MINIMAL DYNAMIC EQUILIBRIA

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Abstract. We demonstrate that prevalent empirical implementations of asset pricing are inconsistent with dynamic equilibria (multiperiod equilibria with no static representations). Specifically, empirical implementations are misspecified with respect to three essential asset pricing questions (TEQ): dependency on higher moments, complexity of risk premia, and mean-variance efficiency of the "market portfolio" (the pricing kernel/SDF). While we already know that "Merton models" and their derivatives differ from static models in all TEQ, we show that this is the case even for Minimal Dynamic Equilibria (dynamic equilibria with the simplest structure).

JEL Codes: G12, G11

Key Words: Minimal, Dynamic, Equilibrium, Higher Moments, Risk Premium, Pricing Kernel, SDF

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1 Introduction

Current asset pricing literature stands on two "legs," static and dynamic. We call models static if they are either single-period or multiperiod with a single-period representation, that is, in each and every period the analysis becomes a single-period one.[1] We call models "dynamic" if they are multiperiod with no single-period representation.

Representatives of the first leg include Markowitz (1952), Sharpe (1963, 1964)-Lintner (1965)-Mossin (1966) single-period CAPM, and multifactor extensions [e.g., Fama and French (1992, 2015)], which, in fact, are single-period linear beta pricing models.[2] Representatives of the second leg include Samuelson (1969), Merton (1971, 1973), Lucas (1980), Cox, Ingersoll, and Ross (1985a,b), Epstein and Zin (1989), Epstein (2001), and Hansen and Sargent (2001). For brevity, we call the latter multiperiod models and their derivatives "Merton models" (MM). MM are multiperiod models with stochastic investment opportunities and with, potentially, exchange, production, capital markets, intermediate consumption, incomplete information, ambiguity, and model uncertainty.[3]

In the context of these two approaches to asset pricing, three essential questions (TEQ) arise:

i. Does the analysis map into a mean-variance (MV) one? Alternatively, is there no dependency on moments higher than mean and variance?

ii. Are risk premia (expected returns[4] in excess of the riskless rate) "simple"? We call risk

[1] The latter are multiperiod models that map (degenerate) into single-period ones, or sequences of these, due to, for example, path independence, dependency on final outcomes only, Martingale representation methods, myopic preferences (including logarithmic preferences, risk neutrality), or periodically independent returns. See Feldman (1992) findings on multiperiod equilibria with optimal myopic decisions.
[2] See also Merton (1972), Black (1972), Roll and Ross (1995), Kandel and Stambaugh (1995), Jagannathan and Wang (1996), Feldman and Reisman (2003), Bick (2004), Ukhov (2006), and Diacogiannis and Feldman (2013).
[3] See also Kreps and Porteus (1978), Dothan and Feldman (1986), Detemple (1986), David (1997), Feldman (2007), Björk, Davis and Landén (2010), Leisen (2016), and Leisen (2018).
[4] For brevity and simplicity, we will use the term "returns" also for "rates of return."

premia "simple" if they are similar to single-period ones, and "complex" if they are similar to those in MM. The latter include additional term(s), for example, terms relating to intertemporal rates of substitution.[5]

iii. Is the pricing kernel/stochastic discount factor (SDF)/market portfolio MV efficient?

The literature characterizes differences in answering these TEQ for static models and only a *subset* of "dynamic" models, the MM. Thus, the disparity/overlap in characterizations between static and other dynamic models, with respect to the TEQ, has not been fully explored. We know that the answers to TEQ for static models are *"yes," "yes,"* and *"yes,"* and for MM are *"no," "no,"* and *"no."* In this paper, we ask whether there exist dynamic models with answers to the TEQ that are different from the answers to MM, thus, more similar to the answers for static models.

Another way to describe the lacuna in the literature is as follows. We know that single-period representations of multiperiod models are sufficient for answering the TEQ with yes, yes, and yes, but the literature has not addressed the question if such a representation of a multiperiod model as a single-period one is necessary for answering the three TEQ affirmatively. We investigate the possibility of a dynamic model (with no single period representation) for which the answers to the TEQ are yes, yes, and yes.

Intuitively, if there exist dynamic models with answers to the TEQ that are different from those of MM and similar to those of static models, they are likely to be among the simplest ones. Thus, we set, as our first objective, to identify a minimal dynamic equilibrium (MDE). Specifically, an MDE is a dynamic model with the simplest structure in terms of number of periods, endowments, risk/stochastic structure, information structure, capital market, and plausible

[5] These are optimal demands induced by stochastic changes in future investment opportunities, which Merton called "hedging demands." Additional "hedging" terms may exist, hedging dynamic precisions of unobservable variables, for example.

preferences. Our second objective is to answer the TEQ with respect to the MDE. Achieving both objectives, we are able to answer the question: Is there an MDE similar to static models in answering the TEQ? Finally, we examine the robustness of our MDE identification with respect to answering the TEQ by modifying it along all relevant directions/dimensions/attributes.

Our results are as follows. We identify an MDE that is minimal in all attributes. It has MV risk-averse representative investors who maximize, over two periods, arithmetic mean returns (possibly of elliptical distribution functions) of investments in numerous risky assets.[6]

The answer to the first TEQ is ***no***: we find that in our MDE there is no riddance of the dependency on higher moments. That is, moments higher than variance do play a prominent role. The relevance of higher moments in risk premia was documented empirically [Harvey and Siddique (2000) and Dittmar (2002), for example]. Some single-period equilibrium models address this issue by defining preferences over higher moments [e.g., Kraus and Litzenberger (1976), Chabi-Yo (2012), and Chabi-Yo, Leisen and Renault (2014)]. Our findings demonstrate, however, that the role of the higher moments in forming equilibrium demands and prices in the face of stochastic investment opportunities is so natural that even within an MDE, they conspicuously appear under MV preferences and elliptical return distributions.

In this context, perhaps it is important to note the danger of misinterpreting the dependency on *only* instantaneous first two moments in continuous time formulations to be like the dependency on two moments in the static case. We must recognize the tradeoff between time and space in the continuous time case. The choice of different functions that instantaneous continuous time first two moments can assume leads to inducing distributions with different specifications of higher

[6] Our analysis is in terms of returns in excess of the riskless rate.

moments, over any finite time interval.[7]

The answer to the second TEQ is ***no***. We find that even MDE risk premia are not simple. They include a term, additional to the one in static models, which depends on the covariance between *prevailing* returns and *future* investment opportunities.

The answer to the third TEQ is ***no***. We find that market portfolios are generally not MV efficient, thus, cannot serve as SDFs.

Furthermore, perhaps an unexpected finding, we identify future market return's variance as a priced factor and a component of the prevailing SDF. This result has been confirmed empirically [see Chabi-Yo (2012)].

We offer insights into our results. The first insight is that while, in general, dynamic equilibria offer a continuum of tradeoffs between income effects and substitution effects where either or neither[8] effect dominates, under our MDE there is only a single such tradeoff within which the substitution effect dominates.

The second insight is that square Sharpe ratios sufficiently characterize future (stochastic) investment opportunities[9] with a "dimension" of square returns.[10] Moreover, covariances between returns and future investment opportunities shape risk premia with a dimension of cubic returns. This, in turn, induces a dependency on higher moments, elaborate risk premia, and MV inefficient market portfolios.

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[7] In Vasicek (1977), for example, over any finite time interval, a common factor is normally distributed. In contrast, in Cox, Ingersoll and Ross (1985b), prices/outputs could, conditionally, have a log normal distribution, and productivity factors a non-central chi-squared one.

[8] No dominating effect is only in the knife-edge case of logarithmic preferences that induce a single unit level of Arrow-Pratt's relative risk aversion (RRA). We rule out the logarithmic case from our MDE choices (please see below) because it induces equilibria that are iid repetitions of single-period equilibria. See, for example, Mossin (1968), Hakansson (1970), and Feldman (1992).

[9] Liu (2007) studies the case where investment opportunities are characterized by Sharpe ratios in a continuous time framework.

[10] For simplicity we use the term "dimension" to describe quadratic and cubic rates of returns, though rates of returns are unitless.

The third insight is our identification of an equilibrium relation between date 2 market expected returns and market return variance, scaled by the market risk-aversion level. This pricing is foreseen by, and has implications for, date 1 demands and prices, implying date 2 market volatility is ex-ante priced in date 1. An increase (decrease) in the covariance between prevailing returns and future investment opportunities results in an increase (decrease) in prevailing expected returns as a pricing adjustment to added (reduced) risk component.

While identifying the MDE, we looked for the simplest dynamic equilibrium; we may also characterize our MDE as a minimal extension of linear beta pricing models (such as the CAPM). However, we conjecture that our MDE choice could be considered as a *first* natural choice even over a larger set of commonly used asset pricing models (including overlapping generations, for example).

Finally, Figure 1 identifies seven spaces that are collectively comprehensive but not mutually exclusive. One large space represents static models. The second large space represents dynamic models. The third represents the overlap between these two spaces, which represents dynamic models with static representation. The fourth (fifth) space (red text) represents MM with (no) static representation. The sixth (seventh, blue text, filled) space represents dynamic models which are not MM with (no) static representation.

For the purpose of our analysis here, we consider the space of dynamic models with static representations to be part of the static model's space, and we do not include it in our analysis. The objective of this paper is to answer the TEQ for the models in the latter (seventh) subspace.

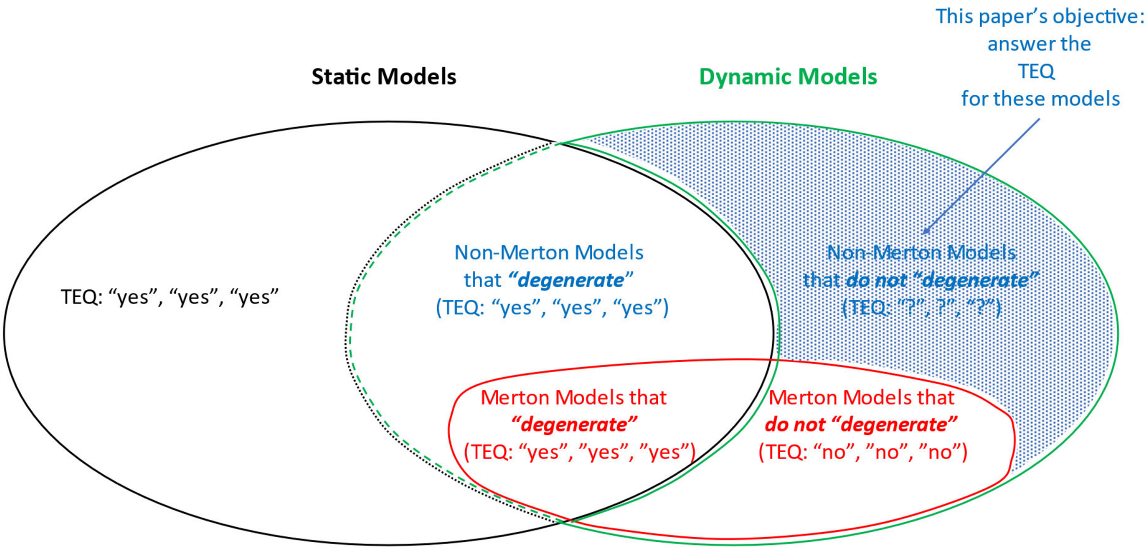

Figure 1. An illustration of model decomposition into subspaces. From the left, 1. static (single period) models, 2. dynamic models, 3. the intersection of the previous two spaces, 4. MM with static representation (red text), 5. MM models with no static representation (red text), 6. non-MM models with static representation (blue text), and 7. non-MM models with no static representation (blue text, filled). We already know the answer to all TEQ for the models in the four former subspaces in the figure. The objective of this paper is to answer the TEQ for the models in the latter (seventh) subspace.

We now point out empirical misspecifications created when applying analysis befitting static models to models with negative answers to TEQ. If the answer to the first TEQ is negative, i.e., demands and prices are functions of higher moments, then ignoring these higher moments results in missing, for example, the contribution of skewness to risk premia, a contribution that has been demonstrated empirically by, for example, Harvey and Siddique (2000). If the answer to the second TEQ is negative, i.e., risk premia are complex, then ignoring the intertemporal component of risk premia, results in demands and prices misspecification, demonstrated empirically by, for example, Bansal and Yaron (2004). If the answer to the third TEQ is negative, i.e., SDF/market portfolio is not MV efficient, then all values of R^2 (including zero) may be obtained "legitimately," rendering the empirical pricing irrelevant. See for example, Roll and Ross (1994), Kandel and

Stambaugh (1995), Jagannathan and Wang (1996), and Diacogiannis and Feldman (2013).

Section 2 identifies the MDE; Section 3 analyzes and characterizes the MDE; Section 4 answers the TEQ; Section 5 discusses the MDE's relevance and implications, including its robustness with respect to answering the TEQ; Section 6 discusses empirical implications; and Section 7 concludes. Appendix A has the mathematical proofs; and Appendix B discusses the MDE choice.

2 MDE Identification

Our first objective is to identify and specify attributes of a model with minimal structure. We begin by introducing concepts of dynamic models, minimal dynamic models, and minimal dynamic equilibria.

Definition. *Dynamic model (DM)*. A multiperiod model with no single-period representation. □

Definition. *Minimal dynamic model (MDM)*. A dynamic model which has the property that changing any aspect of it makes it more complex.[11][12] □

Definition. *Minimal Dynamic Equilibrium (MDE).* Equilibrium in a minimal dynamic model. □

2.1 Characterizing MDM

We will consider all relevant aspects of DM, one by one, and identify their specifications of minimal complexity. It seems that the more similar the specifications are to those of single-period models, such as Markowitz world models and the classical CAPM, the more likely are the

[11] This is an approach that combines relevance with simplicity. The reader may be tempted to define a rigorous mathematical concept of "minimal" model, in the sense of a minimization across a set of models. But we note that such an approach is futile as there is no objective ordering across model characteristics. See our discussion in Appendix B.

[12] The relevant "aspects" in our case and specification of "complexity," are below. This situation might be similar to the case in which US Supreme Court Justice Potter Stewart found that while a concept is hard to define, "I know it when I see it." Jacobellis v. Ohio (1964).

specifications to capture the single period models' properties.[13]

Agents (preferences, endowments, information) are identical.

Preferences. Mean-variance (MV) preferences. MV preferences seem to be the natural choice for the simplest preferences. They seem to least deviate from (static) single period preferences, arising from Markowitz world models and the classical CAPM.

Moreover, a common way to look at expected utility preferences is to carry out Taylor series expansion. In that decomposition, MV preferences come up as the simplest structure.

Preferences that specify intertemporal risk-aversion measures different from cross-sectional ones (e.g., Kreps Porteus, Epstein-Zin–Weil, stochastic differential utilities) are more complex. Preferences that are path-dependent (e.g., habit, non-time-additive) are more complex. Hence, our model aims at the minimal complexity in terms of preferences, and we opt for MV preferences.

With MV preferences, as wealth increases, Arrow-Pratt's absolute risk-aversion measure (ARA) increases as well. This is a property that describes none of us. Thus, in MDM, the MV preferences cannot be related to wealth. Hence, we define preferences for returns rather than wealth. Moreover, we use arithmetic means rather geometric ones because relevant geometric means are path-independent, and thus degenerate to static representations.

Number of periods. Two is the minimal number of periods necessary for a dynamic model.

[13] We do exclude degenerate models that were created by exogenously assumed static mappings, rather than starting from common first principles and demonstrating the mapping to static models. We exclude these models because they do not address our issue of whether there are DM that are similar to MM and, hence, whether empirical implementations are consistent with the prevalent DM. For example, considering a multiperiod problem, Cochrane (2014) aggregated a time series of payoffs into single points in the MV space, transforming the multiperiod model into a static one. (Cochrane's application of the inner product in the Hilbert space of stochastic processes is a useful, elegant transformation that provides interesting insights.)

Capital market. The market consists of multiple (at least three[14]) risky securities, and a riskless one, along with unlimited borrowing and lending at a constant riskless rate.[15]

Risk structure. The probability distributions of security returns are fully captured by MV. We do not explicitly include risks from outside the capital market (e.g., labor market).[16]

Information structure. All information is common knowledge. Thus, information is symmetric; there is no private information. The risk structure is part of the information structure. We do not explicitly model information other than that captured by the risk of financial securities.

No intermediate consumption. In line with simplicity and leaving out non-financial shocks, we do not allow for (intermediate) consumption.

We are not aware, however, of any other multiperiod model which is as minimal as the one here.

2.2 The MDE

Pursuing utmost simplicity, consider a two-period, three-date, t, $t = 0,1,2$, Markowitz world with N risky securities, $N > 2$. To simplify notation, wherever we do not specify the applicable values of t, the applicable values are $t = 1,2$. Risky securities are nonredundant, with finite moments. There is a representative investor with MV preferences and Arrow-Pratt risk-aversion measure $\frac{A}{2}$. Security returns on investments made at date $t - 1$ and realized at date t are

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[14] Allows for a MV inefficient security [see Diacogiannis and Feldman (2013)]. If there are only two of these securities, each of them spans the returns space, all risky assets are MV efficient, and the model becomes implausible. (We later find that having more than two securities comes with "no cost.")

[15] We later show that the model can be easily extended to having a stochastic (one-period default-free) interest rate, introduced at the (endogenous) zero-beta rate, without changing the results.

[16] It is important to stress that the distributions of security returns over the second period cannot be uncorrelated with those of the first period; otherwise, the model would have a static representation. Moreover, because a lack of correlation induces path independence, we note that path independence generally allows single-period representation, which generally induces dependence on end-of-period wealth. Dependence on this wealth generally induces path independence—and so forth—creating a "loop." A highly plausible, sensible, and simple way of avoiding this loop is to allow returns' periodic dependence, which induces path dependency.

$Nx1$ exogenous random variables vectors, R_t, driven by/are conditional on date $t-1$ fundamentals/state variables/productivity factors. For simplicity, security returns are measured in excess of the riskless rate. For simplicity and without loss of generality, we assume that the riskless rate is zero.

The first two moments of security returns μ_{t-1}, $\mu_{t-1} \triangleq \mathrm{E}_{t-1}(R_t)$, and Σ_{t-1}, $\Sigma_{t-1} \triangleq \left\{\mathrm{Cov}_{t-1}(R_{t,i}, R_{t,j})\right\}_{i,j=1,\ldots,N}$ prevail from date $t-1$ to date t and are conditional on date $t-1$ fundamentals. The moments μ_{t-1}, and Σ_{t-1}, are $Nx1$ vectors and NxN matrices, respectively. Let $\mathrm{E}_{t-1}(\cdot)$, $\mathrm{Var}_{t-1}(\cdot)$, and $\mathrm{Cov}_{t-1}(\cdot,\cdot)$ be the expectations, variance, and covariance operators, respectively, conditional on date $t-1$ fundamentals.

The representative investor portfolio's rates of return R_{Pt}, is $R_{Pt} = \theta_{t-1}^{\mathrm{T}} R_t$. Portfolio weights, θ_{t-1}, prevailing from date $t-1$ to date t, are $Nx1$ vectors, and are conditional on date $t-1$ fundamentals.

Preferences are over portfolios' (arithmetic) mean returns over the two periods, $\bar{R}_P \triangleq \frac{R_{P1}+R_{P2}}{2}$. The representative investor trades off mean and variance by choosing portfolio weights $\theta_{t-1},\ t = 1,2$:

$$\underset{\theta_{t-1}, t=1,2}{\mathrm{Max}} \left\{\mathrm{E}_0(\bar{R}_P) - \frac{A}{2}\mathrm{Var}_0(\bar{R}_P)\right\}, \tag{1}$$

where,

$$\begin{aligned} &\mathrm{E}_{t-1}(R_{Pt}) = \theta_{t-1}^{\mathrm{T}} \mu_{t-1}, \\ &\mathrm{Var}_{t-1}(R_{Pt}) = \theta_{t-1}^{\mathrm{T}} \Sigma_{t-1} \theta_{t-1}, \end{aligned} \tag{2}$$

where superscripts T denote the transpose operator. We assume that all conditional expectations of random variables of interest exist.

Let θ_{Mt}, $t = 0,1$ be the market capitalization weights, or the market portfolio weights. The market portfolio returns R_{Mt}, thus are $R_{Mt} = \theta_{Mt-1}^{\mathrm{T}} R_t$.

We are now able to define the equilibrium.

Definition. *Equilibrium*. At both dates, $t = 0{,}1$, the representative investor holds optimal portfolios. □

(The existence of a representative investor implies that, by construction, the market clears, and that the representative investor's optimal portfolio is the market portfolio.)

3 MDE Characterizations

We are now ready to characterize the MDE.

3.1 Second Period Characterization

Proposition 1

Conditional on date 1 fundamentals' realizations,

1. date 1 optimal portfolio weights, or market capitalization, are

$$\theta_1 = \frac{2}{A} \Sigma_1^{-1} \mu_1, \tag{3}$$

and

2. date 1 risk premia, i.e., expected returns (in excess of the riskless rate), are

$$\mu_1 = \frac{A}{2} \Sigma_1 \theta_1. \tag{4}$$

Proof. See Appendix A.

Conditional on date 1 fundamentals' realizations, the date 1 problem becomes the classical single-period one.

We can now use Proposition 1 results to further characterize the date 1 equilibrium, specifically the moments of the market portfolio return.

Corollary 1 to Proposition 1

Conditional on date 1 fundamentals' realizations,

1. date 1 market portfolio's expected return and variance are, respectively,

$$\mathrm{E}_1(R_{M2}) = \frac{2}{A}\mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1, \tag{5}$$

$$\mathrm{Var}_1(R_{M2}) = \left(\frac{2}{A}\right)^2 \mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1, \tag{6}$$

and

2. date 1 market portfolio's square Sharpe ratio, S_1^2, is

$$S_1^2 = \mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1. \tag{7}$$

Proof. See Appendix A.

We are now ready to highlight an equilibrium property of the MDE that relates date 1 market portfolio's Sharpe ratio and volatility. We later demonstrate the implication of this property to the MDE equilibrium's dependence on higher moments.

If we substitute Equation (6) onto Equation (7), we get

$$S_1^2 = \left(\frac{A}{2}\right)^2 \mathrm{Var}_1(R_{M2}), \tag{8}$$

demonstrating that, in equilibrium, the date 1 conditional square Sharpe ratio is equal to a unitless coefficient times the market portfolio return's variance, implying a dimension of square returns. Thus, we proved the following Corollary.

Corollary 2 to Proposition 1

The characterizations of date 1 square Sharpe ratios and (stochastic) investment opportunities have a dimension of "square returns." □

Thus, our MDE has the property that the unitless square Sharpe ratio of the market portfolio return, in equilibrium, becomes one to one with a variable representing square returns.

Another interesting insight conveyed directly by Equation (8) is the equilibrium relation between market Sharpe ratios and volatility. The higher the volatility, the higher the Sharpe ratio required to mitigate its effects on the derived utility. Moreover, this required mitigation is increasing in the representative investor's risk-aversion measure.

3.2 First Period Characterization

To analyze the more interesting period, period 1, we start by defining the covariance between date 1 returns and the future (date 2) investments opportunity set, which we call intertemporal covariance.

Definition. *Intertemporal covariance.* We define c_0, the covariance between date 1 returns and future (date 2) investment opportunity set, as

$$c_0 \triangleq \mathrm{Cov}_0(R_1, S_1^2), \tag{9}$$

that is, $c_0 = \left\{\mathrm{Cov}_0\left(R_{1,i}, S_1^2\right)\right\}_{i=1,\dots,N}$. □

(We note that at date 0, date 1 Sharpe ratio is a random variable.) We characterize date 0, optimal portfolios, market capitalization, and risk premia.

Proposition 2

1. Date 0 optimal portfolio or market capitalization is

$$\theta_0 = \frac{2}{A}\Sigma_0^{-1}(\mu_0 - c_0). \tag{10}$$

2. Date 0 risk premia are

$$\mu_0 = \frac{A}{2}\Sigma_0\theta_0 + c_0. \tag{11}$$

Proof. See Appendix A.

We now use Propositions 1 and 2 results to further characterize date 0 equilibrium, specifically, optimal portfolios' conditional Sharpe ratios and the stochastic conditional investment opportunity set.

Corollary 1 to Proposition 2

1. Conditional on date 1 realizations, the date 1 market portfolio's Sharpe ratio sufficiently characterizes the stochastic investment opportunity set of the MDE.
2. Higher moments than variance play a role in the MDE.
3. There is no degeneration of MDE risk premia to those of single-period models, and MDE risk premia depend on higher moments.
4. The MDE market portfolio is MV inefficient.

Proof. Equations (10) and (11) demonstrate that the MDE date 0 equilibrium demands and prices depend on the future (date 1) only through its future returns and Sharpe ratio (through dependency on c_0).

This proves point 1 of the Corollary.

As c_0 is a covariance between returns and square Sharpe ratios—see the definition in Equation (9)—and as square Sharpe ratios in our MDE are proportional to the market portfolio variance, thus having a dimension of square returns—see Equation (8)—c_0 has a dimension of cubic returns proportional to third moments. As c_0 is integral part of the MDE demands and prices, see Equations (10) and (11), higher moments play a (substantial) role in the MDE.

This proves point 2 of the Corollary.

Equation (11) characterizes the MDE risk premia as a sum of two addends. The first corresponds to single-period risk premia and is similar, for example, to those of the terminal period, date 1, when there are no future opportunities, see Equation (4). The second addend is $-c_{0,}$ which has the dimension of third moments, as does the MDE risk premia.

This proves point 3 of the Corollary.

Finally, Equation (10) demonstrates that the component c_0 takes optimal portfolio rules away from the single-period MV efficient demands. Only in the case in which future investment opportunities are uncorrelated with prevailing ones will the date 0 market portfolio be MV efficient.

This proves point 4 of the Corollary. *QED*

4 Answers to the Three Essential Questions (TEQ)

We note that c_0 corresponds, in Merton's terminology, to demands to hedge changes in future investments opportunities. We highlight how this corollary answers the TEQ. Point 2 established a **"*no*"** regarding the first TEQ (dependency on higher moments) overlap of the MDE and static models.

Equation (11) shows that the MDE risk premia are also functions of c_0, which is an addend to the single-period risk premia. This demonstrates that the MDE risk premia do not degenerate to those of single-period ones. Moreover, we can also say that the MDE risk premia depend on higher moments because c_0 is a function of the third moment of returns. This establishes the second **"*no*"** regarding TEQ overlap.

The MDE optimal demands or portfolio rules are equal those of the single-period ones only if $c_0 = 0$. Under non-zero c_0, the addend to single-period demands in the equation for optimal demands, Equation (10), can be viewed as the, so called, "hedging demands" term, that Merton coined to describe this part of optimal demands. These generally take the representative investor's portfolio away from the MV frontier, rendering their optimal portfolios MV inefficient. However, we may argue, following Merton, that in equilibrium it becomes optimal to "hedge" the changes in future investment opportunities. As these demands take the MDE (date 1) market portfolio away from the MV frontier rendering it "inefficient" and incapable of serving as the SDF, this establishes the third **"*no*"** regarding TEQ overlap.

Still, the MDE optimal demands, as could be expected, are only a special case of demands in general dynamic equilibria. While an increase in risk aversion still reduces risky assets' holdings, it affects the single-period demands and "hedging demands" in equal proportion. Also, a higher positive (negative) covariance between prevailing returns and future investment opportunities would always reduce (increase) holding of risky assets.

Because in the MDE, R_{M1} and R_{M2} are the periodic market portfolio returns, we can specifically identify the SDF and demonstrate that the market portfolio is not the pricing kernel.

Rewriting Equation (11) using Equations (9), and (8), gives

$$\mu_0 = \frac{A}{2}\mathrm{Cov}_0\left(R_1, R_{M1} + \frac{A}{2}\mathrm{Var}_1(R_{M2})\right). \tag{12}$$

We note that the components of the second argument of the covariance operator in Equation (12) are random variables, conditional on date 1 fundamentals' realizations.

Equation (12) identifies, up to a proportionality constant, the SDF as $R_{M1} + \frac{A}{2}\mathrm{Var}_1(R_{M2})$.

We thus proved the following Corollary.

Corollary 2 to Proposition 2

The MDE date $t = 0$, one-period SDF is, up to a proportionality constant, $R_{M1} + \frac{A}{2}\mathrm{Var}_1(R_{M2})$. □

As the SDF includes an addend additional to the market portfolio's return, the market portfolio is not the pricing kernel.

5 MDE Relevance and Implications

We demonstrate the plausibility, relevance, robustness, and generality of the MDE.

5.1 MDE within DM

It might be interesting to study a DM over a variety of preferences, assets, payoffs, strategies, constraints, time structure (discrete versus continuous), agents (short-lived, long-lived, overlapping generations), state space (finite versus infinite, discrete versus a continuum), and markets (exchange, production, contingent claims). In our pioneering study, however, we believe that starting with choices relevant to the most studied and implemented models is the right approach. After all, these models earned their place endogenously, in competition with other models. This is the MDE that we defined in Section 2 and that we characterized in Section 3. It is our conjecture that our MDE will maintain a primary position among DM. As such, we believe the insights from MDE to be highly relevant for dynamic analysis.

5.2 Robustness of MDE

To examine the robustness of our MDE's choice we modify it along its direction/dimensions/attributes and examine where such changes take us in answering the TEQ.

Preferences. In further support of our choice of MV preferences, we note that the influential works Pratt (1966), Arrow (1971), Samuelson (1970) pointed out that MV preferences are a good first approximation to any expected utility preference structure. This supports our choice of MV preferences as the "minimal" ones.[17]

We now consider potential extensions of the preferences attribute. We begin with the following extension. Our MDE preferences are defined over investors' portfolios arithmetic mean returns over the two periods. Introducing preferences over the first period that are separable from the currently assumed ones, which are over the first and second periods together, would not change our results because the currently assumed preferences alone are sufficient for a "no" answer to all TEQ.

As in our Appendix A proof, our MDE date 2 MV optimization is similar to date 1 MV

[17] Later, his analysis was formalized by Judd and Guu (2001); see also Jud and Leisen (2010),and Chabi-Yo, Leisen and Renault (2014).

optimization, except that it must be adjusted by a term that arises from the law of total variance.[18] This leads to an additional term (c_0) that implies a "no" to all TEQ.

We note that weighting differently (an additional) date 1 "utility" and date 2 "utility," does not change the results. The effect of the Law of Total Variance stands.

Similarly, specifying a DM with preferences that exhibit intertemporal risk aversion (intertemporal elasticity of substitution), different from the cross-sectional ones (e.g., Kreps Porteus utility, Epstein-Zin–Weil utility, stochastic differential utilities) falls within the MM and delivers the same answers to the TEQ as those for the MDE.

Finally, considering the MDE assumption of a representative investor, we note such a representation is not restrictive [Magill and Quinzii (2002)],[19] and does not change the answers to the TEQ.

Number of periods. Optimal solutions of dynamic models with more than two periods nests the solution of a two-period model. Therefore, extending the number of periods does not change the answers to the TEQ.

Capital market (Number of Securities). Our structure of at least three securities, one of which is inefficient, already allows for a general MV space [Diacogiannis and Feldman (2013)]. Thus, any number of securities greater than three does not change the answers to the TEQ.

Capital market (Zero beta rate). A DM similar to our MDE but with no riskless rate induces an equilibrium with a zero-beta rate, and the answers to the TEQ remain unchanged. We note that in this case, the zero-beta rate is stochastic (because the MV frontier changes across periods).

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[18] The law of total variance implies that the expectation of a conditional variance of a random variable is *not equal* to the variance of this random variable. (Unlike the case of expectation: the expectation of a conditional expectation of a random variable is *equal* to the expectation of this random variable.)

[19] One can construct a representative investor's preferences from investors' heterogeneous preferences. Under incomplete markets, individual investor's weights might be stochastic.

Risk structure. As preferences are MV, we capture the first two moments of the probability distributions. Capturing higher moments does not affect our results, and the answers to the TEQ remain unchanged. We note that capturing risks from outside our capital market (e.g., labor market) would merely add additional terms that matter for pricing but do not change our results.

Information structure. Equilibria under incomplete or asymmetric information structures are more complex than similar ones under complete symmetric information. However, the former generally nest the latter, thus do not change our results. We illustrate this with the following examples.

Feldman (2004) demonstrates how under incomplete information, equilibrium demand/prices have additional terms to hedge unobservable fundamentals, additional terms to hedge the dynamic precisions of these fundamentals' estimates, and that these precisions are generally stochastic. While we cannot study dynamic learning within static settings, Feldman (2004) shows that the dynamic incomplete information models nest MDE type models. Thus, this added structure does not change the answers to the TEQ.

Brunnermeier (2001), and Kelly and Ljungqvist (2012) study pricing under various asymmetric information situations. Again, the added structures in these situations nest symmetric information equilibria and do not change the answers to the TEQ.

(Intermediate) Consumption. Recall that our MDE preferences depend on both intermediate and terminal date returns. Preference dependency on intermediate returns, in turn, is equivalent to preference dependency on intermediate consumption. Thus, adding intermediate consumption to the MDE does not change our results, and the answers to the TEQ remain unchanged.[20]

[20] Removing the preferences dependency on intermediate returns, in our MDE, would allow a single-period representation of the model (causing it to lose its dynamic nature, according to our definition of dynamic models).

Arithmetic Mean Returns versus Compounded Returns. An alternative specification to our MDE's choice of arithmetic mean (excess) returns is compounded (excess) returns of the form of, say,

$$(1 + R_{P1})(1 + R_{P2}) - 1 = R_{P1} + R_{P2} + R_{P1}R_{P2}. \tag{13}$$

This would require replacing R_{P2} in our second period (conditional optimization, see Appendix A) by $R_{P2}(1 + R_{P1})$, i.e., our results would include a wealth effect from the first period returns. However, this wealth effect does not change our results, and the answers to the TEQ remain unchanged.[21]

Wealth. As discussed in Section 2, MV preferences should be over returns, thus we excluded preferences over wealth. Our preferences over returns imply that the MDE initial wealth level is normalized to one. Choosing a general wealth level would scale quantities[22] but the answers to the TEQ would remain unchanged.

Recall that we searched for the MDE among all plausible theoretically and empirically relevant models. To identify the MDE, the dynamic model of the simplest structure, we studied all relevant model attributes.

We demonstrated in this subsection that modifying the MDE in all relevant nine dimensions does not change the answers to the TEQ. Thus, in the "open" set of models close to the MDE, the answers to the TEQ are ***"no," "no,"*** and ***"no."*** Thus, we have the following proposition.

5.3 Generality of MDE

[21] An interesting empirical question is whether the added term of $R_{P1}R_{P2}$ has significant impact despite being an order of magnitude smaller.
[22] Including the risk-aversion parameter (the tradeoff coefficient between mean and variance).

Proposition. Identifying the plausible relevant dynamic model with the simplest structure, the MDE, we find that

i. The answers to all TEQ within the MDE are no, no and no.

ii. Modifying the MDE along all nine relevant dimensions does not change the answers to the TEQ.

We thus conclude that the answers to all TEQ are no, no, and no for all plausible relevant dynamic models "near" the MDE. □

After the modifications of the MDE along all relevant directions/dimension/attributes and finding it robust in answering the TEQ, and as we otherwise cannot envision plausible relevant dynamic[23] models for which the answers to TEQ are different than for the MDE, we are prepared to state the following conjecture.

Conjecture. *There are no plausible relevant dynamic models with answers to the TEQ, different than "no," "no," and "no."* □

6 Empirical Relevance

Empirical studies based on dynamic models allow studying intertemporal risk premia, dynamic structures, and implications of incomplete information (conditioning). Static models, in contrast, by construction cannot directly or generally capture these. Indeed, a significant segment of asset pricing empirical literature implements MM. Even within MM empirical implementations, the ability to choose an intertemporal risk premium, which is different from the cross-sectional one, results in substantial improvements of positive and predictive powers over original ICAPM

[23] We remind the reader that we define "dynamic models" as multiperiod models with no static representation.

models, where both risk premia are the same. This is demonstrated, for example, by the popularity of Epstein-Zin–Weil models' empirical implementations. In addition, some implementations of static models recognize their lacuna and creatively suggest representing dynamic effects by static variables [e.g., Cochrane (2014)].

Moreover, the vast empirical literature on asset pricing time series of auto regressive processes, e.g., (G)ARCH and their derivatives, are within the realm of dynamic models and not of static ones, facilitating predictability and forecasts.

7 Conclusion

While the object of finance models is a dynamic environment, prevalent asset pricing implementations are static models. Three essential asset pricing questions (TEQ) are dependency on higher moments, complexity of risk premia, and market portfolios being SDFs/pricing kernels MV efficient. We already know that certain dynamic models, including Merton-type models and their various expansions (MM), differ from static ones regarding all TEQ. In this paper, we have aimed to identify dynamic models that retain or capture the static properties regarding the TEQ. For this purpose, within the set of plausible relevant dynamic models, we make the strongest simplifying assumptions that are likely to help capture static models' properties and identify the "simplest/minimal" dynamic equilibria (MDE). We find that within the MDE the answer to the TEQ is ***"no," "no,"*** and ***"no,"*** as in MM and unlike the answers within static models. We confirm that these answers are robust to modifying the MDE along all directions/dimensions/attributes. Furthermore, the future volatility of MDE market portfolios' returns emerges as a pricing factor. Our findings suggest that prevalent empirical asset pricing implementations, such as linear beta pricing, are consistent only with static models.

APPENDIX A: MATHEMATICAL

Proof of Proposition 1

The date 1 problem, conditional on date 1 realizations, is

$$\underset{\theta_1}{\text{Max}}\left\{\mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)\right\}. \tag{A1}$$

Conditional on date 1 fundamentals' realizations, we denote date 1 utility and date 1 derived utility (or indirect utility function) as $U_1(\theta_1)$ and J_1 respectively, we can define

$$U_1(\theta_1) \triangleq \mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{R_{P1}+R_{P2}}{2}\right), \tag{A2}$$

and conditional on date 1 fundamentals' realizations

$$J_1 \triangleq \underset{\theta_1}{\text{Max}}\left\{\mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)\right\} \tag{A3}$$

or

$$J_1 \triangleq \underset{\theta_1}{\text{Max}}\{U_1(\theta_1)\}. \tag{A4}$$

We can rewrite Equation (A1) as

$$J_1 = \underset{\theta_1}{\text{Max}}\left\{\frac{1}{2}R_{P1}+\mathrm{E}_1\left(\frac{1}{2}R_{P2}\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{1}{2}R_{P2}\right)\right\}, \tag{A5}$$

which, for finding optimal portfolio weights, is equivalent to

$$\underset{\theta_1}{\text{Max}}\left\{\mathrm{E}_1\left(\frac{1}{2}R_{P2}\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{1}{2}R_{P2}\right)\right\}. \tag{A6}$$

Thus, conditional on date 1 realizations, the date 1 problem becomes a standard single-period MV problem, and optimal portfolio weights are the argmax of the solution to the following problem:

$$\underset{\theta_1}{\text{Max}}\left\{\mathrm{E}_1\left(\frac{1}{2}\theta_1^{\mathrm{T}}R_2\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{1}{2}\theta_1^{\mathrm{T}}R_2\right)\right\}. \tag{A7}$$

The first-order condition is

$$\frac{\partial}{\partial\theta_1}\left[\mathrm{E}_1\left(\frac{1}{2}\theta_1^{\mathrm{T}}R_2\right)-\frac{A}{2}\mathrm{Var}_1\left(\frac{1}{2}\theta_1^{\mathrm{T}}R_2\right)\right]=0 \tag{A8}$$

or

$$\frac{1}{2}\mu_1 - \frac{A}{4}\Sigma_1\theta_1 = 0 \tag{A9}$$

or

$$\theta_1 = \frac{2}{A}\Sigma_1^{-1}\mu_1. \tag{A10}$$

Because the second-order conditions are satisfied, θ_1, defined in Equation (A10), are date 1 optimal portfolio, or market portfolio, or market capitalization, weights vector.

This proves point 1 of Proposition 1.

Rearranging Equation (A10) yields

$$\mu_1 = \frac{A}{2}\Sigma_1\theta_1, \tag{A11}$$

which are date 1 market risk premia.

This proves point 2 of Proposition 1. *QED*

<u>Proof of Corollary 1 to Proposition 1</u>

In equilibrium, the representative investors' optimal portfolio is the market portfolio.

Date 1 market portfolio's conditional expected (excess) return, then, is

$$\mathrm{E}_1(R_{M2}) = \theta_1^{\mathrm{T}}\mu_1 = \frac{2}{A}\mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1, \tag{A12}$$

where the second equality holds after a substitution using Equation (3).

Date 1 market portfolio's (excess) return variance is

$$\mathrm{Var}_1(R_{M2}) = \theta_1^{\mathrm{T}}\Sigma_1\theta_1 = \left(\frac{2}{A}\right)^2 \mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1. \tag{A13}$$

Again, the second equality holds after a substitution using Equation (3).

This completes the proof of point 1 of the Corollary.

Now, use Equations (A12) and (A13) to obtain

$$S_1^2 = \frac{\left(\mathrm{E}_1(R_{M2})\right)^2}{\mathrm{Var}_1(R_{M2})} = \frac{\left(\mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1\right)^2}{\mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1} = \mu_1^{\mathrm{T}}\Sigma_1^{-1}\mu_1, \tag{A14}$$

which gives Equation (7).

This proves point 2 of the Corollary. *QED*

Proof of Proposition 2

Denoting J_0 as the (total) derived utility, or date 0 derived utility, date 0 problem is

$$J_0 \triangleq \underset{\theta_0,\theta_1}{\mathrm{Max}}\left\{\mathrm{E}_0\left(\frac{R_{P1}+R_{P2}}{2}\right) - \frac{A}{2}\mathrm{Var}_0\left(\frac{R_{P1}+R_{P2}}{2}\right)\right\}, \tag{A15}$$

which is not directly amenable to be solved under the Bellman's principle of optimality.[24] Using the laws of total expectation and total variance, we rewrite the Equation (A15) problem in Equation (A16). [Basak and Chabakauri (2010) and Björk, Murgoci, and Zhou (2014) presented solutions to the problem in a continuous time context. Malamud and Vilkov (2018) use Basak's insights to present a discrete time solution to a similar problem within an overlapping generations model.]

$$\begin{aligned} J_0 = \underset{\theta_0,\theta_1}{\mathrm{Max}}\Bigg\{\mathrm{E}_0\left(\mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right) - \frac{A}{2}\mathrm{Var}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)\right) \\ -\frac{A}{2}\mathrm{Var}_0\left(\mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)\right)\Bigg\}. \end{aligned} \tag{A16}$$

Using Equation (A2), we can rewrite the problem in Equation (A16) as

$$\underset{\theta_0,\theta_1}{\mathrm{Max}}\left\{\mathrm{E}_0(U_1(\theta_1)) - \frac{A}{2}\mathrm{Var}_0\left(\mathrm{E}_1\left(\frac{R_{P1}+R_{P2}}{2}\right)\right)\right\}. \tag{A17}$$

Using the definitions in Equations (A4) and (A15), and date 1 optimal portfolio weights values, θ_1, which we already determined, see Equation (3), we can rewrite Equation (A17) as

$$\underset{\theta_0}{\mathrm{Max}}\left\{\mathrm{E}_0(J_1) - \frac{A}{2}\mathrm{Var}_0\left(\frac{1}{2}\left(R_{P1}+\mathrm{E}_1(R_{P2})\right)\right)\right\}. \tag{A18}$$

[24] Because expectation of a variance is not equal to variance of expectation, the Bellman equation loses its recursive property. See Basak and Chabakauri (2010).

We proceed by calculating the value of each of the two addends of Equation (A18). We will, then, identify the optimal values for date 0 portfolio weights, θ_0.

We identify the first addend of Equation (A18), J_1, using Equation (A5). We substitute into it date 2 optimal portfolio weights values as determined in Equation (A10) and, further, substitute Equations (7) and (8). We have

$$J_1 = \frac{1}{2} R_{P1} + \frac{1}{2}\frac{2}{A} S_1^2 - \frac{1}{2A} S_1^2 = \frac{1}{2} R_{P1} + \frac{1}{2A} S_1^2. \tag{A19}$$

Taking expectation of Equation (A19), we have

$$\mathrm{E}_0(J_1) = \frac{1}{2}\theta_0^{\mathrm{T}}\mu_0 + \frac{1}{2A}\mathrm{E}_0(S_1^2). \tag{A20}$$

Calculating the value of the second addend of Equation (A18) gives

$$\begin{aligned} \mathrm{Var}_0\left(\frac{1}{2}\left(R_{P1} + \mathrm{E}_1(R_{P2})\right)\right) &= \mathrm{Var}_0\left(\frac{1}{2}R_{P1} + \frac{1}{A}S_1^2\right) \\ &= \frac{1}{4}\mathrm{Var}_0(R_{P1}) + \mathrm{Var}_0\left(\frac{1}{A}S_1^2\right) + \mathrm{Cov}_0\left(R_{P1}, \frac{1}{A}S_1^2\right) \\ &= \frac{1}{4}\theta_0^{\mathrm{T}}\Sigma_0^{-1}\theta_0 + \mathrm{Var}_0\left(\frac{1}{A}S_1^2\right) + \frac{1}{A}\theta_0^{\mathrm{T}}c_0. \end{aligned} \tag{A21}$$

The first equality holds because of substitutions following Equations (5) and (7), and the last equality holds because of the use of the definition in Equation (9).

Using the results in Equations (A20) and (A21), the first-order conditions become

$$\begin{aligned} \frac{\partial}{\partial\theta_0}\left[\mathrm{E}_0(J_1) - \frac{A}{2}\mathrm{Var}_0\left(\frac{1}{2}\left(R_{P1} + \mathrm{E}_1(R_{P2})\right)\right)\right] &= \frac{1}{2}\mu_1 - \frac{A}{2}\left(\frac{1}{2}\Sigma_0\theta_0 + \frac{1}{A}c_0\right) \\ &= \frac{1}{2}\mu_0 - \frac{A}{4}\Sigma_0\theta_0 - \frac{1}{2}c_0 = 0. \end{aligned} \tag{A22}$$

Rearranging Equation (A22), gives date 0 optimal portfolio, or market portfolio, or market capitalization, weights vector

$$\theta_0 = \frac{2}{A} \Sigma_1^{-1} (\mu_0 - c_0). \tag{A23}$$

This proves point 1 of Proposition 2.

Solving Equation (A23) for the market risk premia gives

$$\mu_0 = \frac{A}{2} \Sigma_0 \theta_0 + c_0. \tag{A24}$$

This proves point 2 of Proposition 2. *QED*

APPENDIX B: OBJECTIVE MDE

Identification of an MDE must involve numerous quantitative and qualitative attributes. Further, it must be cardinal to facilitate aggregating over attributes. We demonstrate below that ranking criteria are nonunique, subjective, and arbitrary. Thus, defining an MDE is quite illusive.

We demonstrate the illusiveness of defining an MDE by examining the identification of one MDE aspect, preferences. Identifying minimal preferences requires, first, identifying the attributes of various preferences.

- The distribution of coefficients of the Taylor series expansion of utility functions. Consider two distributions. One is $\{0.33+\Delta$, 0.33, $0.33-\Delta\}$, and the other is $\{0.33$, $0.33+2\Delta$, $0.33-2\Delta\}$. Is a Δ advantage in the first and third coefficients enough to reconcile a 2Δ disadvantage in the second and third one? Ranking rules of such sequences are subjective, arbitrary, and nonunique.
- The number of cross-sectional and intertemporal risk-aversion coefficients. MM have one coefficient for any number of periods. Epstein-Zin preferences have two coefficients for any number of periods. Kreps-Porteus preferences have $2n-1$ coefficients for n periods. Ordinal ranking is natural here, but the required cardinal ranking is arbitrary and subjective.
- Whether preferences are time additive, or habit formation type 1, habit formation type 2, etc., even ordinal ranking is nonunique, subjective, and arbitrary.
- The stochastic nature of the utility and the differential nature of the utility. Ranking would be nonunique, subjective, and arbitrary.

Then, one has to create a weighing scheme over various attributes of preferences, which is, again, nonunique, subjective, and arbitrary.

In addition to defining minimal preferences, one has to define other attributes of the MDE and a weighing scheme across the various MDE attributes. We trust that we have demonstrated that such a task is illusive as any outcome would be nonunique, subjective, and arbitrary.

If there exist dynamic models with answers to the TEQ that are different from those of MM and similar to those of static models, they are likely to be among the simplest ones. This led us to identify our MDM by "minimizing" across all relevant attributes (maximal simplification of dynamic models).

Also, practical relevance calls for a minimal extension of static models to dynamic ones. There is a tradeoff, of course, between relevance and simplicity. Naturally, the choice of an MDE is to opt for simplicity.

Therefore, we believe that our approach in identifying the MDE in Section 2 and the robustness analysis in Section 5 are most achievable.

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