frac{1}{N} \sum_{i=1}^N \tilde{r}_i = \left( \frac{1}{N} \sum_{i=1}^N \beta_i \right) \tilde{r}_M + \left( \frac{1}{N} \sum_{i=1}^N \epsilon_i \right).$$

We set

$$\bar{\beta}_B = \frac{1}{N} \sum_{i=1}^N \beta_i$$

to be the average beta of all the stocks, so that

$$\tilde{r}_B = \bar{\beta}_B \cdot \tilde{r}_M + \left( \frac{1}{N} \sum_{i=1}^N \epsilon_i \right).$$

Note that  $\tilde{r}_M$  is the *unobserved* quantity driving the market returns, while  $\tilde{r}_B$  is the *observed* average return. In our work below, we do not require that  $\bar{\beta}_B$  must be exactly equal to 1, but a useful perspective is that when  $N$  is large,  $\bar{\beta}_B$  should be very close to 1 while  $\frac{1}{N} \sum_{i=1}^N \epsilon_i$  is very close to 0, so that  $\tilde{r}_B \approx \tilde{r}_M$ .<sup>3</sup>

We now consider a portfolio manager who has some skill at selecting stocks whose  $\epsilon_i$  are higher than typical. For this, we define certain subsets of the index set  $\{1, \dots, N\}$ . Let  $M$

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<sup>3</sup>Assuming that the market is a greater number of securities,  $N_M$ , but that the benchmark weights securities in a similar fashion to the benchmark (i.e. market capitalization), then as  $N \rightarrow N_M$ , the benchmark beta will approach 1. When the benchmark is all securities in the market weighted in the same way as the market, then the benchmark beta will also equal 1. That is,  $\mathbf{B} = \frac{\boldsymbol{\Sigma} \mathbf{w}_B}{\sigma_B^2}$ , where  $\boldsymbol{\Sigma}$  is the variance-covariance matrix of stock returns, and  $w_B$  is the weights of the benchmark. Thus, the weighted average beta of the benchmark is  $\frac{\mathbf{w}_B' \boldsymbol{\Sigma} \mathbf{w}_B}{\sigma_B^2} = 1$ . For our exercises, we will not assume this.

be the median value of  $\epsilon_1, \dots, \epsilon_N$ . It is convenient to assume that  $N$  is even, so that half of the  $\epsilon_i$  are above  $M$  and the other half are below  $M$ .

Let  $I_U \subseteq \{1, \dots, N\}$  be the set of indices  $i$  for which  $\epsilon_i > M$ , and let  $I_L \subseteq \{1, \dots, N\}$  be the set of indices  $i$  for which  $\epsilon_i < M$ . We have  $|I_U| = |I_L| = N/2$ .

We assume that the portfolio manager will choose a portfolio of  $N_P$  stocks, where  $N_P \leq N$  is a non-random constant. The selection model introduces an integer-valued random variable  $x$ , independent of  $\tilde{r}_M$  and the  $\epsilon_i$ , that counts how many stocks in the portfolio will come from the upper half  $I_U$ . Let  $x$  have mean  $\mu_x$  and variance  $\sigma_x^2$ . Since  $x$  stocks will come from the upper half  $I_U$  and the remaining  $N_P - x$  stocks will come from the lower half  $I_L$ , we require  $x$  to be distributed so that the inequalities  $0 \leq x \leq N/2$  and  $0 \leq N_P - x \leq N/2$  always hold.

The portfolio selection process can be modeled by the following steps: (1) Sample the values of the  $\epsilon_i$  and compute the index sets  $I_U, I_L$ . (2) Sample the value of  $x$ . (3) Choose  $x$  stocks uniformly and without replacement from  $I_U$ . (4) Choose  $N_P - x$  stocks uniformly and without replacement from  $I_L$ . (5) Combine the stocks chosen in steps (3) and (4) into a portfolio  $I_P$ , with  $|I_P| = N_P$ ,  $|I_P \cap I_U| = x$ , and  $|I_P \cap I_L| = N_P - x$ .

Once the portfolio  $I_P$  has been chosen, we can compute its return

$$\tilde{r}_P = \frac{1}{N_P} \sum_{i \in I_P} \tilde{r}_i = \left( \frac{1}{N_P} \sum_{i \in I_P} \beta_i \right) \tilde{r}_M + \left( \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i \right).$$

Define

$$\bar{\beta}_P = \frac{1}{N_P} \sum_{i \in I_P} \beta_i,$$

so that

$$\tilde{r}_P = \bar{\beta}_P \cdot \tilde{r}_M + \left( \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i \right).$$

In this model, we assume that the manager is choosing stocks based only on their knowledge of the idiosyncratic risks  $\epsilon_i$  and without regard to the  $\beta_i$  values. In addition, there is no relationship between the  $\beta_i$  and the  $\epsilon_i$ .

## B.1 Expected Return Derivation

The expected return of the benchmark is

$$\mathbf{E}(\tilde{r}_B) = \bar{\beta}_B \cdot \mathbf{E}(\tilde{r}_M) + 0 = \bar{\beta}_B \cdot \mu_M.$$

To compute the expected return of the portfolio, let  $\mu_U$  be the expected value of an  $\epsilon_i$  drawn uniformly from the upper half of  $\epsilon_1, \dots, \epsilon_N$ , and let  $\mu_L$  be the expected value of an  $\epsilon_i$  drawn uniformly from the lower half of  $\epsilon_1, \dots, \epsilon_N$ . In other words:

$$\mu_U = \mathbf{E} \left( \frac{1}{N/2} \sum_{i \in I_U} \epsilon_i \right), \quad \mu_L = \mathbf{E} \left( \frac{1}{N/2} \sum_{i \in I_L} \epsilon_i \right).$$

Recalling that  $M$  is the median of the  $\epsilon_i$ , we have equivalently

$$\mu_U = \mathbf{E}(\epsilon_i \mid \epsilon_i > M), \quad \mu_L = \mathbf{E}(\epsilon_i \mid \epsilon_i < M).$$

The portfolio return is

$$\begin{aligned} \tilde{r}_P &= \bar{\beta}_P \cdot \tilde{r}_M + \left( \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i \right) \\ &= \bar{\beta}_P \cdot \tilde{r}_M + \frac{1}{N_P} \left( \sum_{i \in I_P \cap I_U} \epsilon_i + \sum_{i \in I_P \cap I_L} \epsilon_i \right), \end{aligned}$$

from which we obtain

$$\mathbf{E}(\tilde{r}_P \mid x) = \bar{\beta}_P \cdot \mu_M + \frac{1}{N_P} [x \cdot \mu_U + (N_P - x) \cdot \mu_L]$$

and

$$\mathbf{E}(\tilde{r}_P) = \bar{\beta}_P \cdot \mu_M + \frac{\mu_x}{N_P} \cdot \mu_U + \left( 1 - \frac{\mu_x}{N_P} \right) \cdot \mu_L.$$

This leads to the formula

$$\mathbf{E}(\tilde{r}_P - \tilde{r}_B) = (\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + \frac{\mu_x}{N_P} \cdot \mu_U + \left( 1 - \frac{\mu_x}{N_P} \right) \cdot \mu_L.$$

We can let  $p^* = \mu_x/N_P$  be the expected fraction of the portfolio that comes from the upper half. Since  $\mu_L = -\mu_U$  in our model, we obtain

$$\begin{aligned}
\mathbf{E}(\tilde{r}_P - \tilde{r}_B) &= (\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + p^* \cdot \mu_U + (1 - p^*) \cdot (-\mu_U) \\
&= (\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + (2p^* - 1) \cdot \mu_U.
\end{aligned}$$

From our derivations in Appendix A.5., we have the approximation

$$\mu_U \approx \sigma_\epsilon \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} \right)$$

which can be substituted in the previous formula to give

$$\mathbf{E}(\tilde{r}_P - \tilde{r}_B) \approx (\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + (2p^* - 1) \cdot \sigma_\epsilon \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} \right).$$

## B.2 Variance and Tracking Error Derivation

We would like to compute  $\text{Var}(\tilde{r}_P)$  and  $\text{Var}(\tilde{r}_P - \tilde{r}_B)$ . The latter is used in the information ratio

$$\text{IR} = \frac{\mathbf{E}(\tilde{r}_P - \tilde{r}_B)}{\text{SD}(\tilde{r}_P - \tilde{r}_B)}.$$

We first see that

$$\text{Var}(\tilde{r}_P) = \text{Var} \left( \bar{\beta}_P \cdot \tilde{r}_M + \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i \right) = \bar{\beta}_P^2 \text{Var}(\tilde{r}_M) + \text{Var} \left( \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i \right).$$

Recall that  $\text{Var}(\tilde{r}_M) = \sigma_M^2$ . In Appendix A, we computed the approximation<sup>4</sup>

$$\begin{aligned} \text{Var}\left(\frac{1}{N_P} \sum_{i \in I_P} \epsilon_i\right) &\approx \sigma_\epsilon^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &\quad + \frac{\sigma_\epsilon^2}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} \text{Var}(\tilde{r}_P) &\approx \bar{\beta}_P^2 \cdot \sigma_M^2 + \sigma_\epsilon^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &\quad + \frac{\sigma_\epsilon^2}{N} \left[ \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right]. \end{aligned}$$

Next, we have

$$\begin{aligned} \text{Var}(\tilde{r}_P - \tilde{r}_B) &= \text{Var}\left((\bar{\beta}_P - \bar{\beta}_B) \cdot \tilde{r}_M + \frac{1}{N_P} \sum_{i \in I_P} \epsilon_i - \frac{1}{N} \sum_{i=1}^N \epsilon_i\right) \\ &= (\bar{\beta}_P - \bar{\beta}_B)^2 \cdot \sigma_M^2 + \text{Var}\left(\frac{1}{N_P} \sum_{i \in I_P} \epsilon_i - \frac{1}{N} \sum_{i=1}^N \epsilon_i\right). \end{aligned}$$

In Appendix A.5., we found that

$$\begin{aligned} \text{Var}\left(\frac{1}{N_P} \sum_{i \in I_P} \epsilon_i - \frac{1}{N} \sum_{i=1}^N \epsilon_i\right) &\approx \sigma_\epsilon^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &\quad + \frac{\sigma_\epsilon^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right]. \end{aligned}$$

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<sup>4</sup>We derived three orders of the approximation. In this section, we use the second order term of the Taylor Series expansion from the derivation in Appendix A.

Therefore,

$$\begin{aligned} \text{Var}(\tilde{r}_P - \tilde{r}_B) &\approx (\bar{\beta}_P - \bar{\beta}_B)^2 \cdot \sigma_M^2 + \sigma_\epsilon^2 \left[ \frac{1}{N_P} \cdot \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{8}{\pi} \right] \\ &\quad + \frac{\sigma_\epsilon^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \cdot \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \cdot \frac{4(4 - \pi)}{\pi} \right]. \end{aligned}$$

### B.3 The Information Ratio

Finally, the information ratio is given by

$$\begin{aligned} \text{IR} &= \frac{\mathbf{E}(\tilde{r}_P - \tilde{r}_B)}{\text{SD}(\tilde{r}_P - \tilde{r}_B)} \\ &\approx \frac{(\bar{\beta}_P - \bar{\beta}_B) \cdot \mu_M + (2p^* - 1) \cdot \sigma_\epsilon \left( \sqrt{\frac{2}{\pi}} + \frac{1}{N} \cdot \frac{4 - \pi}{2\sqrt{2\pi}} \right)}{\sqrt{(\bar{\beta}_P - \bar{\beta}_B)^2 \sigma_M^2 + \sigma_\epsilon^2 \left[ \frac{1}{N_P} \frac{\pi - 2}{\pi} + \frac{\sigma_x^2}{N_P^2} \frac{8}{\pi} \right] + \frac{\sigma_\epsilon^2}{N} \left[ -1 + \frac{2}{\pi} + \frac{1}{N_P} \frac{2(\pi - 3)}{\pi} + \frac{\sigma_x^2}{N_P^2} \frac{4(4 - \pi)}{\pi} \right]}}. \end{aligned}$$

## C A Special Case: Portfolio and Benchmark Returns are Independent but Drawn from Same Distribution

In the main body of the paper we considered the appropriate formulas for excess return, tracking error, and the information ratio when the benchmark stock returns were driven by some external process reflecting a normal distribution,  $N(\mu, \sigma)$ . We also considered a single factor model, in which the returns were driven by a single factor model and the manager had the ability to choose the  $\epsilon$  or  $\alpha$  of stocks from the benchmark.

In this section, we consider a less realistic case, but nevertheless and interesting one, where the stock in the benchmark are chosen from a distribution of stock returns with a given mean and variance  $(\mu, \sigma)$  and the portfolio returns are also chosen from that distribution but independently from the benchmark returns.<sup>5</sup> We derive the expressions for excess return, tracking error, and information ratio in that case. We also consider the case where the weights of the portfolio and benchmark come from a distribution of weights with mean,  $\mu_w$  and  $\sigma_w$ .<sup>6</sup> This is an important addition, since we can consider more than equally weighted portfolios. We first consider the special case of i.i.d. returns and ignore the single factor model.

In order to uncover the relationship between a portfolio manager's skill and their information ratio performance, we use the results of Lemma 9 from Chincarini, Bolshakov, and Jerison (2025). That is, given a portfolio manager with skill level,  $\omega$ , we have an equation that described his information ratio. The information ratio consists of the excess return of the portfolio divided by the tracking error of the portfolio. The principle governing equations are:

$$\mathbb{E}[r_P - r_B] = N_P \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{P,\omega}] - N \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{B,\omega=1}] \quad (5)$$

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<sup>5</sup>One can think of this process as a portfolio manager that specifically chooses stocks outside of the benchmark.

<sup>6</sup>One can think of the process of the portfolio construction as follows: First, the manager decides how many stocks he would like to own in the portfolio. Second, he randomly draws weights of the stocks from a distribution. Next, he renormalizes the weights, so that the portfolio weights sum to 1. That is,  $\tilde{w}_i = \frac{w_i}{\sum_{i=1}^N w_i}$ . We then measure the portfolio's return as  $r_P = \sum_{i=1}^N \tilde{w}_i r_i$ . We also imagine that the portfolio manager is an enhanced index manager which manages to a benchmark. We can then assume that the benchmark weights are either fixed or are also drawn from a random distribution.

where  $\mathbb{E}[\tilde{w}]$  is driven by the weight generation process,<sup>7</sup>  $N$  is the number of stocks included in the benchmark,  $N_P$  is the number of stocks selected for the portfolio, where  $N_P < N$ ,  $\mathbb{E}[\tilde{r}_{P,\omega}]$  is the expected return of the typical portfolio stock based on the portfolio manager's  $\omega$ , and  $\mathbb{E}[\tilde{r}_{B,\omega=1}]$  is the expected return of the typical benchmark selected stock based on “no-skill” (i.e. an  $\omega = 1$ ).

The variance of a portfolio that chooses  $N_P$  stocks out of a total of  $N$  stocks that exist in the benchmark, will be given by (see Lemma 9 from Chincarini, Bolshakov, and Jerison (2025)):

$$\begin{aligned} \text{Var}(r_P) = & \mu_x (\mu_{\tilde{w}}^2 [\sigma_g^2 - \sigma_b^2] + \sigma_{\tilde{w}}^2 [\mu_g^2 + \sigma_g^2 - \mu_b^2 - \sigma_b^2]) + N_P [\mu_{\tilde{w}}^2 \sigma_b^2 + \mu_b^2 \sigma_{\tilde{w}}^2 + \sigma_{\tilde{w}}^2 \sigma_b^2] \quad (6) \\ & - [\sigma_x^2 + \mu_x(\mu_x - 1)] \frac{\mu_g^2 \sigma_{\tilde{w}}^2}{N_P - 1} - [(N_P - \mu_x)^2 + \sigma_x^2 + \mu_x - N_P] \frac{\mu_b^2 \sigma_{\tilde{w}}^2}{N_P - 1} \\ & - 2 [N_P \mu_x - \sigma_x^2 + \mu_x^2] \frac{\mu_g \mu_b \sigma_{\tilde{w}}^2}{N_P - 1} + \mu_{\tilde{w}}^2 \sigma_x^2 (\mu_g - \mu_b)^2. \end{aligned}$$

This is the variance of the portfolio with the selection criterion we described above.<sup>8</sup>

There is a similar formula for the benchmark, that is,

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<sup>7</sup>There are many ways to consider a non-equally weighted portfolio. One method is to assume a fix or slowly moving weight set (e.g. market cap weights). Another method is to randomly select the weights from some distribution. For example, we could choose to draw random weights from a uniform distribution. With random uniform weights,  $\mathbb{E}[\tilde{w}] = \frac{1}{N}$  (see Lemma 1 in Chincarini, Bolshakov, and Jerison (2025)), thus, the expected excess return is equal to  $(p^* - 1/2)(\mu_g - \mu_b)$  as in the equal-weight case.

<sup>8</sup>Neither of these expressions have simple derivations. Chincarini, Bolshakov, and Jerison (2025) show that  $\mu_{\tilde{w}} = \frac{1}{N}$  when  $w \sim U(0,1)$ . For  $\sigma_{\tilde{w}}^2$ , we use an Taylor series approximation that we believed was proposed by Howard Selten.  $\text{Var}\left(\frac{x}{y}\right) \approx \left[ \left( \frac{(\mu_x)^2}{(\mu_y)^2} \right) \left[ \frac{\sigma_x^2}{\mu_x^2} + \frac{\sigma_y^2}{\mu_y^2} - \frac{2\text{Cov}(X,Y)}{\mu_x \mu_y} \right] \right]$ . In our specific case, this becomes  $\text{Var}\left(\frac{w}{\sum_{i=1}^N w_i}\right) \approx \left\{ \left( \frac{(\mu_w)^2}{(\mu_\Sigma)^2} \right) \left[ \frac{\sigma_w^2}{\mu_w^2} + \frac{\sigma_\Sigma^2}{\mu_\Sigma^2} - \frac{2\text{Cov}(w,\Sigma)}{\mu_w \mu_\Sigma} \right] \right\}$ . When using a i.i.d. draws from the  $U(0,1)$  for the weights, then  $\sigma_w^2 = \frac{1}{12}$ ,  $\mu_w = \frac{1}{N}$ ,  $\mu_\Sigma = N\mu_w$ ,  $\sigma_\Sigma^2 = N\sigma_w^2$ , and  $\text{Cov}(w, \Sigma) = \sigma_w^2$ . Thus,  $\text{Var}\left(\frac{w}{\sum_{i=1}^N w_i}\right) \approx \frac{\sigma_w^2}{N^2 \mu_w^2} \left(\frac{N-1}{N}\right)$ . Given the distribution of our weights, this equals  $\frac{N-1}{3N^3}$ . One can also express this variance of the portfolio and similar variance of the benchmark as a matrix,  $(\iota' \Sigma_{N_P \times N_P} \iota)$  and  $(\iota' \Sigma_{N \times N} \iota)$  respectively.

$$\begin{aligned}
\text{Var}(r_B) &= \hat{\mu}_x (\hat{\mu}_{\bar{w}}^2 [\hat{\sigma}_g^2 - \hat{\sigma}_b^2] + \hat{\sigma}_{\bar{w}}^2 [\hat{\mu}_g^2 + \hat{\sigma}_g^2 - \hat{\mu}_b^2 - \hat{\sigma}_b^2]) + N [\hat{\mu}_{\bar{w}}^2 \hat{\sigma}_b^2 + \hat{\mu}_b^2 \hat{\sigma}_{\bar{w}}^2 + \hat{\sigma}_{\bar{w}}^2 \hat{\sigma}_b^2] \\
&\quad - [\hat{\sigma}_x^2 + \hat{\mu}_x(\hat{\mu}_x - 1)] \frac{\hat{\mu}_g^2 \hat{\sigma}_{\bar{w}}^2}{N-1} - [(N - \hat{\mu}_x)^2 + \hat{\sigma}_x^2 + \hat{\mu}_x - N] \frac{\hat{\mu}_b^2 \hat{\sigma}_{\bar{w}}^2}{N-1} \\
&\quad - 2 [N\hat{\mu}_x - \hat{\sigma}_x^2 + \hat{\mu}_x^2] \frac{\hat{\mu}_g \hat{\mu}_b \hat{\sigma}_{\bar{w}}^2}{N-1} + \hat{\mu}_{\bar{w}}^2 \hat{\sigma}_x^2 (\hat{\mu}_g - \hat{\mu}_b)^2.
\end{aligned} \tag{7}$$

where all variables are for the benchmark have hats ( $\hat{\cdot}$ ),  $\hat{\mu}_x = 0.5$  and  $\hat{\sigma}_x^2 = p(1-p)N = \frac{1}{2}(\frac{1}{2})N = \frac{N}{4}$  since the benchmark chooses from the universe of stocks with a probability of  $p$  of picking a good stock and  $1-p$  of picking a bad stock, where  $p = 0.5$  (no skill). This is a binomial distribution.<sup>9</sup>

Thus, the tracking error or variance of difference in returns is given by:

$$\begin{aligned}
TE^2 &= \text{Var}(r_P - r_B) = \text{Var}(r_P) - \text{Var}(r_B) \\
&= \mu_x (\mu_{\bar{w}}^2 [\sigma_g^2 - \sigma_b^2] + \sigma_{\bar{w}}^2 [\mu_g^2 + \sigma_g^2 - \mu_b^2 - \sigma_b^2]) + N_P [\mu_{\bar{w}}^2 \sigma_b^2 + \mu_b^2 \sigma_{\bar{w}}^2 + \sigma_{\bar{w}}^2 \sigma_b^2] \\
&\quad - [\sigma_x^2 + \mu_x(\mu_x - 1)] \frac{\mu_g^2 \sigma_{\bar{w}}^2}{N_P-1} - [(N_P - \mu_x)^2 + \sigma_x^2 + \mu_x - N_P] \frac{\mu_b^2 \sigma_{\bar{w}}^2}{N_P-1} \\
&\quad - 2 [N_P \mu_x - \sigma_x^2 + \mu_x^2] \frac{\mu_g \mu_b \sigma_{\bar{w}}^2}{N_P-1} + \mu_{\bar{w}}^2 \sigma_x^2 (\mu_g - \mu_b)^2 \\
&\quad - \hat{\mu}_x (\hat{\mu}_{\bar{w}}^2 [\hat{\sigma}_g^2 - \hat{\sigma}_b^2] + \hat{\sigma}_{\bar{w}}^2 [\hat{\mu}_g^2 + \hat{\sigma}_g^2 - \hat{\mu}_b^2 - \hat{\sigma}_b^2]) + N [\hat{\mu}_{\bar{w}}^2 \hat{\sigma}_b^2 + \hat{\mu}_b^2 \hat{\sigma}_{\bar{w}}^2 + \hat{\sigma}_{\bar{w}}^2 \hat{\sigma}_b^2] \\
&\quad - [\hat{\sigma}_x^2 + \hat{\mu}_x(\hat{\mu}_x - 1)] \frac{\hat{\mu}_g^2 \hat{\sigma}_{\bar{w}}^2}{N-1} - [(N - \hat{\mu}_x)^2 + \hat{\sigma}_x^2 + \hat{\mu}_x - N] \frac{\hat{\mu}_b^2 \hat{\sigma}_{\bar{w}}^2}{N-1} \\
&\quad - 2 [N\hat{\mu}_x - \hat{\sigma}_x^2 + \hat{\mu}_x^2] \frac{\hat{\mu}_g \hat{\mu}_b \hat{\sigma}_{\bar{w}}^2}{N-1} + \hat{\mu}_{\bar{w}}^2 \hat{\sigma}_x^2 (\hat{\mu}_g - \hat{\mu}_b)^2
\end{aligned} \tag{8}$$

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<sup>9</sup>As we discussed in the main text, the larger the investment portfolio in terms of number of stocks,  $N$ , the lower the variance of the actual returns. That is, if we are drawing from a certain distribution, the  $N$  stock portfolio will have a variance that is  $\sqrt{\frac{\text{Var}(f)}{N}}$  of the one-stock portfolio, where  $f$  in this case represents the one-stock portfolio. This is particularly easy to see when the portfolio manager equal weights the stocks in his portfolio. As the manager chooses more than one stock, say  $N$  stocks, from a return distribution, the precision of estimating the mean or the standard error of the mean estimate becomes smaller at the rate of  $\sqrt{N}$ . Even though this is the theoretical result from taking a sample from a distribution, in the case of a portfolio manager that equal-weights each stock in his portfolio, this has a similar effect. Thus, over many samples of  $N$  stocks, the portfolio's managers variance of the sample mean will be driven by this result.

where we can ignore the covariance in returns by portfolio construction (i.e. random i.i.d returns chosen for benchmark and portfolio independently).

Thus, the information ratio of the portfolio with random weights is given by

$$IR = \frac{N_P \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{P,\omega}] - N \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{B,\omega=1}]}{\sqrt{TE^2}} \quad (9)$$

### C.1 An Equal Weight Portfolio

The expected return for the portfolio and for the benchmark can be obtained a variety of ways. The simplest way is to use the derived probability of the average good stock return and average bad stock return. That is, if the Wallenius or related distribution has an expectation of  $\mu_x$  good stocks and hence  $(1 - \mu_x)$  bad stocks, then the expected return for the portfolio will be given by  $\mathbb{E}[\tilde{r}_{P,\omega}] = p^* \mu_g + (1 - p^*) \mu_b$ , where  $p^* = \frac{\mu_x | \omega}{N_P}$ . For the benchmark, a similar logic applies, except that for the benchmark, the derived probabilities will be  $p^* = (1 - p^*) = 1/2$ . That is,  $\mathbb{E}[\tilde{r}_{B,\omega=1}] = \frac{1}{2} (\mu_g + \mu_b)$ .

Thus, for the equal-weight portfolio, we have

$$\begin{aligned} \mathbb{E}[r_P - r_B] &= N_P \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{P,\omega}] - N \mathbb{E}[\tilde{w}] \mathbb{E}[\tilde{r}_{B,\omega=1}] \\ &= \mathbb{E}[\tilde{r}_{P,\omega}] - \mathbb{E}[\tilde{r}_{B,\omega=1}] \\ &= (p^* - 1/2) (\mu_g - \mu_b) \end{aligned} \quad (10)$$

The tracking error or variance of difference in returns is given by:<sup>10</sup>

$$\begin{aligned} \text{Var}(r_P - r_B) &= \frac{1}{N_p^2} (\mu_x[\sigma_g^2 - \sigma_b^2] + \sigma_x^2[(\mu_g - \mu_b)^2] + N_P\sigma_b^2) \\ &+ \frac{1}{N^2} (\hat{\mu}_x[\sigma_g^2 - \sigma_b^2] + \hat{\sigma}_x^2[(\mu_g - \mu_b)^2] + N\sigma_b^2) \end{aligned} \quad (11)$$

where for the benchmark,  $\hat{\mu}_x = 0.5N$  and  $\hat{\sigma}_x^2 = p^*(1 - p^*)N = \frac{1}{2}(\frac{1}{2})N = \frac{N}{4}$  since the benchmark chooses from the universe of stocks with a probability of  $p$  of picking a good stock (i.e. from the upper half of the distribution) and  $1 - p$  of picking a bad stock (i.e. from the lower half of the distribution), where  $p = 0.5$  (no skill). Also, the equal weighting make  $\mu_{\tilde{w}} = \hat{\mu}_{\tilde{w}} = \frac{1}{N}$  and  $\sigma_{\tilde{w}}^2 = \hat{\sigma}_{\tilde{w}}^2 = 0$ . This is a binomial distribution.

In the case of equal-weighting, the information ratio becomes:

$$IR = \frac{(p^* - 1/2)(\mu_g - \mu_b)}{\sqrt{\frac{1}{N_p^2} (\mu_x[\sigma_g^2 - \sigma_b^2] + \sigma_x^2[(\mu_g - \mu_b)^2] + N_P\sigma_b^2) + \frac{1}{N^2} (\frac{1}{2}[\sigma_g^2 - \sigma_b^2] + (\frac{N}{4})[(\mu_g - \mu_b)^2] + N\sigma_b^2)}} \quad (12)$$

### C.1.1 Example 1: Binary Returns

For the case of equally-weighted portfolio and benchmark with binary returns, the formula simplifies significantly. The average excess return will simplify to  $\mathbb{E}[r_P - r_B] = \mathbb{E}[\tilde{r}_{P,\omega}] - \mathbb{E}[\tilde{r}_{B,\omega=1}] = (p^* - 1/2)(\mu_g - \mu_b)$ , where  $p^* = \frac{\mu_{x|\omega}}{N_p}$ . The denominator of the information ratio (IR) can also be simplified. Since the returns are binary, both the “good” and “bad” stocks have no variance, which means that  $\sigma_g^2 = 0 = \sigma_b^2$ . Thus, the variance or tracking

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<sup>10</sup>See Lemma 9 in Chincarini, Bolshakov, and Jerison (2025).

error squared reduces to  $\frac{1}{N_P^2} \sigma_x^2 (\mu_g - \mu_b)^2$ . Thus, the information ratio for this particular manager is  $IR_\omega = \frac{(p^*-1/2)(\mu_g-\mu_b)}{\frac{1}{N_P} \sigma_x (\mu_g-\mu_b)} = \frac{(p^*-1/2)}{\frac{\sigma_x}{N_P}}$ . This is the exact formula used in Bolshakov and Chincarini (2020).

## C.2 Simulations

The simulation results for this special case simulation are interesting in the sense that the optimal selectivity ratio seems to be slightly less than 50%. This is clear in the table. For example, for the  $\omega = 1.20$ , and  $N = 500$ , the information ratios are 0.7330 and 0.5136 respectively. One can also see the closeness of the exact formula with the simulations.

Table 1: Mapping from Error Returns and Omega to Information Ratio (Portfolio Drawn Independently - Exact Method)

		N= 100.00			N= 500.00			N= 1000.00		
	$\omega$	10%	50%	80%	10%	50%	80%	10%	50%	80%
Theory	1.05	0.0574	0.0879	0.0615	0.1282	0.1963	0.1378	0.1813	0.2775	0.1949
Sims.	1.05	0.0640	0.0854	0.0601	0.1242	0.1949	0.1406	0.1817	0.2766	0.1982
Theory	1.10	0.1122	0.1717	0.1200	0.2505	0.3834	0.2690	0.3542	0.5421	0.3805
Sims.	1.10	0.1051	0.1706	0.1162	0.2479	0.3841	0.2698	0.3600	0.5422	0.3829
Theory	1.15	0.1645	0.2517	0.1759	0.3673	0.5620	0.3941	0.5193	0.7947	0.5575
Sims.	1.15	0.1602	0.2526	0.1737	0.3694	0.5579	0.3986	0.5206	0.7984	0.5574
Theory	1.20	0.2146	0.3282	0.2292	0.4790	0.7330	0.5136	0.6773	1.0364	0.7265
Sims.	1.20	0.2108	0.3230	0.2332	0.4785	0.7241	0.5117	0.6799	1.0393	0.7287
Theory	1.25	0.2625	0.4015	0.2801	0.5861	0.8967	0.6278	0.8287	1.2680	0.8881
Sims.	1.25	0.2653	0.4032	0.2801	0.5890	0.8999	0.6243	0.8323	1.2602	0.8889
Theory	1.30	0.3086	0.4719	0.3288	0.6889	1.0538	0.7370	0.9741	1.4901	1.0426
Sims.	1.30	0.3042	0.4718	0.3265	0.6843	1.0467	0.7468	0.9784	1.4882	1.0415
Theory	1.35	0.3529	0.5395	0.3755	0.7878	1.2048	0.8416	1.1139	1.7036	1.1906
Sims.	1.35	0.3519	0.5447	0.3786	0.7888	1.2119	0.8400	1.1075	1.7114	1.1897
Theory	1.40	0.3955	0.6045	0.4202	0.8829	1.3501	0.9419	1.2484	1.9090	1.3325
Sims.	1.40	0.3977	0.6063	0.4197	0.8785	1.3501	0.9353	1.2431	1.9100	1.3379
Theory	1.80	0.6885	1.0502	0.7201	1.5369	2.3456	1.6149	2.1731	3.3167	2.2847
Sims.	1.80	0.6890	1.0525	0.7251	1.5394	2.3501	1.6089	2.1687	3.3145	2.2753
Theory	2.00	0.8102	1.2345	0.8393	1.8086	2.7573	1.8830	2.5572	3.8988	2.6640
Sims.	2.00	0.8110	1.2368	0.8336	1.8050	2.7541	1.8907	2.5594	3.8782	2.6658
Theory	3.00	1.2693	1.9238	1.2420	2.8331	4.2968	2.7913	4.0057	6.0758	3.9501
Sims.	3.00	1.2684	1.9222	1.2389	2.8308	4.2947	2.7974	4.0000	6.0677	3.9662

*Note:* This table shows the simulated and theoretical results of information ratio based on a Wallenius picking procedure by a manager with a given  $\omega$  for 100,000 simulations. The returns of the underlying errors are  $N(0.12, 0.20)$ . Selectivity levels are listed for 10, 50, and 80%.