Tomorrow Never Dies: The Impact of Measuring Portfolio Risk Ex-ante and Ex-post*

Christoph Frey[a] and Björn Uhl[b]

[a] *Lancaster University, Pinechip Capital GmbH*
[b] *Pinechip Capital GmbH*

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Abstract

Portfolio risk can be estimated in various ways. The different methodologies can be classified as either using only the current portfolio weights (ex-ante) or using the complete history of portfolio weights (ex-post). Although the differences are typically negligible, at times, there can be substantial discrepancies between the two. If a portfolio is actively managed using such risk measures, the subsequent impact on performance can also be significant. In this study, we formalize these two concepts and discuss their properties in the context of portfolio risk and performance. In an empirical illustration, we show that the risk measure choice can have a significant impact on portfolio performance and characteristics.

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1 Introduction

Measuring portfolio risk is a central element in managing a portfolio. However, there are various ways of estimating the risk of a portfolio. There are two groups of estimators: Those that are based on the historical returns of the portfolio, which are typically referred to as *ex-post* risk measures, and those that are based on the current portfolio allocation and a historical estimate of the covariance matrix of market returns, which are typically referred to as *ex-ante* risk measures. Such risk measures are often interpreted as forecasts and frequently used to steer a portfolio and for regulatory purposes.

As both measures are valid risk measures, the conceptual differences between ex-post and ex-ante volatility estimates are as profound as they are often overlooked. Ex-ante risk measures have the distinct advantage of not requiring historical information about the portfolio, i.e. past portfolio weights or returns. It only requires the current market allocations and an estimate of the market covariance matrix. While ex-ante and ex-post volatility figures differ in size and dynamics, it is less understood how they result in very different portfolio strategies if the risk estimate is used to manage the portfolio. Hence, in this paper we not only consider the difference in measuring the risk of a given portfolio, ex-ante vs. ex-post, but we also, and primarily, try to understand the impact of the different measurements on portfolio performance.

The literature concerning ex-post and ex-ante risk measures is relatively scarce. While Steiner (2013) derives an attribution scheme to explain the difference in ex-post and ex-ante volatility based on portfolio weights and covariance estimates, Satchell et al. (2001) show that the ex-post tracking error of a portfolio is always larger than the ex-ante tracking error, since portfolio weights are ex-post stochastic. Clarke et al. (2002) derive ex-ante and ex-post correlation relationships that facilitate the performance analysis of constrained portfolios using the fundamental law of active portfolio management. Also, Simlai (2014) use a factor model to estimate the ex-ante market that uses an aggregate dividend yield, the default spread, the term spread as well as the short-term interest rate.

As many portfolio strategies are risk-targeting by nature, they require the timely estimation of the variance-covariance matrix. This includes variance overlay strategies and gearing schemes. Hence, a differentiation between ex-ante and ex-post volatility can also help to explain certain portfolio investment rules (Johnson et al., 2007). Eventually, this paper is also related to Siburg et al. (2015) who investigate the predictive power of Value-at-Risk models and Wang and Yan (2021), who show that downside risk measures portfolios outperform traditional volatility target portfolios.

The following example illustrates the potential differences between both methodological approaches and motivates our subsequent analyses. In particular, in section 2 we formalize both concepts and derive some properties. In section 3 we apply both approaches to typical systematic investment strategies and discuss their impact. The final section concludes.

Motivating example

Let us consider a simple example: JB is a portfolio manager and runs a long-only vol-targeting fund that wants to achieve a volatility level of 10% annually. JB rebalances the strategy at the end of each month and he can take leverage up to a certain degree. In January 2022, his manager M asks him why his portfolio runs at such a high risk with a Value-at-Risk of 35.6% at the one percent significance level calculated based on the last 260 trading days. However, JB wonders and replies that the actual VaR, he measures, is only 23.97%, only slightly above his target of 23.33 percent ($2.33 \times 10\%$). So where does this discrepancy come from? It can be explained by the risk estimation methodology.

M uses the current portfolio allocation to calculate the VaR (ex-ante VaR), because this gives the most accurate estimate of the current level of risk. M believes his estimate is correct as it measures the risk in the current portfolio. JB instead uses the actual portfolio weights at each point in time in the past year to calculate the VaR (ex-post VaR). He argues his estimate is correct as his portfolio was less leveraged when the market volatility was high and would be so in the future. He says: "It's like measuring a safe driver only by his current speed. If he is just driving slowly because traffic is heavy or there is a sharp corner, you might think that he must be a safe driver. But how does he behave when there are no corners or other cars? If he always drives at the same speed, then the current speed is enough to decide whether you want to drive along. Otherwise, one should always look at how fast the driver has driven in the past."

The same holds for the leverage or gearing of a portfolio to measure its riskiness. So what happened to JBs' portfolio at the beginning of 2022? A sharp increase in market volatility in November of 2021 led to a slight decrease in portfolio leverage. While the ex-post VaR in the left panel of Figure 1 does not change dramatically, the ex-ante VaR measures increased quickly in November 2021 to over 30%.[1] In fact, whereas the

[1] Note that in the plot, the Value-at-Risk decreases as it is a negative-valued risk measure.

ex-ante VaR can change very quickly when the volatility of the market changes, the ex-post VaR measure is much more stable, only sharply decreasing in March 2021 when the Corona shock 260 days earlier faded out. Another perspective on this is visible in the right panel of 1: The actual daily returns produced from the ex-post measure are much less volatile than the returns of the ex-ante measure produced from the actual portfolio weights in January 2022.

Figure 1 to go here.

Measuring the riskiness of a portfolio can have a severe impact on for example capital requirements. Hence, the use of ex-ante or ex-post risk metrics might be very important from a risk management perspective. Nevertheless, we argue here that this choice also renders the performance attribution of volatility-targeting portfolios. We find that the use of an ex-post instead of an ex-ante measured risk target can not stabilize portfolio allocations, but can also increase portfolio performance overall; especially in market times of high volatility.

2 Ex-post vs. ex-ante volatility

Let us now get a better understanding of the conceptual differences and similarities between ex-post and ex-ante volatility and formalize a theoretical framework. The general difference between ex-ante and ex-post risk estimators is whether only the current portfolio weights are used or the sample history of weights.

Let us assume that an investor can invest in N risky financial assets with arithmetic return $r_{t,n}$, $n = 1, \ldots, N$. We denote the vector of time-t returns as $\mathbf{r}_t = (r_{t,1}, \ldots, r_{t,N})$ be the $(N \times 1)$ with mean vector $\boldsymbol{\mu}$ and $(N \times N)$ variance-covariance matrix $\boldsymbol{\Sigma}$. Let $\boldsymbol{\omega}_t = (\omega_{t,1}, \ldots, \omega_{t,N})$ be portfolio allocation in the N assets at time t. We do not constrain $\boldsymbol{\omega}_t$ to add up to 1 so that the investor does not need to be fully invested or can take leverage. We also allow for taking short positions but the results do not depend on it.

The portfolio return at time t is calculated by

$$r_t^p = \sum_{n=1}^{N} \omega_{t-1,n} \cdot r_{t,n} = \boldsymbol{\omega}_{t-1}' \mathbf{r}_t \tag{1}$$

and the estimated empirical portfolio variance over the period from $t = 1$ to $t = T$ is then given by

$$\hat{\sigma}^2_{expost} = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_t^p - \bar{r}^p\right)^2, \text{ where } \quad \bar{r}^p = \frac{1}{T}\sum_{t=1}^{T} r_t^p. \tag{2}$$

It proves useful to rewrite this expression as

$$\begin{aligned}\hat{\sigma}^2_{expost} &= \frac{1}{T-1}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\omega_{t-1,n}\cdot r_{t,n} - \frac{1}{T}\sum_{t=1}^{T}\sum_{n=1}^{N}\omega_{t-1,n}\cdot r_{t,n}\right)^2 \\ &= \frac{1}{T-1}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\omega_{t-1,n}\left(r_{t,n} - \bar{r}_n\right)\right)^2 \end{aligned}\tag{3}$$

where $\bar{r}_n = \frac{1}{T}\sum_{t=1}^{T} r_{t,n}$. This is also called *ex-post* volatility estimate, as it considers the variability portfolio weights over time. Let us further denote the matrix of all portfolio weights by

$$\mathbf{\Omega} = \begin{bmatrix} \omega_{0,1} & \cdots & \omega_{T-1,1} \\ \vdots & \ddots & \vdots \\ \omega_{0,N} & \cdots & \omega_{T-1,N} \end{bmatrix}_{(N\times T)}. \tag{4}$$

With this, we can show that the ex-post volatility given in equation (3) is the scaled trace of the quadratic matrix product $\mathbf{\Omega}'\mathbf{\Sigma}\mathbf{\Omega}$.

Proposition 1. *The ex-post volatility of a portfolio given in equation (3) can be written in matrix notation as*

$$\hat{\sigma}^2_{expost} = \frac{1}{T}\operatorname{tr}\left(\mathbf{\Omega}'\hat{\mathbf{\Sigma}}\mathbf{\Omega}\right) + \rho, \tag{5}$$

where $\hat{\mathbf{\Sigma}} = \frac{1}{T-1}\tilde{\mathbf{R}}'\tilde{\mathbf{R}}$, $\tilde{\mathbf{R}}$ *is the matrix of all de-meaned asset returns of dimension* $T \times N$*,*

$\mathbf{\Omega}$ is given in equation (4) and ρ is a quadratic form given by

$$\rho = \frac{1}{T-1} vec\left(\mathbf{\Omega}\right)' \cdot \underbrace{\begin{bmatrix} \tilde{\mathbf{r}}_1\tilde{\mathbf{r}}_1' - \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}' - \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T \end{bmatrix}}_{=:\Phi} \cdot vec\left(\mathbf{\Omega}\right)$$

$$= \frac{1}{T-1} vec\left(\mathbf{\Omega}\right)' \Phi \, vec\left(\mathbf{\Omega}\right), \tag{6}$$

where $\tilde{\mathbf{r}}_t = (r_{t,1} - \bar{r}_1, \ldots, r_{t,N} - \bar{r}_N)'$ $\forall t = 1, \ldots, T$.

The proof of Proposition 1 is in Appendix A.

This is an interesting result since the trace of a matrix is also equivalent to the sum of its eigenvalues, which is also the variance of the first k principal components of the matrix $\mathbf{\Omega}'\hat{\mathbf{\Sigma}}\mathbf{\Omega}$. Similar results involving the Wishart traces are used for the derivation of the distribution of mean-variance style portfolio weights in Okhrin and Schmid (2006). While it holds that $\hat{\sigma}^2_{expost}$ is a consistent and unbiased estimator for the portfolio variance, it can also be shown that the $\lim_{t\to\infty} \rho = 0$ by applying a law of large numbers for weighted sums (Jamison et al., 1965).

On the contrary, *ex-ante* volatility is typically determined based only on a current (forward-looking) weight vector $\boldsymbol{\omega}_T = (\omega_{T,1}, \ldots, \omega_{T,N})'$, i.e. the current composition of the portfolio, and by assuming (or knowing) the true forward-looking variance-covariance matrix of the asset returns $\mathbf{\Sigma}$. Then the ex-ante variance of a portfolio at time T is given by

$$\sigma^2_{exante} = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_{T,i}\omega_{T,j}\sigma_{i,j} = \boldsymbol{\omega}_T'\mathbf{\Sigma}\boldsymbol{\omega}_T. \tag{7}$$

In other words, ex-ante volatility refers to a forward-looking estimation of a portfolio's volatility, considering the current portfolio positioning while anticipating covariances deemed representative for the specified forecasting period of the investment. However, the definition of ex-ante volatility in equation (7) also appears unconventional. The term *constant* portfolio/asset weights implies an ideal scenario where the portfolio is rebalanced flawlessly to its initial weighting at any point during the investment period is a very impractical assumption.

Therefore, it's crucial to recognize that the ex-ante volatility does not describe a *buy-and-hold* portfolio strategy. In reality, if assets are initially purchased and then held, their weights will naturally drift based on each asset's relative return compared to the portfolio return. The opposite of this is a continuously rebalanced portfolio.

Corollary 1. *The ex-post volatility of a portfolio given in equation (5) equals the ex-ante volatility in equation (7) if (i) portfolio weights are constant over time (continuous rebalancing) and (ii) the historical variance-covariance matrix is an unbiased estimator for the true (forward-looking) variance-covariance matrix.*

The proof of Corollary 1 is given in Appendix A.

Distributional considerations

Besides equivalence between the two variance measurements, one could also ask about the distributional properties of the ex-post and ex-ante variance. For this, let us assume that $\mathbf{r}_t \overset{i.i.d.}{\sim} \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for all t. Then it holds that the estimate of the covariance follows a Wishart distribution with scaling parameter $\boldsymbol{\Sigma}$ and degrees of freedom T, $\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}(\boldsymbol{\Sigma}, T-1)$, (Muirhead, 2009, Corollary 3.2.2). Given a portfolio allocation $\boldsymbol{\omega} \in \mathbb{R}^N$ such that $\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \neq 0$, the estimated ex-ante portfolio variance now follows a Chi-squared distribution with T degrees of freedom, i.e. $\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}/\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \sim \chi^2_T$ (Muirhead, 2009, Corollary 3.2.9).

The situation is somewhat more tricky for the ex-post variance as we need to derive distributional properties for a trace of a Wishart distribution. First of all, given that $\boldsymbol{\Omega}$ is non-random and has full column rank, it holds that $\boldsymbol{\Omega}\mathbf{S}\boldsymbol{\Omega}' \sim \mathcal{W}(\boldsymbol{\Omega}\boldsymbol{\Sigma}\boldsymbol{\Omega}', T)$, where $T > N-1$ is the degrees of freedom (Muirhead, 2009, Theorem 3.2.5). Kourouklis and Moschopoulos (1985) show some properties for the trace of a Wishart-distributed random matrix derived from the Normal distribution. A closed-form solution for the distribution of the trace only exists under specific circumstances (Pham-Gia et al., 2015) and Glueck and Muller (1998) show that the trace of a Wishart equals a weighted sum of non-central chi-squared random variables and constants. It holds from the definition of the chi-squared distribution that the sum of independent chi-squared variables is also chi-squared distributed with the degree of freedom is the sum of the degrees of freedom of the independent chi-squared variables. Hence, by construction, we would expect that the dispersion of the distribution of the ex-post variance is greater than the estimated ex-ante variance.

In the following simulation, we compare the distribution of the ex-post and ex-ante variances for two different portfolio strategies: A long-only risk-parity portfolio and a long-short momentum type portfolio.[2] Figure 2 shows the simulated distribution of monthly annualized portfolio volatility for different points in time, namely a low (30.03.2018) and a high (31.03.2020) volatility market environment.

Figure 2 to go here.

The top panel compares the distributions of the ex-ante and ex-post portfolio volatility for the long-only risk-parity portfolio. It shows that both distributions do not differ significantly, no matter the market environment. This is because the portfolio weights in the long-only case are more stable and the actual ex-ante and ex-post portfolio strategies are very similar in-sample. In contrast, the lower row portrays the same comparison for the long-short momentum portfolio and illustrates a considerably lower dispersion for the ex-ante volatility distribution. While this holds for all market environments, note that the ex-post distributions for the long-only as well as the long-short are very similar. These results fit the distributional considerations from above that the ex-post volatility has a wider dispersion than the ex-ante. All in all, these results are a strong indicator that long-short portfolio management based on ex-ante volatility might differ from an ex-post strategy.

3 Empirical application

3.1 Data

We use daily data for 55 futures markets, which cover a typical liquid multi-asset universe, see e.g. Pedersen et al. (2021). All data are retrieved from Bloomberg. The sample period is January 1980 to September 2023. The futures and some basic empirical properties are summarized in Table 1.

Table 1 to go here.

We compute arithmetic returns based on the rolled ratio-adjusted time series of prices. All futures contracts are rolled either on their last trading day or on the first day of the expiration month whichever is earlier. As futures are traded on margin accounts that earn interest, the market returns can be interpreted as excess returns. All returns are in local currency.

[2] The strategies and data will be introduced in detail in the empirical section 3.

At first, we inspect the correlations between all markets. To account for timezone discrepancies in Figure 3 we show the correlations of the monthly returns. The sector clusters can be easily identified.

Figure 3 to go here.

3.2 Setup

As shown above, if a portfolio is managed passively, i.e. $\omega_t \equiv \omega$, then ex-ante and ex-post volatilities coincide. Consequently, here we consider two standard actively managed portfolios:

1. Volatility-scaled long-only ('long'), e.g Harvey et al. (2018) and
2. Time series momentum ('TSMOM'), e.g Moskowitz et al. (2012).

Whilst the long-only portfolio simply allocates to each instrument inversely proportionally to a forecast of its volatility, for the latter a forecast of the future market returns based on past returns is required. In its simplest form Levine and Pedersen (2016) define a TSMOM signal as the past 1-year return, i.e.

$$s_{n,t}^{\text{plain mom}} = r_{n,t}^{1Y}. \tag{8}$$

As this signal is not comparable across different markets, a normalization is required. A generic implementation which divides the plain signal in (8) by an estimate of the volatility is suggested by Harvey et al. (2021), so that we define[3]

$$s_{n,t}^{\text{mom}} = \frac{r_{n,t}^{1Y}}{\hat{\sigma}_{n,t}^{1Y}}. \tag{9}$$

We now distinguish scaled and unscaled weights. The latter are the portfolio weights pre-gearing whilst the former are the portfolio weights that are intended to achieve the volatility target. For both investment strategies cases, the portfolio weights are obtained by scaling each market's signal by an estimate of its volatility, i.e.

$$\omega_{n,t}^{\text{unscaled}} = \frac{s_{n,t}}{\hat{\sigma}_{n,t}}. \tag{10}$$

[3] It should be noted that this definition includes the common normalization of using the sign function if $\hat{\sigma}_{n,t}^{1Y}$ is defined as $|r_{n,t}^{1Y}|$, see Moskowitz et al. (2012). The definition using a generic volatility estimate is favorable because using the sign function dictates to flip from a full long-position to a fully short and vice versa.

It should be noted that the volatility estimates in (9) and (10) serve different purposes and need not be the same. The former makes signals comparable whilst the latter translates this comparability to the risk that is taken in each instrument.[4] As these are volatilities of individual instruments, there is no distinction between ex-ante and ex-post volatility. For both we use exponentially weighted moving averages (ewma),

$$\hat{\sigma}^2_{n,t} = \lambda\, \hat{\sigma}^2_{n,t-1} + (1-\lambda)\, r^2_{n,t}. \tag{11}$$

The parameter $0 < \lambda < 1$ controls how much weight is given to the most recent information.

So far these portfolio weights are not designed to attain a target level of risk. For this, an estimate of the portfolio volatility is needed which could be either an ex-ante or an ex-post measure as described above. For the ex-post volatility estimate, we need the unscaled portfolio returns, i.e.

$$r_t^{\text{p, unscaled}} = \boldsymbol{\omega}_{t-1}^{\text{unscaled}\prime}\, \mathbf{r}_t. \tag{12}$$

For the ex-ante risk, an estimate of the market returns covariance matrix, $\boldsymbol{\Sigma}_t$, is required. Differences in the trading hours can cause spuriously low correlations. In practice, this has been solved by using overlapping average returns. Pedersen et al. (2021) suggest to estimate volatilities and the correlation matrix $\mathbf{C}$ separately, i.e. $\widehat{\boldsymbol{\Sigma}}_t = \widehat{\mathbf{C}}_t \circ \hat{\boldsymbol{\sigma}}_t \hat{\boldsymbol{\sigma}}_t'$, where $\circ$ denotes the (element-wise) Hadamard product.

Pedersen et al. (2021) suggest that volatility estimates are the center of mass of 60 days and the correlation estimates of 150 days.[5] The unified parameter entails the advantage of having comparable estimates across ex-ante and ex-post measures.[6] Also following again Pedersen et al. (2021), the correlation matrix is estimated using 3-day

[4]If momentum signals are defined as the (unscaled) return over the preceding year the signal magnitude depends on the volatility of the underlying instrument, e.g. a 10% return in New York Natural Gas is not five times as extreme as a 2% return in a Treasury future. Consequently, these need to be related to each instrument's volatility. A similar argument applies to positions. If, for example, the last year's return in the aforementioned Nat Gas future was one standard deviation and the same for the Treasury future, the cash allocated to each Nat Gas future would need to be about five times smaller than for the Treasury futures to make the equal signals to equal risk. In summary, a five times higher last year's return in Nat Gas needs to be mapped to a five times smaller position to deploy the correct amount of risk.

[5]The center of mass is defined as $com = \sum (1-\lambda)\, \lambda^i\, i$ and relates to the more common measure half-live hl as follows: $com = \sum_{i=1}^{\infty} (1-\lambda)\lambda^i i = \lambda/(1-\lambda)$ and $hl = -1/\log_2(\lambda)$.

[6]For ex-post only the volatility is estimated whilst for ex-ante the volatilities and correlations are estimated separately.

overlapping returns to mitigate time-zone differences across the futures. We use the same pre-averaging for the returns that are used for volatility estimates to ensure the comparability between ex-ante and ex-post estimates.

The desired target volatility for the portfolio $\sigma^{\text{target}} = 10\%$ p.a. is achieved by adjusting the unscaled weighted in (10) by the ratio of the target volatility and the current estimate of the portfolio volatility, i.e.

$$\boldsymbol{\omega}_t^{\text{scaled}} = \boldsymbol{\omega}_t^{\text{unscaled}} \frac{\sigma^{\text{target}}}{\hat{\sigma}_t}. \tag{13}$$

For the investment strategies above we compute the historical time series of portfolio weights both with the ex-ante and with the ex-post gearing methodology. Rather than merely considering performance, we are interested in the risk properties of the two methodologies. First, we test whether the volatility target is attained. To this end, we compute the corresponding portfolio returns and statistically test for the difference between the overall volatility and the volatility target. Second, we test how accurate value-at-risk (VaR) forecasts are. We test both parametric and nonparametric VaR forecasts. Given the time series of unscaled portfolio weights $\omega_t^{\text{unscaled}}$ we estimate ex-ante and ex-post volatilities as laid out above and forecast VaR as

$$VaR_t^{\text{ex ante, param}} = c_\alpha \cdot \hat{\sigma}_t^{\text{ex ante}} \quad \text{and} \tag{14}$$

$$VaR_t^{\text{ex post, param}} = c_\alpha \cdot \hat{\sigma}_t^{\text{ex post}}, \tag{15}$$

respectively, where c_α is the $(1-\alpha)$-percentile of the Normal distribution. For the nonparametric VaR, we use the percentiles of the realized returns. Similar to the definition of the ex-ante and ex-post volatility estimators the nonparametric VaRs are defined as

$$VaR_t^{\text{ex ante, non-param}} = \sum_{n=1}^{N} \omega_{n,t} Q_{1-\alpha}\left(\mathbf{r}_{n\,1:t}\right) \quad \text{and} \tag{16}$$

$$VaR_t^{\text{ex post, non-param}} = Q_{1-\alpha}\left(\mathbf{r}_{1:t}^{p}\right), \tag{17}$$

where $Q_{1-\alpha}$ denotes the $(1-\alpha)$-percentile of the empirical distribution. By $(\mathbf{r}_{n\,1:t}$ we denote the vector of market returns for the nth market from time 1 to t and similarly by $\mathbf{r}_{1:t}^{p}$ we denote the vector of (unscaled) portfolio returns until time t.
For all methods, we statistically test whether the frequency of breaching the forecasted

VaR deviates significantly from their respective expected values.

3.3 Results

A summary of the historical performance measures is shown in Table 3. The differences are generally small. For both the long-only and the momentum strategy, the mean performance is higher for the ex-ante portfolio. However, it also overshoots the target volatility significantly for the momentum case. On the other side, the ex-post volatility measured portfolio under-shoots the volatility target of ten percent p.a. for both strategies. While the Sharpe ratio is higher for the ex-ante volatility target strategies, the turnover is lower for the ex-post portfolio.

Table 3 to go here.

Table 4 reports the correlation between the ex-ante and ex-post returns, between the corresponding portfolio volatilities, and the correlation between the gearing factors for each portfolio, i.e. high correlation implies practically no difference between the ex-ante and ex-post portfolios. However, while we see almost no differences for the long-only portfolio, for which the correlations between all portfolio returns, volatilities, and gearing factors are above 95%, the picture looks different for the momentum strategy. Although the portfolio returns are also highly correlated with 0.97, this correlation drops when looking at the volatilities to 0.92 and even further to 0.81 for the gearing factors. This indicates that in terms of portfolio positioning, both strategies can differ significantly at times; underlining the importance of this research.

Table 4 to go here.

These dynamics could also be concluded for the time series plots in Figures 4 and 5. Here, volatility and gearing are displayed for the total sample and also specifically for the years 2007 and 2008, the Great Financial Crisis (GFC). It is apparent in both plots that the ex-post volatility and therefore the ex-post gearing factor are much less volatile than their ex-ante counterparts. This is due to the moving average effects in the ex-post estimation methods as each portfolio weight only differs slightly from the previous and so produces a more stable return distribution.

Figure 4 and Figure 5 to go here.

The unleveraged portfolio weights in Figure 6 for the long-only and momentum strategies eventually show the difference in portfolio characteristics. The short position and higher portfolio variability in the lower panel for the momentum signals should lead to more distinguishable results.

Figure 6 to go here.

Testing the realized portfolio volatilities for differences to their target, we find that, in all cases, the realized volatility is close to the target. The results are displayed in Table 5. Here, we can see that the ex-post geared portfolio undershoots the target significantly (at different significant levels for the two strategies). In contrast, the ex-ante portfolio overshoots the target volatility for the momentum case. Note that the fact that small deviations are statistically significant is owed to the daily frequency of the data which provides us with a large number of observations.

Figure 4 to go here.

For many investors, a key metric is value-at-risk (VaR). Here, we test how frequently the predicted VaR has been breached compared to the VaR level. The results for the parametric VaR are shown in Table 6. There are three columns for each ex-ante and ex-post VaR with the empirical frequency and the bootstrapped 5th and 95th percentiles as confidence bands for both the long-only and the momentum portfolios. First, the ex-ante portfolio overshoots the target risk only at the 1% level for the long-only portfolio but leads to a statistically significantly higher rate of VaR breaches at all levels for the momentum portfolio. The ex-post strategy, in contrast, is conservative for both long-only and momentum at the 5% level, with significantly lower rates of VaR breaches and only overshoots at the 1% level for the momentum portfolio.

Table 6 to go here.

The results for the non-parametric VaRs are shown in Table 7. The most interesting observation can be made for the ex-ante non-parametric VaR for the momentum portfolio. The VaR is breached at all levels more than 30% of the time, i.e. the forecasts tend to be substantially too low. This is caused by the property of the ex-ante estimate to only depend on the current weight. If the portfolio weights adapt according to the market behavior and there is a change in the market regime the estimate can be substantially off.

Table 7 to go here.

4 Conclusion

Portfolio risk can be estimated in various ways. The main distinction between the methodologies is whether only the current portfolio weights are used together with a historical estimate of the market returns covariance matrix or whether each market return is matched with its corresponding historical weight. We find that, in many cases, the differences are small and the impact of key performance metrics is unaffected. However, in some cases where the portfolio strategy depends on the market regime the ex-ante risk measure can be misleading as it does not account for the change in the portfolio weights. This affects primarily value-at-risk forecasts, especially for the non-parametric or historical simulation version. As this version is often used for regulatory purposes in risk monitoring, this paper gives valuable insights into the question of which portfolios such a measure is not suitable.

In addition, this paper also studies the theoretical properties of both volatility measures, derives an applicable formula for the ex-post volatility, and proves the differences to the ex-ante volatility in a simulation study.

Finally, we can also note that volatility target portfolios with dynamic levels of gearing or leverage are especially prone to errors in risk estimation and should be managed with the right metrics. Hence, our study also relates to other non-momentum and overlay portfolio strategies which not only invest in future contracts but also in other asset classes in which transaction costs can be sustainably higher. For such portfolios, the difference in performance can even be bigger and more costly.

A Proofs

Proof of Proposition 1. Using the definition of $\mathbf{\Omega}$ from equation (4), we can write the portfolio variance in equation (3) in matrix notation as

$$
\begin{aligned}
\hat{\sigma}^2_{expost} &= \frac{1}{T-1}\sum_{t=1}^{T} \underbrace{\left(\sum_{n=1}^{N} \omega_{t-1,n}\left(r_{t,n}-\bar{r}_n\right)\right)^2}_{=\left(\omega_{t-1,1},\ldots,\omega_{t-1,N}\right)'\left(r_{t,1}-\bar{r}_1,\ldots,r_{t,N}-\bar{r}_N\right)} \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\left(\boldsymbol{\omega}'_{t-1}\tilde{\mathbf{r}}_t\right)^2 = \frac{1}{T-1}\sum_{t=1}^{T}\boldsymbol{\omega}'_{t-1}\tilde{\mathbf{r}}_t\tilde{\mathbf{r}}'_t\boldsymbol{\omega}_{t-1}, \quad \text{where } \tilde{\mathbf{r}}_t = \mathbf{r}_t - \bar{\mathbf{r}} \\
&= \frac{1}{T-1}\left[\boldsymbol{\omega}'_0,\ldots,\boldsymbol{\omega}'_{T-1}\right]\cdot\begin{bmatrix}\tilde{\mathbf{r}}_1\tilde{\mathbf{r}}'_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}'\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{\omega}_0\\ \vdots \\ \boldsymbol{\omega}_{T-1}\end{bmatrix} \\
&= \frac{1}{T-1}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{r}}_1\tilde{\mathbf{r}}'_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}'\end{bmatrix}\cdot\operatorname{vec}\left(\mathbf{\Omega}\right) \\
&= \frac{1}{T-1}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T\end{bmatrix}\cdot\operatorname{vec}\left(\mathbf{\Omega}\right) \\
&\quad + \frac{1}{T-1}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{r}}_1\tilde{\mathbf{r}}'_1-\tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}'-\tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T\end{bmatrix}\cdot\operatorname{vec}\left(\mathbf{\Omega}\right) \\
&= \frac{1}{T-1}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T\end{bmatrix}\cdot\operatorname{vec}\left(\mathbf{\Omega}\right) + \rho \\
&= \frac{1}{T-1}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\left(\mathbf{I}_T\otimes\left(\tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T\right)\right)\cdot\operatorname{vec}\left(\mathbf{\Omega}\right) + \rho \\
&\overset{*}{=} \frac{1}{T-1}\frac{1}{T}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\operatorname{vec}\left(\tilde{\mathbf{R}}'\tilde{\mathbf{R}}\cdot\mathbf{\Omega}\right) + \rho = \frac{1}{T}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\operatorname{vec}\left(\frac{\tilde{\mathbf{R}}'\tilde{\mathbf{R}}}{T-1}\cdot\mathbf{\Omega}\right) + \rho \\
&= \frac{1}{T}\operatorname{vec}\left(\mathbf{\Omega}\right)'\cdot\operatorname{vec}\left(\hat{\mathbf{\Sigma}}\cdot\mathbf{\Omega}\right) + \rho = \frac{1}{T}\operatorname{tr}\left(\mathbf{\Omega}'\hat{\mathbf{\Sigma}}\mathbf{\Omega}\right) + \rho
\end{aligned} \tag{18}
$$

where $\mathbf{I}_T$ is the identify matrix of dimension T, $\otimes$ denotes the Kronecker product, $\boldsymbol{\iota} = (1, \dots, 1)'$ is a vector of ones with length T and tr the trace of a (square) matrix. $*$ is the result of $\text{vec}\,(A \cdot B) = (\mathbf{I}_T \otimes A) \cdot \text{vec}\,(B)$. □

Proof of Corollary 1. For (ii) we note that the true forward-looking variance-covariance matrix $\boldsymbol{\Sigma}$ is usually unknown, so we can only estimate the ex-ante variance of a portfolio by replacing $\boldsymbol{\Sigma}$ in equation (7) by the unbiased estimator $\hat{\boldsymbol{\Sigma}} = \frac{1}{T-1}\tilde{\mathbf{R}}'\tilde{\mathbf{R}}$. Then

$$\begin{aligned}
\hat{\sigma}^2_{exante} &= \frac{1}{T-1}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\omega_{T-1,n}\left(r_{t,n}-\bar{r}_n\right)\right)^2 \qquad \Big|\omega_{t,n}=\omega_{s,n}=\omega_{T-1,n}\,\forall t\neq s \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\left(\boldsymbol{\omega}'_{T-1}\tilde{\mathbf{r}}_t\right)^2 = \boldsymbol{\omega}'_{T-1}\left(\frac{1}{T-1}\sum_{t=1}^{T}\tilde{\mathbf{r}}_t\tilde{\mathbf{r}}'_t\right)\boldsymbol{\omega}_{T-1} \\
&= \boldsymbol{\omega}'_{T-1}\left(\frac{1}{T-1}\tilde{\mathbf{R}}'\tilde{\mathbf{R}}\right)\boldsymbol{\omega}_{T-1} = \boldsymbol{\omega}'_{T-1}\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}_{T-1}.
\end{aligned} \tag{19}$$

As $\hat{\boldsymbol{\Sigma}}$ is unbiased, so is the ex-ante estimate: $\mathrm{E}\left(\hat{\sigma}^2_{exante}\right) = \sigma^2_{exante}$. Rewriting $\boldsymbol{\Omega} = \boldsymbol{\omega}\cdot\boldsymbol{\iota}'$ from equation (4) for $\boldsymbol{\omega}_s = \boldsymbol{\omega}_t = \boldsymbol{\omega}\,\forall t \neq s$, $\hat{\sigma}^2_{expost}$ in (5) can be rewritten as

$$\begin{aligned}
\hat{\sigma}^2_{expost} &= \frac{1}{T}\,\mathrm{tr}\left(\boldsymbol{\Omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\Omega}\right) + \rho = \frac{1}{T}\,\mathrm{tr}\left(\boldsymbol{\iota}\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}\boldsymbol{\iota}'\right) + \rho \\
&= \frac{1}{T}\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}\,\mathrm{tr}\left(\boldsymbol{\iota}\boldsymbol{\iota}'\right) + \rho = \frac{1}{T}\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}\cdot T + \rho \\
&= \boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega} + \rho.
\end{aligned} \tag{20}$$

Now we can focus on ρ and find that

$$\begin{aligned}
\rho &= \frac{1}{T-1}\mathrm{vec}\left(\boldsymbol{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{r}}_1\tilde{\mathbf{r}}'_1 - \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}' - \tilde{\mathbf{R}}'\tilde{\mathbf{R}}/T\end{bmatrix}\cdot\mathrm{vec}\left(\boldsymbol{\Omega}\right) \\
&= \underbrace{\frac{1}{T-1}\mathrm{vec}\left(\boldsymbol{\Omega}\right)'\cdot\begin{bmatrix}\tilde{\mathbf{r}}_1\tilde{\mathbf{r}}'_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbf{r}}_T\tilde{\mathbf{r}}_T{}'\end{bmatrix}\cdot\mathrm{vec}\left(\boldsymbol{\Omega}\right)}_{=\frac{1}{T-1}\sum_{t=1}^{T}\boldsymbol{\omega}'\tilde{\mathbf{r}}_t\tilde{\mathbf{r}}_t{}'\boldsymbol{\omega}=\boldsymbol{\omega}'\left(\frac{1}{T-1}\sum_{t=1}^{T}\tilde{\mathbf{r}}_t\tilde{\mathbf{r}}_t{}'\right)\boldsymbol{\omega}=\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}} - \underbrace{\frac{1}{T}\,\mathrm{tr}\left(\boldsymbol{\Omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\Omega}\right)}_{=\boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega}} \\
&= \boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega} - \boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega} = 0.
\end{aligned} \tag{21}$$

Hence, it holds that $\hat{\sigma}^2_{expost} = \boldsymbol{\omega}'\hat{\boldsymbol{\Sigma}}\boldsymbol{\omega} + 0 = \hat{\sigma}^2_{exante}$. This completes the proof. □

B Figures

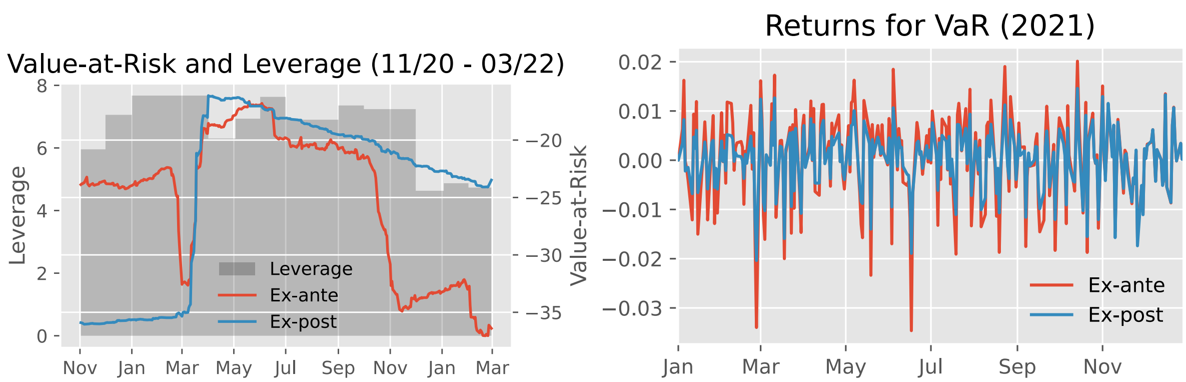

Figure 1: Portfolio Value-at-Risk measures and leverage

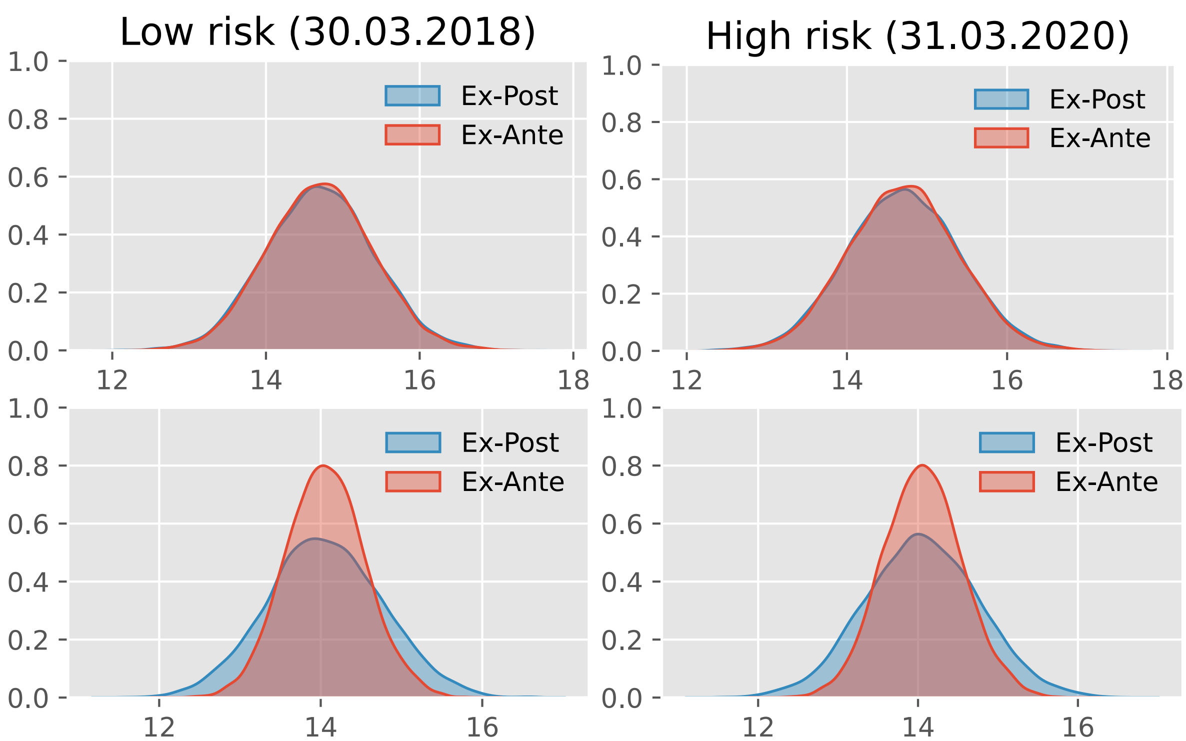

Figure 2: Distribution of annualized portfolio volatilities

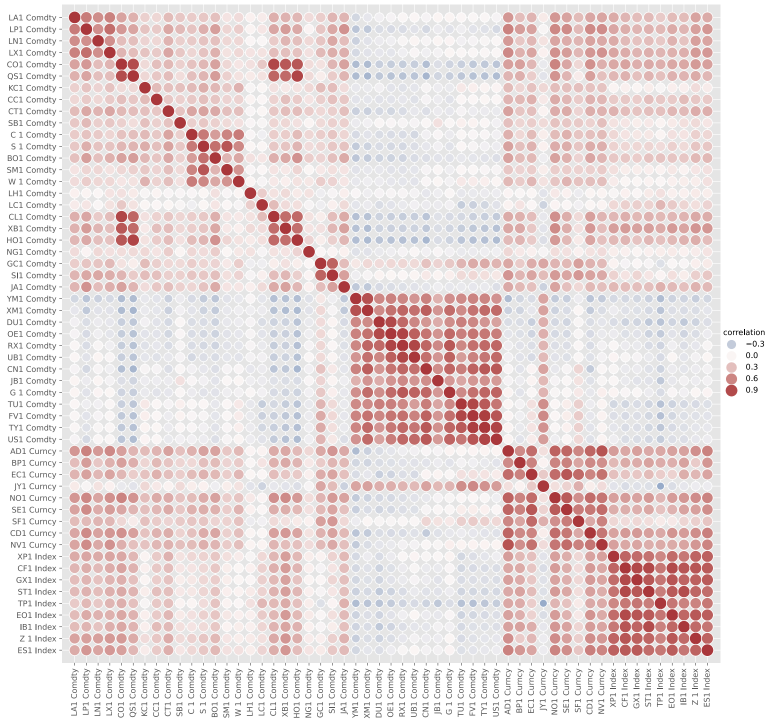

Figure 3: Correlations of monthly future returns

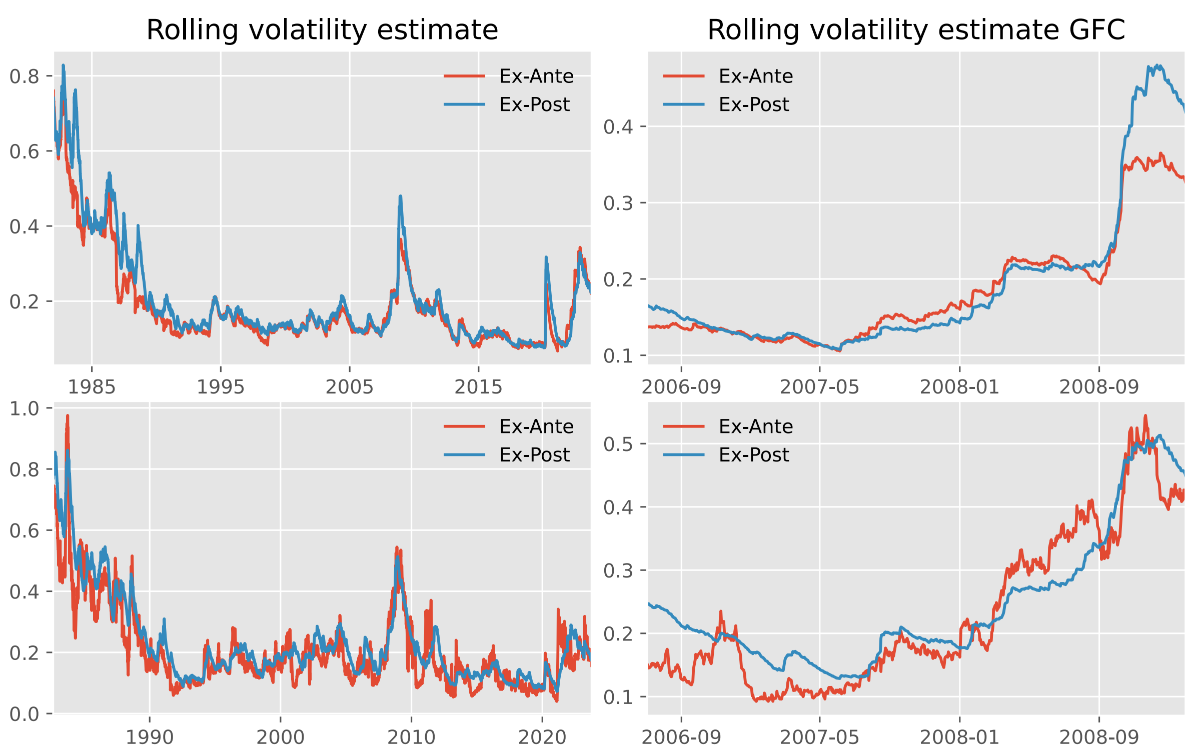

Figure 4: Rolling volatility estimate

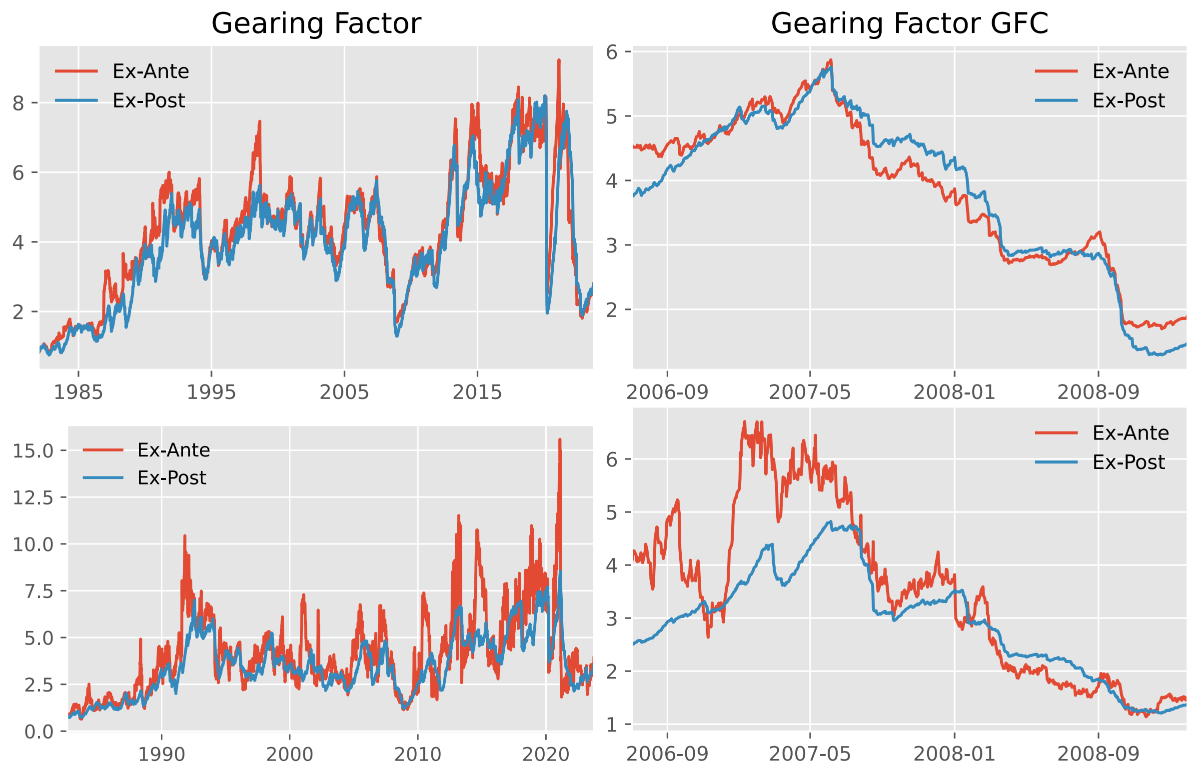

Figure 5: Portfolio Gearing Factor

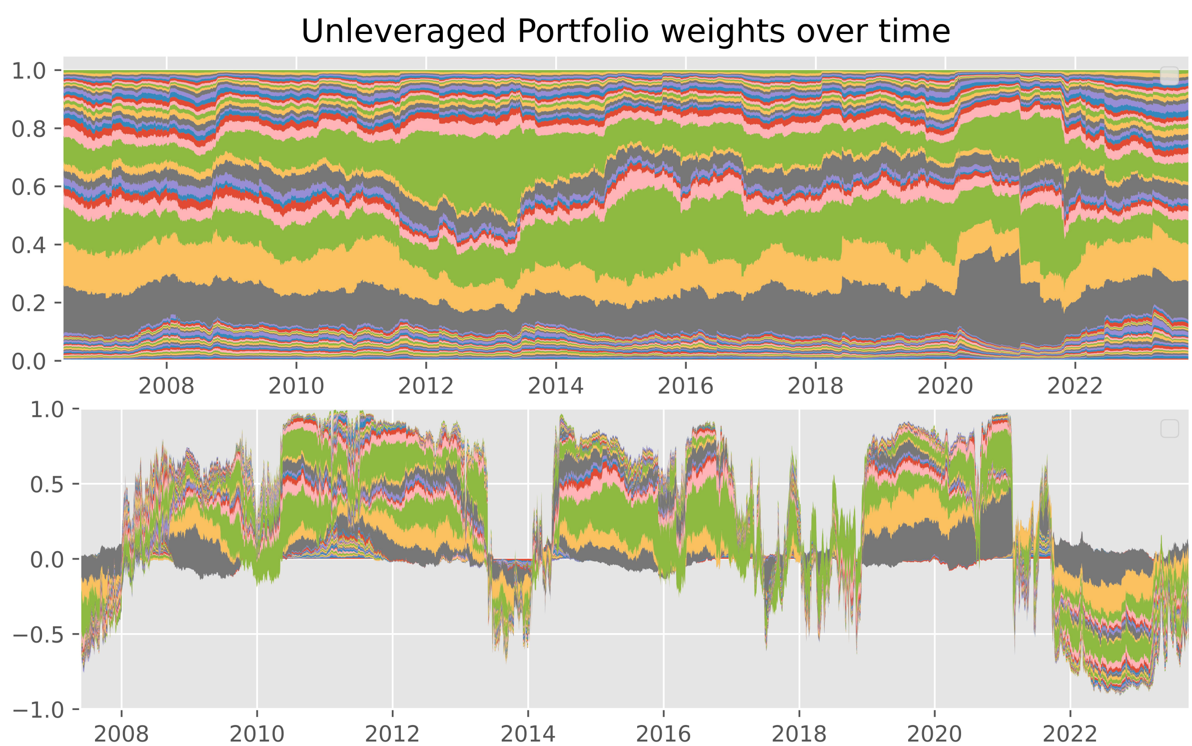

Figure 6: Portfolio weights over time

C Tables

Table 1: Summary of futures markets (equity, bonds, and currencies).

Ticker	Name	Sector	μ	σ	Start date
XP1 Index	SFE S&P ASX Price Index	Equities	5.3	16.4	2000-05-03
CF1 Index	Euronext CAC 40 Index Future	Equities	6.7	21.7	1988-12-08
GX1 Index	Eurex DAX Index Future	Equities	7.3	22.2	1990-11-26
ST1 Index	FTSE/MIB Index Future	Equities	6.0	23.3	2004-03-23
TP1 Index	TSE TOPIX Tokyo Price Index	Equities	2.9	22.2	1990-05-17
EO1 Index	Euronext Amsterdam Index	Equities	7.9	20.7	1989-01-03
IB1 Index	MEFF Madrid IBEX 35 Index	Equities	8.0	23.0	1992-07-21
Z 1 Index	FTSE 100 Index Future	Equities	5.0	17.8	1988-02-29
ES1 Index	S&P 500 mini-Future	Equities	7.1	19.7	1997-09-10
YM1 Comdty	SFE 3 Year Australian Bond	Bonds	0.6	1.2	1989-12-11
XM1 Comdty	SFE 10 Year Australian Bond	Bonds	0.4	1.2	1987-09-21
DU1 Comdty	Eurex 2 Year Euro SCHATZ	Bonds	0.5	1.3	1997-03-10
OE1 Comdty	Eurex 5 Year Euro BOBL	Bonds	2.0	3.3	1991-10-07
RX1 Comdty	Eurex 10 Year Euro BUND	Bonds	3.2	5.7	1990-11-26
UB1 Comdty	Eurex 30 Year Euro BUXL	Bonds	3.1	12.3	1998-10-05
CN1 Comdty	Montreal Exchange 10 Yr. Fut.	Bonds	2.8	6.2	1989-09-18
JB1 Comdty	JGB Future	Bonds	2.6	4.5	1985-10-21
G 1 Comdty	Long Gilt Future	Bonds	2.1	7.7	1982-11-19
TU1 Comdty	CBOT 2 Yr. US Treasury Note	Bonds	1.0	1.6	1990-06-26
FV1 Comdty	CBOT 5 Yr. US Treasury Note	Bonds	2.1	4.0	1988-05-23
TY1 Comdty	CBOT 10 Yr. US Treasury Note	Bonds	3.7	6.6	1982-05-04
US1 Comdty	CBOT US Treasure Bond Fut.	Bonds	3.8	11.2	1980-01-01
AD1 Curncy	CME Australian Dollar	Curncy	2.7	11.6	1987-01-13
BP1 Curncy	CME British Pound	Curncy	1.1	9.8	1986-05-28
EC1 Curncy	CME Euro Foreign Exchange	Curncy	-0.6	9.4	1998-05-20
JY1 Curncy	CME Japanese Yen	Curncy	-1.4	10.7	1986-05-23
NO1 Curncy	CME Norwegian Krone	Curncy	0.1	12.6	2002-05-17
SE1 Curncy	CME Swedish Krona	Curncy	-0.1	11.9	2002-05-17
SF1 Curncy	CME Swiss Franc	Curncy	1.1	11.3	1986-04-07
CD1 Curncy	CME Canadian Dollar	Curncy	0.9	7.5	1986-04-04
NV1 Curncy	CME New Zealand Dollar	Curncy	2.3	12.7	1997-05-08

Notes: All statistics are computed over the respective full sample. μ is the annualized mean return of each future and σ is the volatility in percent. Data source: Bloomberg.

Table 2: Summary of futures markets (commodities).

Ticker	Name	Sector	μ	σ	Start date
QS1 Comdty	ICE Gas Oil Future	Agris.	14.6	33.9	1989-07-04
KC1 Comdty	NYBOT CSC C Coffee Future	Agris	-1.6	34.6	1980-01-01
CC1 Comdty	NYBOT CSC Cocoa Future	Agris	-2.9	28.7	1980-01-01
CT1 Comdty	NYBOT CTN Nr. 2 Cotton Fu	Agris	0.9	24.2	1980-01-01
SB1 Comdty	NYBOT CSC Nr. 11 World Su	Agris	3.0	36.1	1980-01-01
C 1 Comdty	CBOT Corn Future	Agris	-3.4	23.4	1980-01-01
S 1 Comdty	CBOT Soybean Future	Agris	3.7	21.9	1980-01-01
BO1 Comdty	CBOT Soybean Oil Future	Agris	0.2	23.6	1980-01-01
SM1 Comdty	CBOT Soybean Meal Future	Agris	8.2	24.0	1980-01-01
W 1 Comdty	CBOT Wheat Future	Agris	-5.6	26.8	1980-01-01
LH1 Comdty	CME Lean Hogs Future	Agris	3.6	24.7	1986-04-02
LC1 Comdty	CME Live Cattle Future	Agris	3.3	15.1	1980-01-01
LA1 Comdty	LME Primary Aluminum Future	Metals	-0.9	21.2	1997-07-24
LP1 Comdty	LME Copper Future	Metals	7.5	25.0	1997-06-30
LN1 Comdty	LME Nickel Future	Metals	11.5	38.1	1997-07-24
LX1 Comdty	LME Zinc Future	Metals	3.0	28.5	1997-07-24
GC1 Comdty	COMEX Gold 100 Troy Oz.	Metals	0.1	18.3	1980-01-01
SI1 Comdty	COMEX Silver Future	Metals	-1.1	30.1	1980-01-01
JA1 Comdty	OSE Platinum Future	Metals	4.0	23.5	1984-01-27
CO1 Comdty	ICE Brent Crude Oil Future	Energy	14.6	35.5	1988-06-24
CL1 Comdty	NYMEX Light Sweet Crude Oil	Energy	10.0	38.3	1983-03-31
XB1 Comdty	NYMEX Reformulated Gasoline	Energy	15.6	39.9	2005-10-04
HO1 Comdty	NYMEX NY Harbor ULSD Fut.	Energy	16.3	35.3	1986-07-01
NG1 Comdty	NYMEX Natural Gas	Energy	-8.1	50.9	1990-04-04

Notes: All statistics are computed over the respective full sample. μ is the annualized mean return of each future and σ is the volatility in percent. Data source: Bloomberg.

Table 3: Portfolio performance for long-only and TSMOM strategy.

Measure	Long ex-ante	Long ex-post	Momentum ex-ante	Momentum ex-post
Return (%)	6.91	6.08	12.72	10.57
Volatility (%)	10.04	9.34	10.69	9.85
Sharpe	0.69	0.65	1.19	1.07
MDD (%)	-27.32	-26.32	-20.67	-21.88
Turnover	12.18	11.41	71.86	63.50

Note: The table reports annualized mean results for portfolio returns. 'MDD' denotes the maximum drawdown.

Table 4: Correlation for long and momentum strategy for future asset returns

	Returns	Volatility	Gearing
Long	0.99	0.97	0.95
Momentum	0.97	0.92	0.81

Note: The table reports the overall correlation between ex-ante and ex-post portfolio returns volatilities, and gearing factors for each strategy.

Table 5: Portfolio volatilities for ex-ante and ex-post risk methods.

Strategy	risk method	vol (%)	p-value	5% conf (%)	95% conf (%)
long	ex-ante	10.04	68.96	9.88	10.20
	ex-post	9.34	0.00	9.18	9.48
momentum	ex-ante	10.69	0.00	10.53	10.86
	ex-post	9.85	14.40	9.68	10.02

Table 6: Testing parametric Value-at-Risk for long and momentum future strategy.

Strategy	Level	ex-ante	5% conf	95% conf	ex-post	5% conf	95% conf
long	1.0%	1.44	1.26	1.63	1.11	0.95	1.27
	2.5%	2.56	2.32	2.80	2.13	1.91	2.36
	5.0%	4.87	4.54	5.21	3.91	3.61	4.22
momentum	1.0%	2.13	1.90	2.37	1.71	1.51	1.92
	2.5%	3.59	3.29	3.88	2.70	2.44	2.95
	5.0%	5.43	5.07	5.80	4.55	4.21	4.88

Table 7: Testing non-parametric Value-at-Risk for long and momentum future strategy

Strategy	Level	ex-ante	p-val	5%	95%	ex-post	p-val	5%	95%
long	1.0%	0.02	0.0	0.00	0.05	1.41	0.00	1.24	1.60
	2.5%	0.04	0.0	0.01	0.07	2.95	0.40	2.68	3.21
	5.0%	0.09	0.0	0.05	0.14	5.47	3.44	5.09	5.82
momentum	1.0%	28.89	0.0	28.17	29.61	1.50	0.00	1.31	1.70
	2.5%	29.59	0.0	28.86	30.34	3.01	0.48	2.74	3.28
	5.0%	30.84	0.0	30.10	31.58	5.49	2.88	5.13	5.84

Note: Non-Parametric VaR: rolling 260d percentile weighted by current weights (ex-ante) and rolling 260d percentile for portfolio returns (ex-post).

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