Segmentation Premia

Soohun Kim* Robert A. Korajczyk† Andreas Neuhierl‡

December 8, 2023

*College of Business, Korea Advanced Institute of Science and Technology
†Kellogg School of Management, Northwestern University
‡Olin Business School, Washington University in St. Louis

Segmentation Premia

Abstract

We present a novel approach to discern discrepancies in compensation for shared risk factors across different markets by leveraging firm-specific characteristics on factor loadings. Our methodology is tested on simulated factor economies and a vast dataset of international stock returns. The results demonstrate the efficacy of our method in both simulated and real-world scenarios. Specifically, we identify numerous instances where pricing for common risks differs across countries during certain periods. We also propose a portfolio strategy that capitalizes on these temporary market segmentations.

JEL classification codes: G12, C58, C38
Keywords: arbitrage, characteristic based model, factor pricing

1 Introduction

Market integration has been a key driver of human history, fostering cooperation, and promoting the exchange of ideas, goods, and services. In his book *Sapiens*, Harari (2015) provides a unique perspective on the role of market integration in shaping human history, arguing that the integration of markets has been a fundamental force driving human integration. Harari's views are supported by the evidence from history, where we see the emergence of vast trade networks that spanned continents, connecting people and cultures from different regions. The Silk Road is a prime example of such a network, which linked China, Central Asia, and the Mediterranean world, facilitating the exchange of goods, technology, and culture.

Agreement on the value of traded assets is a key indicator of market integration. This agreement can be achieved through the actions of market participants who buy and sell assets across different regions and cultures. For instance, when the Mediterranean trade began, Indian merchants might have considered gold to be a useless stone, and it would have had little value in their local markets. However, Mediterranean merchants recognized the value of gold and saw an opportunity to buy it at a low price in India and sell it at a higher price in the Mediterranean region. This trade led to the gradual convergence of the value of gold across the two geographically segmented areas.

The financial markets have been a valuable test bed for investigating the process of market integration, given their unique characteristics. Unlike other markets, the financial markets do not require the physical movement of assets, making them highly accessible to investors across different regions and countries. Furthermore, the barriers to cross-country investment have been steadily melting away over the past few decades, creating new opportunities for investors to participate in global financial markets. As a result, the financial markets have become an essential tool for testing theories of market integration and exploring the factors that drive it.

The finance literature has a long history of investigating international market integration, with many studies focused on analyzing the degree of cross-market correlation in asset prices and the effectiveness of international portfolio diversification. These studies have shown that there is a high degree of correlation between asset prices in different markets, suggesting that the financial markets are highly integrated. Furthermore, these studies have demonstrated that international portfolio diversification can be an effective way of reducing risk and maximizing returns.

This paper's primary contribution is the introduction of a novel concept of market integration, which involves the agreement of common factor prices across two countries, with the assumption that the returns of each country have a factor structure where individual stocks can be driven by common as well as country-specific factors. The proposed method involves three steps:

In the first step, common factors are identified by applying principal component analysis (PCA) to individual assets in each country, obtaining all systematic factors, including common and country-specific factors. Canonical correlation analysis (CCA) is then used to find the combination of within-country factors that maximizes cross-country co-movement and recovers the true common factors.

In the second step, factor loadings on common factors are measured, which is known to be a daunting task. The approach used follows Kim, Korajczyk, and Neuhierl (2021), where the information in firm characteristics is exploited to estimate the factor loadings. Returns are regressed on the interaction between firm characteristics and common factors, identifying the linear combination of firm characteristics that explains the returns through the common factors.

In the last step, the difference in the prices of common risk is identified by using the classical pricing equation, where the coefficient on factor loadings represents the prices of risk. A cross-sectional regression of average returns on the estimated factor loading is performed, reflecting the country-specific risk prices and factor randomness. By subtracting the estimated risk prices, differences in risk prices across countries can be recovered. This approach cancels out factor randomness and allows comparison of risk prices even with short time series data. Furthermore, we also show that when there is a disagreement on risk prices of common factors, one can construct an arbitrage portfolio that exploits such segmentation.

2 The Model

We consider a pair of two countries, indexed by $g = 1, 2$. We assume that there exists a large number of securities in each country and the return generating processes for those individual securities are stable over a short horizon $t = 1, \cdots, T$.

We specify the return generating process of individual securities in each country g. The returns of individual stocks in country g follow a K_g-factor model in which the factors are unobservable, latent factors. Among K_g factors, K^c factors are common across the two countries and K_g^s factors are country-specific. Thus, it holds that $K_g = K^c + K_g^s$. In particular, the excess return of i-th asset in country at time t is generated by the following model: for $g = 1, 2,\ i = 1, \cdots, N_g$ and $t = 1, \cdots, T$,

$$R_{git} = \boldsymbol{\beta}'_{gi} \left(\boldsymbol{\lambda}_g^c + \mathbf{f}_t \right) + \boldsymbol{\delta}'_{gi} \left(\boldsymbol{\lambda}_g^s + \mathbf{g}_{gt} \right) + e_{git}, \tag{2.1}$$

where $\boldsymbol{\beta}'_{gi} = [\beta_{gi1} \ \cdots \ \beta_{giK^c}]'$ are the $(K^c \times 1)$ factor loadings of the i-th asset to the common factors, $\boldsymbol{\lambda}_g^c = \left[\lambda_{g1}^c \ \cdots \ \lambda_{gK^c}^c\right]'$ is the $(K^c \times 1)$ vector of risk premium in country g on the exposure to common systematic factors, $\mathbf{f}_t = [f_{g1} \ \cdots \ f_{gK^c}]'$ is the $(K^c \times 1)$ systematic zero-mean common factor realization in period t, $\boldsymbol{\delta}'_{gi} = \left[\delta_{gi1} \ \cdots \ \delta_{giK_g^s}\right]'$ is the $\left(K_g^s \times 1\right)$ factor loadings of the i-th asset to the country-specific factors in country g, $\boldsymbol{\lambda}_g^s = \left[\lambda_{g1}^s \ \cdots \ \lambda_{gK_g^s}^s\right]'$ is the $\left(K_g^s \times 1\right)$ vector of risk premium in country g on the exposure to country-specific systematic factors, $\mathbf{g}_{gt} = \left[g_{g1} \ \cdots \ g_{gK_g^s}\right]'$ is the $\left(K_g^s \times 1\right)$ systematic zero-mean country-specific factor realization in period t, and e_{git} is the zero-mean idiosyncratic residual return of asset i at time t.[1] Throughout, we use $\mathbf{0}_m$, $\mathbf{1}_m$, and $\mathbf{0}_{m \times l}$ denote the $(m \times 1)$ vectors of zeros and ones and the $(m \times l)$ matrix of zeros, respectively. The return generating process of (2.1) is expressed compactly in matrices: for $g = 1, 2$,

$$\mathbf{R}_g = \underbrace{\mathbf{B}_g \left(\boldsymbol{\lambda}_g^c \mathbf{1}'_T + \mathbf{F}' \right)}_{\text{shared across countries}} + \underbrace{\mathbf{D}_g \left(\boldsymbol{\lambda}_g^s \mathbf{1}'_T + \mathbf{G}'_g \right)}_{\text{country specific}} + \mathbf{E}_g, \tag{2.2}$$

where the (i, t) element of the $(N_g \times T)$ matrix $\mathbf{R}_g$ is R_{git}, the i-th row of the $(N_g \times K^c)$ matrix $\mathbf{B}_g$ is $\boldsymbol{\beta}'_{gi}$, the t-th row of the $(T \times K^c)$ matrix $\mathbf{F}$ is $\mathbf{f}'_t$, the i-th row of the $\left(N_g \times K_g^s\right)$ matrix $\mathbf{D}_g$ is $\boldsymbol{\delta}'_{gi}$, the t-th row of the $\left(T \times K_g^s\right)$ matrix $\mathbf{G}_g$ is $\mathbf{g}'_t$, and the (i, t) element of the $(N_g \times T)$ matrix $\mathbf{E}_g$ is e_{git}.

Next, we provide the economic interpretations of (2.2). The first term of RHS in (2.2) is related to the common factors, i.e. factors that shared across the two countries. The price of risk for the common factors, is $\boldsymbol{\lambda}_g^c$. Our main question is whether the

[1] We can add a mispricing term to the return generating process (2.1) and derive identical results, our objective is to identify the market segmentation through differences in the risk compensation and hence we do not include the mispricing term to simplify the exposition.

compensation to the common factors is identical across the two countries, i.e. is the compensation for the same risk identical across countries. We therefore index the prices of risk by g (2.2) as our main empirical question will be if $\boldsymbol{\lambda}_1^c \neq \boldsymbol{\lambda}_2^c$. The bulk of this section will detail the structure and assumptions on the data generating process, before we detail our estimation procedure to estimate the difference in the prices of risk between two countries.

First, let us consider a hypothetical case. If we knew $[\mathbf{B}_g \ \mathbf{D}_g]$, the exposures to the systematic factors in each country, we could have estimated $\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'$ from straightforward cross-sectional regressions in each country under reasonable assumptions.[2] Then, we might have recovered any differences in $\boldsymbol{\lambda}_1^c$ and $\boldsymbol{\lambda}_2^c$ as the differences in the estimated $\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'$ because $\mathbf{F}'$, the random realization of common factors, is cancelled out. However, it is well known in the literature that beta estimates for a large cross-section over a short horizon entail substantial estimation errors. Hence, we exploit the intuition above by extending the approach in Fan et al. (2016) and Kim et al. (2021) to the two country setup. The crucial difference from Kim et al. (2021) is that we aim to identify the differences in compensation by extracting the information on the factor loadings in characteristics while Kim et al. (2021) focuses on extracting the mispricing embedded in characteristics.

To that end, we allow the systematic risk $[\mathbf{B}_g \ \mathbf{D}_g]$ to be functions of asset-specific characteristics in each country. Let $\mathbf{x}_{gi} = \begin{bmatrix} x_{gi1} & \cdots & x_{giL_g} \end{bmatrix}'$ be the $(L_g \times 1)$ vector of the characteristics associated with stock i in country g. Then, define the $(N_g \times L_g)$ matrix of $\mathbf{X}_g$, the i-th row of which is $\mathbf{x}_{gi}'$. We assume the following structure for $[\mathbf{B}_g \ \mathbf{D}_g]$:

$$\mathbf{B}_g = \mathbf{X}_g \Theta_g^c + \Gamma_g^c \tag{2.3}$$

and

$$\mathbf{D}_g = \mathbf{X}_g \Theta_g^s + \Gamma_g^s, \tag{2.4}$$

where Θ_g^c is the $(L_g \times K^c)$ matrix, Θ_g^s is the $\left(L_g \times K_g^s\right)$ matrix, and the $(N_g \times K^c)$ matrix, Γ_g^c and the $\left(N_g \times K_g^s\right)$ matrix, Γ_g^s are cross-sectionally orthogonal to the characteristic space of $\mathbf{X}_g$. We call the two matrices of Θ_g^c and Θ_g^s as factor loading matrices because they relate characteristics to factor loadings to the common and country-specific

[2]The object of $\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'$ is closely related to the concept of expost risk premia discussed in Shanken (1992) and Kim and Skoulakis (2018).

systematic factors, respectively. The two terms of Γ_g^c and Γ_g^s represent the sources of beta that are not attributable to the characteristics. While the factor loading matrices, Θ_g^c and Θ_g^s can be consistently estimated in the large N/small T setting used here, consistent estimates of Γ_g^c and Γ_g^s are not available. Therefore, our procedure does not attempt to exploit the gammas, just their orthogonality to the characteristics in each country. Furthermore, although we restrict the relation between factor loadings and characteristics to be linear, there are various approaches to incorporate non-linearity. For example, we would have chosen $\mathbf{X}_g$ to be a large set of characteristics, possibly containing suitable polynomials of some underlying characteristics, $\mathbf{X}_g^*$. Incorporating (2.3) and (2.4) into (2.2), we have that:

$$\mathbf{R}_g = \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\left(\boldsymbol{\lambda}_g^c\mathbf{1}_T' + \mathbf{F}'\right) + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\left(\boldsymbol{\lambda}_g^s\mathbf{1}_T' + \mathbf{G}_g'\right) + \mathbf{E}_g. \tag{2.5}$$

We highlight some advantages of our approach. First, we can learn about beta through $\mathbf{X}_g\Theta_g^c$ and $\mathbf{X}_g\Theta_g^s$ even when data are relatively infrequently observed (such as monthly) over short horizon (such as a year) by instrumenting characteristics of $\mathbf{X}_g$. This is a strong advantage over other factor loadings extraction methods requiring long time series or high frequency observations. Second, because we set T as a short horizon, the process in (2.5) and can be treated as a local approximation of a conditional model over a long horizon model. Third, our rolling estimation of (2.5) enables us to study the *temporal* relation of characteristics to risk. Many empirical approaches (e.g. Kelly et al. (2019), Ferson and Harvey (1999), Ghysels (1998)) construct conditional model by allowing the characteristics to change period-by-period but holding the cross-sectional relation between characteristics and risk *constant*, which might not be suitable for detecting dynamic interplay between firm characteristics and risk. By estimating (2.5) over rolling-windows, we can learn about the dynamics of $\mathbf{X}_g\Theta_g^c$ and $\mathbf{X}_g\Theta_g^s$. Lastly, we do not need to necessarily have all important characteristics for risks in (2.5). Because any information in the missing characteristics is captured by Γ_g^c and Γ_g^s, our model already incorporates the possibility of misspecifying the set of characteristics. Hence, if some important characteristics are missing, we may lose some precision but will not generate spurious results.

Note that the Arbitrage Pricing Theory (APT, Ross (1976)) implies that the compensation for the common risk should be identical across all assets in the economy of the two countries, which will become clear later. Hence, in an economy without an arbitrage, it supposes to hold that $\boldsymbol{\lambda}_1^c = \boldsymbol{\lambda}_2^c$. Allowing for a market segmentation across the two countries implies that the norm of $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$ may be nonzero:

Assumption 1. *It holds that* $(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c)' (\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c) \geq 0$.

From (2.5), note that $\mathbf{X}_1\Theta_1^c$ affects $\mathbf{R}_1$ individual stocks in country 1 by $\boldsymbol{\lambda}_1^c$ plus a realization of common factors and that $\mathbf{X}_2\Theta_2^c$ affects $\mathbf{R}_2$ individual stocks in country 2 by $\boldsymbol{\lambda}_2^c$ plus a realization of common factors. Hence, we can identify $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$ by observing the differences in the relation between $\mathbf{X}_1\Theta_1^c$ and $\mathbf{R}_1$ and that between $\mathbf{X}_2\Theta_2^c$ and $\mathbf{R}_2$. It is beyond the scope of this paper to examine the underlying cause of such a disagreement in compensation.[3] Also note that Assumption 1 does not imply that markets are always segmented. The main objective of this paper is to provide a method to detect the market segmentation if it exists. Furthermore, it will be shown that such information allows us to form portfolios that yield abnormal returns (if $(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c)' (\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c) > 0$) while not having any exposure to systematic risk both to the country-specific or common factors.

Next, we assume standard regularity conditions on the characteristics and residual returns in the two countries.

Assumption 2. *In each country* $g = 1, 2$*, as* $N_g \to \infty$*, it holds that*
(i) $\frac{\mathbf{R}_g'\mathbf{R}_g}{N_g} \xrightarrow{p} \mathbf{V}_{R_g}$ *and* $\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g} \to \mathbf{V}_{X_g}$*, where* $\mathbf{V}_{R_g}$, $\mathbf{V}_{X_g}$ *are positive definite matrices,*
(ii) $\frac{\mathbf{X}_g'\Gamma_g^c}{N_g} \xrightarrow{p} \mathbf{0}_{L_g \times K^c}$, $\frac{\mathbf{X}_g'\Gamma_g^s}{N_g} \xrightarrow{p} \mathbf{0}_{L_g \times K_g^s}$*, and* $\frac{\mathbf{X}_g'\mathbf{E}_g}{N_g} \xrightarrow{p} \mathbf{0}_{L_g \times T}$,
(iii) $\frac{N_1}{N_1+N_2} \to n_1 < 1$.

Condition (i) simply states that the cross section of returns and characteristics are not redundant but well-spread over individual stocks in each country. Condition (ii) imposes the cross-sectional orthogonality conditions between the characteristics of $\mathbf{X}_g$ factor loading regression residuals of Γ_g^c and Γ_g^s, and residual returns of $\mathbf{E}_g$ in each country. Lastly, condition (iii) imposes that the numbers of individual stocks are comparable across the two countries.

Lastly, we assume mild restrictions to separately identify factor loading matrices Θ_g^c and Θ_g^s. We introduce the $(T \times T)$ matrix $\mathbf{J}_T = \mathbf{I}_T - \frac{1}{T}\mathbf{1}_T\mathbf{1}_T'$, which corresponds to time-series demeaning, and the $(T \times T)$ matrix $\mathcal{P}^c = \mathbf{J}_T\mathbf{F}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\mathbf{F}'\mathbf{J}_T$, which corresponds to the projection to the demeaned common factors. Also, we let Θ_g and $\mathbf{H}_g$ be $\left[\Theta_g^c \ \Theta_g^s\right]$ and $\left[\mathbf{F} \ \mathbf{G}_g^s\right]$, respectively.

[3]See xxx among many for potential causes of market segmentation.

Assumption 3. *In each country $g = 1, 2$, as $N_g \to \infty$, it holds that*

(i) $\frac{\Theta_g' \mathbf{X}_g' \mathbf{X}_g \Theta_g}{N_g} = \begin{bmatrix} \frac{\Theta_g^{c\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^c}{N_g} & \frac{\Theta_g^{c\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^s}{N_g} \\ \frac{\Theta_g^{s\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^c}{N_g} & \frac{\Theta_g^{s\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^s}{N_g} \end{bmatrix} \to \begin{bmatrix} \Theta_g^{c\prime} \mathbf{V}_{X_g} \Theta_g^c & \Theta_g^{c\prime} \mathbf{V}_{X_g} \Theta_g^s \\ \Theta_g^{s\prime} \mathbf{V}_{X_g} \Theta_g^c & \Theta_g^{s\prime} \mathbf{V}_{X_g} \Theta_g^s \end{bmatrix} = \mathbf{V}_{\Theta_g}$, *where* $\mathbf{V}_{\Theta_g}$ *is a* $(K_g \times K_g)$ *positive definite matrix,*

(ii) $\frac{\Theta_g^{c\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^s}{N_g} \to \Theta_g^{c\prime} \mathbf{V}_{X_g} \Theta_g^s = \mathbf{0}_{K^c \times K_g^s}$,

(iii) $\frac{\mathbf{H}_g' \mathbf{J}_T \mathbf{H}_g}{T} = \begin{bmatrix} \frac{\mathbf{F}' \mathbf{J}_T \mathbf{F}}{T} & \frac{\mathbf{F}' \mathbf{J}_T \mathbf{H}_g}{T} \\ \frac{\mathbf{H}_g' \mathbf{J}_T \mathbf{F}}{T} & \frac{\mathbf{H}_g' \mathbf{J}_T \mathbf{H}_g}{T} \end{bmatrix} = \begin{bmatrix} \Sigma_F & \Sigma_{FH_g} \\ \Sigma_{FH_g}' & \Sigma_{H_g} \end{bmatrix} = \Sigma_{H_g}$, *where* Σ_{H_g} *is a* $(K_g \times K_g)$ *positive denifite matrix,*

(iv) $(\mathbf{I}_T - \mathcal{P}^c)\,[\mathbf{G}_1 \ \mathbf{G}_2]$ *is a full rank* $(T \times (K_1^s + K_1^s))$ *matrix.*

Condition (i) implies that each column of $\mathbf{X}_g \Theta_g$ provides non-redundant information. In a similar vein, condition (iii) posits that factor realizations are not redundant in each country. Later, these limits will explicitly appear for identification restrictions.[4] Condition (ii) restricts that $\mathbf{X}_g \Theta_g^c$ are cross-sectionally orthogonal to $\mathbf{X}_g \Theta_g^s$. This assumption is without loss of generality. If there is any correlation between $\mathbf{X}_g \Theta_g^c$ and $\mathbf{X}_g \Theta_g^s$, the correlated component can be assigned to the country-specific component $\mathbf{X}_g \Theta_g^s$ by shifting country-specific part accordingly (Connor et al. (2012) and Kim et al. (2021) utilize a similar orthogonality condition for identification between mispricing and factor loadings). Condition (iv) implies that each country specific factors across the two countries are not redudant after orthogonalizing against the common factors. This condition is natural given that country-specific factors in a country should reflect some shocks independent of the factors in the other country. Also, this property plays an important role later in separating common factors from country-specific factors.

2.1 Methodology

Our procedure for detecting market segmentation involves Projected-PCA (PPCA) procedure, projecting returns on cross-sectional information for short time-series samples, and Canonical Correlation Analysis (CCA), finding a pair of linear combination from each group (country in our framework) so as to maximize the correlation. Fan et al. (2016) show that the estimated factor loadings using such an approach converges to the

[4]Without this restriction, we cannot identify factor loadings functions because of the rotational indeterminacy of latent factor models. For example, $\mathbf{X}_g \Theta_g \mathbf{F}_g' = \mathbf{X}_g \Theta_g \mathcal{M} \mathcal{M}^{-1} \mathbf{F}_g'$ for any invertible matrix $\mathcal{M}$.

true factor loadings as the cross-sectional sample increases, even for small time-series samples. Kim et al. (2021) extend the PPCA estimator to not only estimate factors, but also the mispricing function. We propose a method to detect market segmentation by applying the PPCA estimator along with CCA estimator for a pair of countries.

We achieve the goal of detecting market segmentation in three steps. In the first step, we estimate (demeaned) common factors $\mathbf{F}'\mathbf{J}_T$ by applying Asymptotic Principal Components (APC) to demeaned projected returns in each country, (Connor and Korajczyk (1986)) and selecting the common factors from CCA. In the second step, we estimate factor loadings for the common factors $\mathbf{X}_g\Theta_g^c$. To this end, we (i) orthogonalize returns against estimated demeaned common factors $\mathbf{F}'\mathbf{J}_T$ from the first step, (ii) project the orthogonalized returns on the characteristics and apply APC to obtain the country specific factor loadings, $\mathbf{X}_g\Theta_g^s$., and (iii) finally regress demeaned returns on the product of demeaned common factors $\mathbf{J}_T\mathbf{F}$ and characteristics $\mathbf{X}_g$, revealing $\mathbf{X}_g\Theta_g^c$. In the third step, we regress the average returns of two countries, $[\mathbf{R}_1\ \mathbf{R}_2]\,\frac{\mathbf{1}_T}{T}$ on the stacked factor loadings $[\mathbf{X}_1\Theta_1^c\ \mathbf{X}_2\Theta_2^c]$ and the stacked loadings times a country-1 dummy $[\mathbf{X}_1\Theta_1^c\ \mathbf{0}_{N_2\times K^c}]$. Then, it turns out that the coefficients on the second set of regressors will consistently estimate $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$. Furthermore, if $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$ is not zero, we exploit this market segmentation across Home and Foreign countries and suggest a portfolio which delivers the profit of δ without any exposure to the systematic risks of either country.

Step 1: estimation of common factors The first step of our procedure is the estimation of $\mathbf{F}'\mathbf{J}_T$ (upto rotation) from returns in the two coutries. Recall that the observed returns in (2.5) are driven both by the risk premium $\boldsymbol{\lambda}_g^c$ and $\boldsymbol{\lambda}_g^s$, and realization of $\mathbf{F}$ and $\mathbf{H}_g$. Because we want to learn the factor realization not the risk premium in this step, we eliminate the effect of the risk premium by multiplying $\mathbf{J}_T\left(=\mathbf{I}_T - \frac{1}{T}\ \mathbf{1}_T\mathbf{1}_T\right)$, or demeaning the observed returns:

$$\begin{aligned}\mathbf{R}_g\mathbf{J}_T &= \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\left(\boldsymbol{\lambda}_g^c\mathbf{1}_T' + \mathbf{F}'\right)\mathbf{J}_T \\ &\quad + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\left(\boldsymbol{\lambda}_g^s\mathbf{1}_T' + \mathbf{G}_g'\right)\mathbf{J}_T + \mathbf{E}_g\mathbf{J}_T \\ &= \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\mathbf{F}'\mathbf{J}_T + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\mathbf{G}_g'\mathbf{J}_T + \mathbf{E}_g\mathbf{J}_T, \end{aligned} \tag{2.6}$$

where the last equality is from the property of $\mathbf{1}_T'\mathbf{J}_T = \mathbf{1}_T'\left(\mathbf{I}_T - \frac{1}{T}\ \mathbf{1}_T\mathbf{1}_T'\right) = \mathbf{1}_T' - \frac{T}{T}\mathbf{1}_T' = \mathbf{0}_T'$. For further isolation of $\mathbf{X}_g\Theta_g^c$ and $\mathbf{X}_g\Theta_g^s$, we project the demeaned returns of (2.1) on the (linear) span of $\mathbf{X}_g$ by premultiplying by the projection matrix $\mathbf{P}_g = \mathbf{X}_g \left(\mathbf{X}_g'\mathbf{X}_g\right)^{-1} \mathbf{X}_g'$. Then, we get

$$
\begin{aligned}
\widehat{\mathbf{R}}_g \equiv & \mathbf{P}_g \mathbf{R}_g \mathbf{J}_T \\
= & \left(\mathbf{P}_g \mathbf{X}_g \Theta_g^c + \mathbf{P}_g \Gamma_g^c\right) \mathbf{F}' \mathbf{J}_T + \left(\mathbf{P}_g \mathbf{X}_g \Theta_g^s + \mathbf{P}_g \Gamma_g^s\right) \mathbf{G}_g' \mathbf{J}_T + \mathbf{P}_g \mathbf{E}_g \mathbf{J}_T \\
= & \left(\mathbf{X}_g \Theta_g^c + \mathbf{P}_g \Gamma_g^c\right) \mathbf{F}' \mathbf{J}_T + \left(\mathbf{X}_g \Theta_g^s + \mathbf{P}_g \Gamma_g^s\right) \mathbf{G}_g' \mathbf{J}_T + \mathbf{P}_g \mathbf{E}_g \mathbf{J}_T,
\end{aligned} \tag{2.7}
$$

where the last equality is from $\mathbf{P}_g\mathbf{X}_g = \mathbf{X}_g$. Furthermore, exploiting the orthogonality of Γ_g^c and Γ_g^s with respect to $\mathbf{X}_g$ and the limits in Assumption 2(ii), $\mathbf{P}_g\Gamma_g^c$, $\mathbf{P}_g\Gamma_g^s$, and $\mathbf{P}_g\mathbf{E}_g$ will become negligible for large N_g. Hence, with a large N_g, it follows that $\widehat{\mathbf{R}}_g = \mathbf{P}_g\mathbf{R}_g\mathbf{J}_T \approx \left[\mathbf{X}_g\Theta_g^c \; \mathbf{X}_g\Theta_g^s\right]\left[\mathbf{F}'\mathbf{J}_T \; \mathbf{G}_g'\mathbf{J}_T\right] = \mathbf{X}_g\Theta_g\mathbf{H}_g'\mathbf{J}_T$, where $\Theta_g = \left[\Theta_g^c \; \Theta_g^s\right]$ and $\mathbf{H}_g = \left[\mathbf{F} \; \mathbf{G}_g\right]$. Finally, as in Fan et al. (2016), we estimate $\mathbf{H}_g\mathbf{J}_T$ by applying standard principal component analysis to $\widehat{\mathbf{R}}_g$.

Theorem 2.1. *Let $\widehat{\mathbf{H}}_g$ denote the $(T \times K_g)$ matrix, the k-th column of which is the eigenvector of $\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g}$ corresponding to the k-th largest eigenvalue of $\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g}$, where $\widehat{\mathbf{R}}_g$ is given by (2.7). Under Assumptions 2 and 3, as N_g increases, $\widehat{\mathbf{H}}_g \xrightarrow{p} \mathbf{J}_T\mathbf{H}_g\mathcal{O}_g$, where the $(K_g \times K_g)$ matrix $\mathcal{O}_g$ is given in Lemma A.1.*

Note that we identify the (demeaned) factors only up to rotation. We choose the matrix $\mathcal{O}_g$ so that $\mathcal{O}_g'\Sigma_{F_g}\mathcal{O}_g$ is an identity matrix and $\mathcal{O}_g^{-1}\mathbf{V}_{\Theta_g}\mathcal{O}_g^{-1\prime}$ is a diagonal matrix. To provide some intuition of the above theorem, recall that $\widehat{\mathbf{R}}_g$ converges (as $N_g \to \infty$) to $\mathbf{X}_g\Theta_g\mathbf{H}_g'\mathbf{J}_T = \mathbf{X}_g\Theta_g\mathcal{O}_g^{-1\prime}\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T$. Therefore, $\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g}$ converges to $\mathbf{J}_T\mathbf{H}_g\mathcal{O}_g\left(\mathcal{O}_g^{-1}\frac{\Theta_g'\mathbf{X}_g\mathbf{X}_g\Theta_g}{N_g}\mathcal{O}_g^{-1\prime}\right)\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T$. Recall the given property of $\mathcal{O}_g$, $\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T\mathbf{H}_g\mathcal{O}_g = \mathbf{I}_{K_g}$. Furthermore, with Assumption 3(i), $\mathcal{O}_g^{-1}\frac{\Theta_g'\mathbf{X}_g\mathbf{X}_g\Theta_g}{N_g}\mathcal{O}_g^{-1\prime}$ converges a diagonal matrix as a property of $\mathcal{O}_g$. Hence, with a large N_g, each column of $\mathbf{J}_T\mathbf{H}_g\mathcal{O}_g$ and each diagonal element of $\mathcal{O}_g^{-1}\frac{\Theta_g'\mathbf{X}_g\mathbf{X}_g\Theta_g}{N_g}\mathcal{O}_g^{-1\prime}$ can be interpreted as an eigenvector and an eigenvalue of $\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g}$, respectively. Resorting to these observations, we attempt to recover $\mathbf{J}_T\mathbf{H}_g\mathcal{O}_g$ as stated in Theorem 2.1.

Up to now, we have recovered the factors that are common across the countries and the country specific factors. In the next steps, we will separate the common from the country specific factors. The main analytical tool for this purpose is Canonical Correlation Analysis (CCA). Define the $(K_1 \times K_1)$ matrix $\widehat{\Sigma}_{H_1}$, the $(K_2 \times K_2)$ matrix $\widehat{\Sigma}_{H_2}$, and the $(K_1 \times K_2)$ matrix $\widehat{\Sigma}_{H_{12}}$ as $\widehat{\Sigma}_{H_1} = \frac{\widehat{\mathbf{H}}_1'\widehat{\mathbf{H}}_1}{T}$, $\widehat{\Sigma}_{H_2} = \frac{\widehat{\mathbf{H}}_2'\widehat{\mathbf{H}}_2}{T}$, and $\widehat{\Sigma}_{H_{12}} = \frac{\widehat{\mathbf{H}}_1'\widehat{\mathbf{H}}_2}{T}$.

Then, we propose an estimator $\widehat{\mathbf{F}}_1$, constructed as $\widehat{\mathbf{H}}_1\widehat{\mathbf{W}}_1$ where k-th column of the $(K_1 \times K^c)$ matrix $\widehat{\mathbf{W}}_1$ is the canonical directions associated with the k-th largest eigenvalues obtained from eigen-decomposition of $\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}'_{H_{12}}$ and $\widehat{\mathbf{W}}'_1\widehat{\Sigma}_{H_1}\widehat{\mathbf{W}}_1 = \mathbf{I}_{K_c}$. Similarly, we define $\widehat{\mathbf{F}}_2$ as $\widehat{\mathbf{H}}_2\widehat{\mathbf{W}}_2$ where k-th column of the $(K_2 \times K^c)$ matrix $\widehat{\mathbf{W}}_2$ is the cannonical directions associated with the k-th largest eigenvalues obtained from eigen-decomposition of $\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}'_{H_{12}}\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}$ and $\widehat{\mathbf{W}}'_2\widehat{\Sigma}_{H_2}\widehat{\mathbf{W}}_2 = \mathbf{I}_{K_c}$. The next theorem shows that we can recover the linear space generated by the demeaned common factors from either $\widehat{\mathbf{F}}_1$ or $\widehat{\mathbf{F}}_2$. [this formulation is from the group factor model by Ghysel. But, I think the singular value decomposition is simpler.]

Theorem 2.2. *Under Assumptions 2 and 3, it holds that* $\widehat{\mathbf{F}}_g \xrightarrow{p} \mathbf{J}_T\mathbf{F}\mathcal{S}$ *for some* $\mathcal{S}$ *such that* $\mathcal{S}\mathcal{S}' = \Sigma_F^{-1}$ *for* $g = 1, 2$.

Note that CCA suggests to pair a linear combination of $\widehat{\mathbf{H}}_1$ with a linear combination of $\widehat{\mathbf{H}}_2$ so that the correlation between the linear combinations is maximized. Given that both $\widehat{\mathbf{H}}_1$ and $\widehat{\mathbf{H}}_2$ asymptotically recover the linear space spanned by the common factors but that the systematic factor of one country cannot recover the country-specific factors of the other country (Assumption 3(iv)), we can filter out the country-specific factors by applying CCA to the $\widehat{\mathbf{H}}_1$ and $\widehat{\mathbf{H}}_2$. Then, we use the following corollary to pin down the estimator for common factors:

Corollary 2.1. *Under Assumptions 2 and 3, it holds* $\widehat{\mathbf{F}}\left(\equiv \frac{1}{2}\left(\widehat{\mathbf{F}}_1 + \widehat{\mathbf{F}}_2\right)\right) \xrightarrow{p} \mathbf{J}_T\mathbf{F}\mathcal{S}$ *for some* $\mathcal{S}$ *such that* $\mathcal{S}\mathcal{S}' = \Sigma_F^{-1}$.

So far, we have recovered the information on the common factors. In the next step, we estimate Θ_g^c, the factor loading matrix with respect to the common factors.

Step 2: estimation of factor loading matrices on common factors Next, we proceed to estimate Θ_g^c, the factor loading matrices with respect to the common factors. In Step 1, we introduce $\widehat{\mathbf{R}}_g \equiv \mathbf{P}_g\mathbf{R}_g\mathbf{J}_T$ and exploit the following property of $\widehat{\mathbf{R}}_g$:

$$\widehat{\mathbf{R}}_g \approx \mathbf{X}_g\Theta_g^c\mathbf{F}'\mathbf{J}_T + \mathbf{X}_g\Theta_g^s\mathbf{G}'_g\mathbf{J}_T.$$

Because we already have a consistent estimator $\widehat{\mathbf{F}}' \xrightarrow{p} \mathcal{S}'\mathbf{F}'\mathbf{J}_T$ from Step 1, we might attempt to recover the information on Θ_g^c by regressing $\widehat{\mathbf{R}}_g$ on the interaction between $\mathbf{X}_g$ and $\widehat{\mathbf{F}}'$.

However, such an approach does not accurately isolate $\mathbf{X}_g\Theta_g^c$ because of the country-specific factors. Even when the country-specific factors are orthogonal to common factors in the population, the country-specific factors can be correlated to the common factors over a short horizon. Hence, such regression confounds factor loadings to common factors with factor loadings to the country-specific factors.[5] Hence, we propose to identify Θ_g^s first and estimate Θ_g^c by imposing cross-sectional orthogonality between $\mathbf{X}_g\Theta_g^s$ and $\mathbf{X}_g\Theta_g^c$ (Assumption 3(ii)).

For that reason, we eliminate the terms related to common factors by projecting $\widehat{\mathbf{R}}_g$ to the space orthogonal to $\widehat{\mathbf{F}}$:

$$\begin{aligned}\widetilde{\mathbf{R}}_g &\equiv \widehat{\mathbf{R}}_g\left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) && (2.8)\\ &\approx \mathbf{X}_g\Theta_g^c\mathbf{F}'\mathbf{J}_T\left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) + \mathbf{X}_g\Theta_g^s\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right)\\ &\approx \mathbf{X}_g\Theta_g^s\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) && (2.9)\end{aligned}$$

where $\widehat{\mathcal{P}}^c = \widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}'\widehat{\mathbf{F}}\right)^{-1}\widehat{\mathbf{F}}'$. Then, given that $\widehat{\mathbf{F}} \xrightarrow{p} \mathbf{J}_T\mathbf{F}\mathcal{S}$ from Step 1, it follows $\widehat{\mathcal{P}}^c \xrightarrow{p} \mathcal{P}^c = \mathbf{J}_T\mathbf{F}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\mathbf{F}'\mathbf{J}_T$. Hence, the first term of RHS in (2.8) will become negligible and it holds that $\widetilde{\mathbf{R}}_g \approx \mathbf{X}_g\Theta_g^s\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)$. Then, we estimate Θ_g^s by utilizing principal component analysis to $\widetilde{\mathbf{R}}_g$.

Theorem 2.3. *Let $\widehat{\mathbf{D}}_g$ denote the $\left(N_g \times K_g^s\right)$ matrix, the k-th column of which is $\sqrt{N_g}$ times the eigenvector of $\frac{\widetilde{\mathbf{R}}_g\widetilde{\mathbf{R}}_g'}{N_g}$ corresponding to the k-th largest eigenvalue of $\frac{\widetilde{\mathbf{R}}_g\widetilde{\mathbf{R}}_g'}{N_g}$, where $\widetilde{\mathbf{R}}_g$ is given by (2.8). Define $\widehat{\Theta}_g^s$ as $\widehat{\Theta}_g^s = \left(\mathbf{X}_g'\mathbf{X}_g\right)^{-1}\left(\mathbf{X}_g'\widehat{\mathbf{D}}_g\right)$. Under Assumptions 2 and 3, it holds that $\widehat{\Theta}_g^s \xrightarrow{p} \Theta_g^s\mathcal{D}_g$, where $\mathcal{D}_g$ is given by Lemma A.7.*

The intuition of the above theorem is similar to that of Theorem 2.1. The matrix $\mathcal{D}_g$ in Lemma A.7 is designed to make $\mathcal{D}_g'\left(\frac{\Theta_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\Theta_g^s}{N_g}\right)\mathcal{D}_g$ converge to an identity matrix and to let $\mathcal{D}_g^{-1}\left(\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\right)\mathcal{D}_g^{-1\prime}$ be a diagonal matrix. Recall that $\widetilde{\mathbf{R}}_g$ converges (as $N_g \to \infty$) to $\left(\mathbf{X}_g\Theta_g^s\mathcal{D}_g\right)\left(\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\right)$. Therefore, as $N_g \to \infty$ $\frac{\widetilde{\mathbf{R}}_g\widetilde{\mathbf{R}}_g'}{N_g}$ converges to $\left(\frac{\mathbf{X}_g\Theta_g^s}{\sqrt{N_g}}\mathcal{D}_g\right)\left(\mathcal{D}_g^{-1}\left(\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\right)\mathcal{D}_g^{-1\prime}\right)\left(\frac{\mathbf{X}_g\Theta_g^s}{\sqrt{N_g}}\mathcal{D}_g\right)'$. From Assumptions 3(i) and a property of $\mathcal{D}_g$, $\mathcal{D}_g'\frac{\Theta_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\Theta_g^s}{N_g}\mathcal{D}_g \to \mathbf{I}_{K_g^s}$, so each column of $\frac{\mathbf{X}_g\Theta_g^s}{\sqrt{N_g}}\mathcal{D}_g$ can be treated as an eigenvector. Hence, the eigenvector of $\frac{\widetilde{\mathbf{R}}_g\widetilde{\mathbf{R}}_g'}{N_g}$ recovers

[5] More precisely, we need the orthogonality between $(\mathbf{X}_g \otimes \mathbf{F})$ and $(\mathbf{X}_g \otimes \mathbf{G}_g)$.

$\frac{\mathbf{X}_g\Theta_g^s}{\sqrt{N_g}}\mathcal{D}_g$, which in turn justifies rescaling the eigenvector by $\sqrt{N_g}$ to obtain $\widehat{\mathbf{D}}_g$ and regressing $\mathbf{X}_g$ on $\widehat{\mathbf{D}}_g$ for $\widehat{\Theta}_g^s$.

Next, we proceed to estimate Θ_g^c. Given the consistent estimator $\Theta_g^s\mathcal{D}_g$ from Theorem 2.3 and the orthogonality between $\mathbf{X}_g\Theta_g^c$ and $\mathbf{X}_g\Theta_g^s$ given by Assumption 3(ii), we identify Θ_g^c by regressing demeaned returns on the interaction between $\mathbf{X}_g$ and $\widehat{\mathbf{F}}$ such that the estimated $\mathbf{X}_g\Theta_g^c$ is cross-sectionally orthogonal to the estimated $\mathbf{X}_g\widehat{\Theta}_g^s$. The following theorem establishes that we can recover Θ_g^c.

Theorem 2.4. *Let the $(L_g \times K^c)$ matrix of $\widehat{\Theta}_g^c$ be the solution of the following constrained optimization problem:*

$$\widehat{\Theta}_g^c = \arg\min_{\Theta_g^c} \|\mathbf{R}_g\mathbf{J}_T - \left(\mathbf{X}_g\Theta_g^c\right)\widehat{\mathbf{F}}'\| \tag{2.10}$$
$$\textit{subject to} \quad \widehat{\Theta}_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\Theta_g^c = \mathbf{0}_{K_g^s\times K^c},$$

where $\widehat{\Theta}_g^s$ is given by Theorem 2.3. Then, under Assumptions 2 and 3, it holds that $\widehat{\Theta}_g^c \xrightarrow{p} \Theta_g^c\mathcal{S}'^{-1}$ for some $\mathcal{S}$ such that $\mathcal{S}\mathcal{S}' = \Sigma_F^{-1}$.

The problem in the above theorem can be transformed into a conventional ordinary least square problem with linear equality constraints and the closed form solution is easily obtained.

Step 3: detecting the market segmentation In the third step, we detect the market segmentation by revealing whether $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$ is zero or not. In contrast to the previous two steps which eliminate risk premia by demeaning, we need to keep the information of risk premia and hence take the time-series average of returns. Define a $(m\times 1)$ vector $\overline{\mathbf{A}}$ and a $(1\times m)$ vector $\overline{\mathbf{B}}$ as $\frac{1}{T}\mathbf{A}\mathbf{1}_T$ for a $(m\times T)$ matrix $\mathbf{A}$ and $\frac{1}{T}\mathbf{1}_T'\mathbf{B}$ for a $(T\times m)$ matrix $\mathbf{B}$, respectively. Let $\overline{\mathbf{R}}$ be the $(N_1+N_2)\times 1$ vector of $\left[\overline{\mathbf{R}}_1' \; \overline{\mathbf{R}}_2'\right]'$. Note that $\overline{\mathbf{R}}_g$ can be rewritten as

$$\begin{aligned}\overline{\mathbf{R}}_g &= \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\left(\boldsymbol{\lambda}_g^c + \overline{\mathbf{F}}'\right) + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\left(\boldsymbol{\lambda}_g^s + \overline{\mathbf{G}}_g'\right) + \overline{\mathbf{E}}_g,\\ &= \mathbf{X}_g\Theta_g^c\mathcal{S}'^{-1}\mathcal{S}'\left(\boldsymbol{\lambda}_g^c + \overline{\mathbf{F}}'\right) + \Gamma_g,\end{aligned} \tag{2.11}$$

where $\mathbf{u}_g = \Gamma_g^c\left(\boldsymbol{\lambda}_g^c + \overline{\mathbf{F}}'\right) + \Gamma_g^s\left(\boldsymbol{\lambda}_g^s + \overline{\mathbf{G}}_g'\right) + \overline{\mathbf{E}}_g + \mathbf{X}_g\Theta_g^s\left(\boldsymbol{\lambda}_g^s + \overline{\mathbf{G}}_g'\right)$. From Assumption 2(ii), $\mathbf{X}_g\Theta_g^c$ is orthogonal to the first three terms of $\mathbf{u}_g$. Furthermore, Assumption

3(ii) confirms the orthogonality of $\mathbf{X}_g\Theta_g^c$ against the last term of $\mathbf{u}_g$. By stacking the expressions of (2.11) over the two countries, we have the

$$\begin{aligned}\overline{\mathbf{R}} = \begin{bmatrix} \overline{\mathbf{R}}_1 \\ \overline{\mathbf{R}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{X}_1\Theta_1^c\mathcal{S}'^{-1}\mathcal{S}'\left(\boldsymbol{\lambda}_1^c + \overline{\mathbf{F}}'\right) \\ \mathbf{X}_2\Theta_2^c\mathcal{S}'^{-1}\mathcal{S}'\left(\boldsymbol{\lambda}_2^c + \overline{\mathbf{F}}'\right) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{X}_1\Theta_1^c\mathcal{S}'^{-1} \\ 0_{N_2\times K_c} \end{bmatrix}\mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) + \begin{bmatrix} \mathbf{X}_1\Theta_1^c\mathcal{S}'^{-1} \\ \mathbf{X}_2\Theta_2^c\mathcal{S}'^{-1} \end{bmatrix}\mathcal{S}'\left(\boldsymbol{\lambda}_2^c + \overline{\mathbf{F}}'\right) + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}.\end{aligned}$$

As described before, the residual term of $[\Gamma_1'\ \Gamma_2']'$ is orthogonal to the infeasible regressors of $\begin{bmatrix} \mathbf{X}_1\Theta_1^c\mathcal{S}'^{-1} & \mathbf{X}_1\Theta_1^c\mathcal{S}'^{-1} \\ 0_{N_2\times K_c} & \mathbf{X}_2\Theta_2^c\mathcal{S}'^{-1} \end{bmatrix}$. Hence, if we have the true values of $\Theta_1^c\mathcal{S}^{-1\prime}$ and $\Theta_1^c\mathcal{S}^{-1\prime}$, we can consistently estimate the coefficients $\mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)$. Instead, we have $\widehat{\Theta}_1^c \xrightarrow{p} \Theta_1^c\mathcal{S}$ and $\widehat{\Theta}_2^c \xrightarrow{p} \Theta_2^c\mathcal{S}$ from Theorem 2.4. The next theorem shows that we can learn about the differences in the risk premia $\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c$ by exploiting the feasible estimators $\widehat{\Theta}_g^c$ from Step 2.

Theorem 2.5. *Define* $\widehat{\boldsymbol{\lambda}}_\Delta = [\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K^c}]\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}'\overline{\mathbf{R}}$, *where the* $(N_1+N_2)\times(2K^c)$ *matrix* $\widehat{\mathbf{B}}$ *is given by* $\begin{bmatrix} \mathbf{X}_1\widehat{\Theta}_1^c & \mathbf{X}_1\widehat{\Theta}_1^c \\ 0_{N_2\times K_c} & \mathbf{X}_2\widehat{\Theta}_2^c \end{bmatrix}$. *Then, under Assumptions 2 and 3, it holds that* $\widehat{\boldsymbol{\lambda}}_\Delta \xrightarrow{p} \mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)$.

The above theorem is the punchline of this paper: an investor can identify the market segmentation by regressing the average returns of the two countries countries $\overline{\mathbf{R}}$ on $\widehat{\mathbf{B}}$, constructed by estimated factor loadings to common factors, and selecting the first K^c estimated coefficients.

Lastly, we justify our market segmentation test from the perspective of APT by confirming that once an investor identifies the condition of $\boldsymbol{\lambda}_1^c \neq \boldsymbol{\lambda}_2^c$, s/he can form a portfolio which delivers $\delta = \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)'\Sigma_F^{-1}\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)$ without any exposure to risks.

Theorem 2.6. *Define a portfolio vector* $\widehat{\mathbf{w}}$ *as* $\widehat{\mathbf{w}} = \widehat{\boldsymbol{\lambda}}_\Delta'[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K^c}]\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}'$, *where the* $(K^c\times 1)$ *vector of* $\widehat{\boldsymbol{\lambda}}_\Delta$ *and the* $((N_1+N_2)\times(2K^c))$ *matrix* $\widehat{\mathbf{B}}$ *are given in Theorem 2.5 and* $\mathbf{R} = [\mathbf{R}_1'\ \mathbf{R}_2']'$. *Then, under Assumptions 2 and 3, it holds that* $\widehat{\mathbf{w}}\mathbf{R} \xrightarrow{p} \delta\mathbf{1}_T'$.

The above theorem reveals some potentially practical benefits from identifying market segmentation. An investor can consistently recover the positive profits, should they exist, as the number of securities grows large. Note that the construction of this portfolio does not require large T. Hence, we can estimate $\mathbf{w}$ over one sample and calculate out-of-sample returns over a subsequent sample.

A Proofs

Lemma A.1. *Let* $\mathbf{L}_g$ *be a lower triangular matrix such that* $\mathbf{V}_{\Theta_g} = \mathbf{L}_g\mathbf{L}_g'$*. From the eigendecompotion of* $\mathbf{L}_g'\Sigma_{H_g}\mathbf{L}_g$*, find* $\mathbf{U}_g$ *such that* $\mathbf{L}_g'\Sigma_{H_g}\mathbf{L}_g = \mathbf{U}_g\mathbf{D}\mathbf{U}_g'$*, where* $\mathbf{D}$ *is a diagonal matrix and* $\mathbf{U}_g'\mathbf{U}_g = \mathbf{I}_{K_g}$*. Define* $\mathcal{O}_g$ *as* $\mathbf{L}_g\mathbf{U}_g$*. Then, it holds*
(i) $\mathcal{O}_g^{-1}\mathbf{V}_{\Theta_g}\mathcal{O}_g^{-1\prime}$ *is an identity matrix, and*
(ii) $\mathcal{O}_g'\Sigma_{H_g}\mathcal{O}_g$ *is a diagonal matrix.*

Proof First, we show (i) $\mathcal{O}_g^{-1}\mathbf{V}_{\Theta_g}\mathcal{O}_g^{-1\prime}$ is an identity matrix. Note that $\mathcal{O}_g^{-1} = \mathbf{U}_g^{-1}\mathbf{L}_g^{-1} = \mathbf{U}_g'\mathbf{L}_g^{-1}$, which in turn gives,

$$\begin{aligned}\mathcal{O}_g^{-1}\mathbf{V}_{\Theta_g}\mathcal{O}_g^{-1\prime} &= \mathbf{U}_g'\mathbf{L}_g^{-1}\mathbf{V}_{\Theta_g}\mathbf{L}_g^{-1\prime}\mathbf{U}_g = \mathbf{U}_g'\mathbf{L}_g^{-1}\mathbf{L}_g\mathbf{L}_g'\mathbf{L}_g^{-1\prime}\mathbf{U}_g \\ &= \mathbf{U}_g'\mathbf{U}_g = \mathbf{I}_{K_g},\end{aligned}$$

where the last equality is from the property of $\mathbf{U}_g$.

We move to the next claim. Note that

$$\mathcal{O}_g'\Sigma_{H_g}\mathcal{O}_g = \mathbf{U}_g'\mathbf{L}_g'\Sigma_{H_g}\mathbf{L}_g\mathbf{U}_g = \mathbf{U}_g'\mathbf{U}_g\mathbf{D}\mathbf{U}_g'\mathbf{U}_g = \mathbf{D},$$

where the second equality is from the eigendecompotion of $\mathbf{L}_g'\Sigma_{H_g}\mathbf{L}_g$ and the last equality is from the property of $\mathbf{U}_g$. This completes the proof of the lemma. □

Proof of Theorem 2.1 The following three steps complete the proof.

Step 1. $\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g} \xrightarrow{p} \mathbf{J}_T\mathbf{H}_g\mathbf{V}_{\Theta_g}\mathbf{H}_g'\mathbf{J}_T$: From (2.7), we have that

$$\widehat{\mathbf{R}}_g = j_1 + j_2 + j_3 + j_4, \tag{A.1}$$

where $j_1 = \mathbf{X}_g\Theta_g\mathbf{H}_g'\mathbf{J}_T$, $j_2 = \mathbf{P}_g\Gamma_g^c\mathbf{F}'\mathbf{J}_T$, $j_3 = \mathbf{P}_g\Gamma_g^s\mathbf{G}_g'\mathbf{J}_T$, and $j_4 = \mathbf{P}_g\mathbf{E}_g\mathbf{J}_T$. Then,

$$\frac{\widehat{\mathbf{R}}_g'\widehat{\mathbf{R}}_g}{N_g} = \sum_{k,l=1,2,3,4}\frac{j_k'j_l}{N_g}. \tag{A.2}$$

Note that

$$\frac{j_1'j_1}{N_g} = \mathbf{J}_T\mathbf{H}_g\frac{\Theta_g'\mathbf{X}_g'\mathbf{X}_g\Theta_g}{N_g}\mathbf{H}_g'\mathbf{J}_T \xrightarrow{p} \mathbf{J}_T\mathbf{H}_g\mathbf{V}_{\Theta_g}\mathbf{H}_g'\mathbf{J}_T, \tag{A.3}$$

where the limit is from Assumption 3(i), and that

$$\frac{j_2'j_2}{N_g} = \frac{\mathbf{J}_T\mathbf{F}\Gamma_g^{c\prime}\mathbf{P}_g\Gamma_g^c\mathbf{F}'\mathbf{J}_T}{N_g} = \mathbf{J}_T\mathbf{F}\frac{\Gamma_g^{c\prime}\mathbf{X}_g}{N_g}\left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\right)^{-1}\frac{\mathbf{X}_g'\Gamma_g^c}{N_g}\mathbf{F}'\mathbf{J}_T \xrightarrow{p} \mathbf{0}_{T\times T}, \tag{A.4}$$

where the limit is from Assumptions 2(i) and 2(ii), and that

$$\frac{j_3'j_3}{N_g} = \frac{\mathbf{J}_T\mathbf{G}_g\Gamma_g^{s\prime}\mathbf{P}_g\Gamma_g^s\mathbf{G}_g'\mathbf{J}_T}{N_g} = \mathbf{J}_T\mathbf{G}_g\frac{\Gamma_g^{s\prime}\mathbf{X}_g}{N_g}\left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\right)^{-1}\frac{\mathbf{X}_g'\Gamma_g^s}{N_g}\mathbf{G}_g'\mathbf{J}_T \xrightarrow{p} \mathbf{0}_{T\times T}, \tag{A.5}$$

where the limit is from Assumptions 2(i) and 2(ii), and that

$$\frac{j_4'j_4}{N_g} = \frac{\mathbf{J}_T\mathbf{E}_g'\mathbf{P}_g\mathbf{E}_g\mathbf{J}_T}{N_g} = \mathbf{J}_T\frac{\mathbf{E}_g'\mathbf{X}_g}{N_g}\left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\right)^{-1}\frac{\mathbf{X}_g'\mathbf{E}_g}{N_g}\mathbf{J}_T \xrightarrow{p} \mathbf{0}_{T\times T}, \tag{A.6}$$

where the limit is from Assumptions 2(i) and 2(ii). From (A.3)-(A.6) and the submultiplicativity of Frobenius norm, we have that

$$\|\frac{j_1'j_l}{N_g}\| \le \|\frac{j_1}{\sqrt{N_g}}\|\|\frac{j_l}{\sqrt{N_g}}\| = \sqrt{\mathrm{tr}\left(\frac{j_1'j_1}{N_g}\right)\mathrm{tr}\left(\frac{j_l'j_l}{N_g}\right)} \xrightarrow{p} 0 \tag{A.7}$$

for $l = 2, 3, 4$ and that

$$\|\frac{j_k'j_l}{N_g}\| \le \|\frac{j_k}{\sqrt{N_g}}\|\|\frac{j_l}{\sqrt{N_g}}\| = \sqrt{\mathrm{tr}\left(\frac{j_k'j_k}{N_g}\right)\mathrm{tr}\left(\frac{j_l'j_l}{N_g}\right)} \xrightarrow{p} 0 \tag{A.8}$$

for $k, l = 2, 3, 4$. By plugging (A.3)-(A.8) into (A.2), we confirm the claim of Step 1.

Step 2. The k-th column of $\mathbf{J}_T\mathbf{H}_g\mathcal{O}_g$, where $\mathbf{J}_T\mathbf{H}_g$ is given by Lemma A.1, is the k-th eigenvector of $\mathbf{J}_T\mathbf{H}_g\mathbf{V}_{\Theta_g}\mathbf{H}_g'\mathbf{J}_T$: Note that

$$\mathbf{J}_T\mathbf{H}_g\mathbf{V}_{\Theta_g}\mathbf{H}_g'\mathbf{J}_T = \mathbf{J}_T\mathbf{H}_g\mathcal{O}_g\left(\mathcal{O}_g^{-1}\mathbf{V}_{\Theta_g}\mathcal{O}_g^{-1\prime}\right)\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T. \tag{A.9}$$

Given the properties of $\mathcal{O}_g$ in Lemma A.1, the claim directly follows. Step 3. $\widehat{\mathbf{H}}_g \xrightarrow{p} \mathbf{J}_T\mathbf{H}_g\mathcal{O}_g$: The claim holds due to the continuity of the eigendecomposition. This completes the proof of the theorem. □

Lemma A.2. *Define* $\widehat{\Sigma}_{H_g}$ *and* $\widehat{\Sigma}_{H_{12}}$ *as* $\frac{\mathbf{H}_g'\mathbf{J}_T\mathbf{H}_g}{T}$ *and* $\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}$*, respectively, where* $\mathbf{H}_g$ *is given in Theorem 2.1. Let* $\widehat{\mathbf{L}}_{H_g}$ *be a lower triangular matrix such that* $\widehat{\Sigma}_{H_g} = \widehat{\mathbf{L}}_{H_g}\widehat{\mathbf{L}}_{H_g}'$ *for* $g = 1, 2$*. From the singular value decomposition of* $\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1}$*, we have* $\widehat{\mathbf{V}}_1$*,* $\widehat{\mathbf{V}}_2$*, and* $\widehat{\Sigma}$ *such that*

$$\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1} = \widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\mathbf{V}}_2', \tag{A.10}$$

where $\widehat{\mathbf{V}}_1'\widehat{\mathbf{V}}_1 = \mathbf{I}_{K_1}$*,* $\widehat{\mathbf{V}}_2'\widehat{\mathbf{V}}_2 = \mathbf{I}_{K_2}$*, and* $\widehat{\Sigma}$ *is a rectangular diagonal matrix with nonnegative diagonal elements in a decreasing order. Let* $\widehat{\Sigma}_{kk}$ *be the* k*-th diagonal element of* $\widehat{\Sigma}$*. Then, it holds that*

(i) the k*-th column of the* $(K_1 \times K^c)$ *matrix* $\widehat{\mathbf{W}}_1$ *is the* k*-th column of* $\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_1^s}\right]'$ *and the* k*-th lagest eigenvalue of* $\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}_{H_{12}}'$ *is* $\left(\widehat{\Sigma}_{kk}\right)^2$*,*

(ii) the k*-th column of the* $(K_2 \times K^c)$ *matrix* $\widehat{\mathbf{W}}_2$ *is the* k*-th column of* $\widehat{\mathbf{L}}_{H_2}'^{-1}\widehat{\mathbf{V}}_2\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_2^s}\right]'$ *and the* k*-th lagest eigenvalue of* $\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}_{H_{12}}'\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}$ *is* $\left(\widehat{\Sigma}_{kk}\right)^2$*.*

Proof Recall that $\widehat{\mathbf{W}}_1$ is defined by (a) the k-th column of $\widehat{\mathbf{W}}_1$ is the eigenvector of $\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\Sigma}_{H_2}\widehat{\Sigma}_{H_{12}}'$ corresponding to the k-th largest eigenvalues obtained and (b) $\widehat{\mathbf{W}}_1'\widehat{\Sigma}_{H_1}\widehat{\mathbf{W}}_1 = \mathbf{I}_{K_c}$.

From $\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1} = \widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\mathbf{V}}_2'$, we have that

$$\left(\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1}\right)\left(\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1}\right)' = \left(\widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\mathbf{V}}_2'\right)\left(\widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\mathbf{V}}_2'\right)',$$

which, in conjunction with $\widehat{\mathbf{V}}_2'\widehat{\mathbf{V}}_2 = \mathbf{I}_{K_2}$, gives

$$\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1}\widehat{\mathbf{L}}_{H_2}^{-1}\widehat{\Sigma}_{H_{12}}'\widehat{\mathbf{L}}_{H_1}'^{-1} = \widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\Sigma}'\widehat{\mathbf{V}}_1',$$

yielding, along with $\widehat{\mathbf{V}}_1'\widehat{\mathbf{V}}_1 = \mathbf{I}_{K_1}$, that

$$\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}_{H_{12}}'\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1 = \widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\Sigma}'\widehat{\mathbf{V}}_1'\widehat{\mathbf{V}}_1 = \widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1\widehat{\Sigma}\widehat{\Sigma}',$$

This shows that the k-th column of $\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1$ is the eigenvector of $\widehat{\Sigma}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\Sigma}_{H_2}^{-1}\widehat{\Sigma}_{H_{12}}'$ corresponding to the eigenvalue in the k-th diagonal element of $\widehat{\Sigma}\widehat{\Sigma}'$. This confirms the condition (a) for $\widehat{\mathbf{W}}$. The condition (b) is satisfied due to that $\widehat{\mathbf{V}}_1'\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_1}\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1 =$

$\widehat{\mathbf{V}}_1'\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\mathbf{L}}_{H_1}\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{L}}_{H_1}'^{-1}\widehat{\mathbf{V}}_1 = \widehat{\mathbf{V}}_1'\widehat{\mathbf{V}}_1 = \mathbf{I}_{K_1}$. Hence, the claim (i) holds.

The claim (ii) follows due to symmetry. This completes the proof of the lemma. □

The next lemma reveals the N-limit behavior of CCA in our setup.

Lemma A.3. *Let $\mathbf{V}_g$ and $\mathbf{L}_{H_g}$ be the N-limits of $\widehat{\mathbf{V}}_g$ and $\widehat{\mathbf{L}}_{H_g}$, respectively, where $\widehat{\mathbf{V}}_g$ and $\widehat{\mathbf{L}}_{H_g}$ are given in Lemma A.2. Let $\mathbf{L}_F$ be a lower triangular matrix such that $\frac{\mathbf{F}'\mathbf{J}_T\mathbf{F}}{T} = \mathbf{L}_F\mathbf{L}_F'$. Then, it holds that*

$$\mathbf{V}_g \left[\mathbf{I}_{K^c} \ \mathbf{0}_{K^c\times K_g^s}\right]' = \mathbf{L}_{H_g}'\mathcal{O}_g^{-1}\left[\mathbf{I}_{K^c} \ \mathbf{0}_{K^c\times K_g^s}\right]'\mathbf{L}_F'^{-1}\mathcal{R}$$

for a rotation matrix $\mathcal{R}$.

Proof From Theorem 2.1, we have that

$$\widehat{\mathbf{L}}_{H_1}^{-1}\widehat{\Sigma}_{H_{12}}\widehat{\mathbf{L}}_{H_2}'^{-1} \xrightarrow{p} \mathbf{L}_{H_1}^{-1}\mathcal{O}_1'\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\mathcal{O}_2\mathbf{L}_{H_2}'^{-1}. \tag{A.11}$$

From the continuity of singular value decomposition, it also holds that

$$\mathbf{L}_{H_1}^{-1}\mathcal{O}_1'\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\mathcal{O}_2\mathbf{L}_{H_2}'^{-1} = \mathbf{V}_1\Sigma\mathbf{V}_2', \tag{A.12}$$

where $\mathbf{V}_1'\mathbf{V}_1 = \mathbf{I}_{K_1}$, $\mathbf{V}_2'\mathbf{V}_2 = \mathbf{I}_{K_2}$, and Σ is a rectangular diagonal matrix with nonnegative diagonal elements in a non-increasing order.

Note that the elements in the diagonal matrix $\widehat{\Sigma}$ reveal the correlations. Hence, because $\mathbf{F}$ is in both $\mathbf{H}_1$ and $\mathbf{H}_2$ and country-specific factors are not redundant (Assumption 3(iv)), it holds that $\Sigma_{kk} = 1$ for $k = 1, \cdots, K^c$ and $\Sigma_{kk} < 1$ for $k > K^c$.

Define $\mathcal{Y}_g$ as follows:

$$\mathcal{Y}_g = \mathbf{L}_{H_g}'\mathcal{O}_g^{-1}\left[\mathbf{I}_{K^c} \ \mathbf{0}_{K^c\times K_g^s}\right]'\mathbf{L}_F'^{-1}\mathcal{R}. \tag{A.13}$$

Then, the claim of the lemma will be shown by confirming that

$$\mathcal{Y}_g'\mathcal{Y}_g = \mathbf{I}_{K^c} \text{ for } g = 1, 2, \tag{A.14}$$

$$\mathcal{Y}_1'\mathbf{L}_{H_1}^{-1}\mathcal{O}_1'\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\mathcal{O}_2\mathbf{L}_{H_2}'^{-1}\mathcal{Y}_2 = \mathbf{I}_{K^c}. \tag{A.15}$$

First, note that

$$
\begin{aligned}
\mathcal{Y}_g'\mathcal{Y}_g &= \mathcal{R}'\mathbf{L}_F^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]'\mathcal{O}_g^{-1\prime}\mathbf{L}_{H_g}\mathbf{L}_{H_g}'\mathcal{O}_g^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]'\mathbf{L}_F'^{-1}\mathcal{R} \\
&= \mathcal{R}'\mathbf{L}_F^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]'\mathcal{O}_g^{-1\prime}\mathcal{O}_g'\left(\frac{\mathbf{H}_g'\mathbf{J}_T\mathbf{H}_g}{T}\right)\mathcal{O}_g\mathcal{O}_g^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]'\mathbf{L}_F'^{-1}\mathcal{R} \\
&= \mathcal{R}'\mathbf{L}_F^{-1}\left(\frac{\mathbf{F}'\mathbf{J}_T\mathbf{F}}{T}\right)\mathbf{L}_F'^{-1}\mathcal{R} = \mathcal{R}'\mathbf{L}_F^{-1}\mathbf{L}_F\mathbf{L}_F'\mathbf{L}_F'^{-1}\mathcal{R} = \mathcal{R}'\mathcal{R} = \mathbf{I}_{K^c},
\end{aligned}
$$

which confirms (A.14).

Next, note that

$$
\begin{aligned}
&\mathcal{Y}_1'\mathbf{L}_{H_1}^{-1}\mathcal{O}_1'\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\mathcal{O}_2\mathbf{L}_{H_2}'^{-1}\mathcal{Y}_2 \\
=&\mathcal{Y}_1'\mathbf{L}_{H_1}^{-1}\mathcal{O}_1'\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\mathcal{O}_2\mathbf{L}_{H_2}'^{-1}\mathcal{Y}_2 \\
=&\mathcal{R}'\mathbf{L}_F^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_1^s}\right]\left(\frac{\mathbf{H}_1'\mathbf{J}_T\mathbf{H}_2}{T}\right)\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_2^s}\right]'\mathbf{L}_F'^{-1}\mathcal{R} \\
=&\mathcal{R}'\mathbf{L}_F^{-1}\left(\frac{\mathbf{F}'\mathbf{J}_T\mathbf{F}}{T}\right)\mathbf{L}_F'^{-1}\mathcal{R} = \mathcal{R}'\mathcal{R} = \mathbf{I}_{K^c},
\end{aligned}
$$

which confirms (A.15). This completes the proof of the lemma. □

Proof of Theorem 2.2 Note that

$$
\begin{aligned}
\widehat{\mathbf{F}}_g' = \widehat{\mathbf{W}}_g'\widehat{\mathbf{H}}_g' &= \left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]\widehat{\mathbf{V}}_g'\widehat{\mathbf{L}}_{H_g}'^{-1}\widehat{\mathbf{H}}_g' \\
&\xrightarrow{p} \left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]\mathbf{V}_g'\mathbf{L}_{H_g}'^{-1}\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T \\
&= \mathcal{R}'\mathbf{L}_F^{-1}\left[\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K_g^s}\right]'\mathcal{O}_g'^{-1}\mathcal{O}_g'\mathbf{H}_g'\mathbf{J}_T = \mathcal{R}'\mathbf{L}_F^{-1}\mathbf{F}'\mathbf{J}_T,
\end{aligned}
$$

where the second equality is from Lemma A.2, the limit is from Lemma A.3 and Theorem 2.1 and the third equality is from Lemma A.3.

Finally, set $\mathcal{S}' = \mathcal{R}'\mathbf{L}_F^{-1}$. Note that $\mathcal{S}\mathcal{S}' = \mathbf{L}_F'^{-1}\mathcal{R}\mathcal{R}'\mathbf{L}_F^{-1} = \mathbf{L}_F'^{-1}\mathbf{L}_F^{-1} = \Sigma_F^{-1}$. This completes the proof of the theorem. □

The following several lemmas are useful for the proof of Theorem 2.3.

Lemma A.4. *Under Assumptions 2 and 3, as N_g increases, $\widehat{\mathcal{P}}^c \xrightarrow{p} \mathcal{P}^c$, where $\widehat{\mathcal{P}}^c = \widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}'\widehat{\mathbf{F}}\right)^{-1}\widehat{\mathbf{F}}$ and $\mathcal{P}^c = \mathbf{J}_T\mathbf{F}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\mathbf{F}'\mathbf{J}_T$.*

Proof The claim directly follows from $\widehat{\mathbf{F}} \xrightarrow{p} \mathbf{J}_T\mathbf{F}\mathcal{S}$ from Theorem 2.1. This completes the proof of the lemma. $\square$

Lemma A.5. *Let* $\mathbf{Y}$ *be a* $(N \times T)$ *matrix. Assume that the first* K *eigenvalues of* $\mathbf{Y}'\mathbf{Y}$ *are distinct and strictly positive. Define* $\widehat{\mathbf{F}}$ *and* $\mathbf{D}$ *such that the k-th column of the* $(N \times K)$ *matrix* $\widehat{\mathbf{F}}$ *is the eigenvector of* $\mathbf{Y}'\mathbf{Y}$ *corresponding to the k-th largest eigenvalue of* $\mathbf{Y}'\mathbf{Y}$ *and the k-th diagonal element of the* $(K \times K)$ *diagonal matrix* $\mathbf{D}$ *is the k-th largest eigenvalue of* $\mathbf{Y}'\mathbf{Y}$*. Define* the $(N \times K)$ *matrix* $\widehat{\boldsymbol{\Lambda}}$ *such that the k-th column of* $\widehat{\boldsymbol{\Lambda}}$ *is the eigenvector of* $\mathbf{Y}\mathbf{Y}'$ *corresponding to the k-th largest eigenvalue of* $\mathbf{Y}\mathbf{Y}'$*. Let* $\widetilde{\boldsymbol{\Lambda}} = \mathbf{Y}\widetilde{\mathbf{F}}\left(\widetilde{\mathbf{F}}'\widetilde{\mathbf{F}}\right)^{-1}$*, where* $\widetilde{\mathbf{F}} = \widehat{\mathbf{F}}\mathbf{D}^{1/2}$*. Then, it holds that*

$$\widehat{\boldsymbol{\Lambda}} = \widetilde{\boldsymbol{\Lambda}}.$$

Proof This lemma is identical to Lemma 1 of Kim et al. (2021). $\square$

Lemma A.6. *Consider the* $\left(T \times K_g^s\right)$ *matrix* $\widehat{\mathbf{G}}_g$ *and the* $\left(K_g^s \times K_g^s\right)$ *diagonal matrix* Δ_g *such that the k-th column of* $\widehat{\mathbf{G}}_g$ *is the eigenvector of* $\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g}$ *corresponding to the k-th largest eigenvalue, which is the k-th diagonal element of* Δ_g*.*

Define $\widetilde{\mathbf{D}}_g = \widetilde{\mathbf{R}}_g\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}$*, where* $\widetilde{\mathbf{G}}_g = \widehat{\mathbf{G}}_g\Delta_g^{1/2}$*. Then, it holds that*

(i) $\widehat{\mathbf{D}}_g = \widetilde{\mathbf{D}}_g$

(ii) $\mathbf{P}_g\widehat{\mathbf{D}}_g = \widehat{\mathbf{D}}_g$,

where $\widehat{\mathbf{D}}_g$ *is given by Theorem 2.3.*

Proof Note that $\frac{\widetilde{\mathbf{R}}_g\widetilde{\mathbf{R}}_g'}{N_g} = \left(\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}\right)\left(\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}\right)'$ and $\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g} = \left(\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}\right)'\left(\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}\right)$ and that $\widetilde{\mathbf{D}}_g = \sqrt{N_g}\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}$. Hence, by setting $N = N_g$ and $\mathbf{Y} = \frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}}$, the claim (i) directly follows from Lemma A.5.

Turn to (ii). Because

$$\begin{aligned}
\mathbf{P}_g\widetilde{\mathbf{D}}_g &= \mathbf{P}_g\widetilde{\mathbf{R}}_g\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1} \\
&= \mathbf{P}_g\mathbf{P}_g\mathbf{R}_g\mathbf{J}_T\left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{F}}_g^s\left(\widetilde{\mathbf{F}}_g^{s\prime}\widetilde{\mathbf{F}}_g^s\right)^{-1} = \mathbf{P}_g\mathbf{R}_g\mathbf{J}_T\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1} \\
&= \widetilde{\mathbf{R}}_g\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1} = \widetilde{\mathbf{D}}_g,
\end{aligned}$$

(ii) is also true from (i). This completes the proof of the lemma. $\square$

Lemma A.7. *Let* $\mathbf{L}_{\Theta_g^s}$ *be a lower triangular matrix such that* $\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s = \mathbf{L}_{\Theta_g^s}\mathbf{L}'_{\Theta_g^s}$. *Then, consider the* $\left(K_g^s \times K_g^s\right)$ *matrix* $\mathbf{U}_{\Theta_g^s}$ *and the* $\left(K_g^s \times K_g^s\right)$ *diagonal matrix* $\Delta_{\Theta_g^s}$ *such that*

$$\mathbf{L}'_{\Theta_g^s}\frac{\mathbf{G}'_g\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g}{T}\mathbf{L}_{\Theta_g^s} = \mathbf{U}_{\Theta_g^s}\Delta_{\Theta_g^s}\mathbf{U}'_{\Theta_g^s}, \tag{A.16}$$

where $\Delta_{\Theta_g^s}$ *is a diagonal matrix and* $\mathbf{U}'_{\Theta_g^s}\mathbf{U}_{\Theta_g^s} = \mathbf{I}_{K_g^s}$. *Define* $\mathcal{D}_g$ *as* $\left(\mathbf{U}'_{\Theta_g^s}\mathbf{L}'_{\Theta_g^s}\right)^{-1}$. *Then, it holds that*
(i) $\mathcal{D}'_g\left(\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\right)\mathcal{D}_g$ *is an identity matrix and,*
(ii) $\mathcal{D}_g^{-1}\frac{\mathbf{G}'_g\mathbf{J}_T(\mathbf{I}_T-\mathcal{P}^c)\mathbf{J}_T\mathbf{G}_g}{T}\mathcal{D}_g^{-1\prime} = \Delta_{\Theta_g^s}$.

Proof First, we show the claim (i) that $\mathcal{D}'_g\left(\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\right)\mathcal{D}_g$ is an identity matrix. From $\mathbf{U}_{\Theta_g^s} = \mathbf{U}'^{-1}_{\Theta_g^s}$, note that $\mathcal{D}_g = \left(\mathbf{L}_{\Theta_g^s}\right)^{-1\prime}\mathbf{U}_{\Theta_g^s}$, which in turn gives,

$$\begin{aligned}\mathcal{D}'_g\left(\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\right)\mathcal{D}_g &= \mathbf{U}'_{\Theta_g^s}\left(\mathbf{L}_{\Theta_g^s}\right)^{-1}\left(\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\right)\left(\mathbf{L}_{\Theta_g^s}\right)^{-1\prime}\mathbf{U}_{\Theta_g^s}\\ &= \mathbf{U}'_{\Theta_g^s}\left(\mathbf{L}_{\Theta_g^s}\right)^{-1}\mathbf{L}_{\Theta_g^s}\mathbf{L}'_{\Theta_g^s}\left(\mathbf{L}_{\Theta_g^s}\right)^{-1\prime}\mathbf{U}_{\Theta_g^s} = \mathbf{U}'_{\Theta_g^s}\mathbf{U}_{\Theta_g^s} = \mathbf{I}_{K_g^s},\end{aligned}$$

verifying the claim (i).

We move to the next claim (ii). Note that

$$\begin{aligned}&\mathcal{D}_g^{-1}\frac{\mathbf{G}'_g\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g}{T}\mathcal{D}_g^{-1\prime} = \mathbf{U}'_{\Theta_g^s}\mathbf{L}'_{\Theta_g^s}\frac{\mathbf{G}'_g\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g}{T}\mathbf{L}_{\Theta_g^s}\mathbf{U}_{\Theta_g^s}\\ =&\mathbf{U}'_{\Theta_g^s}\mathbf{U}_{\Theta_g^s}\Delta_{\Theta_g^s}\mathbf{U}'_{\Theta_g^s}\mathbf{U}_{\Theta_g^s} = \Delta_{\Theta_g^s},\end{aligned}$$

where the second equality is from (A.16). This completes the proof of the lemma. □

Lemma A.8. *It holds that*

$$\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}} = l_1 + l_2 + l_3 + l_4,$$

where

$$
\begin{aligned}
l_1 &= \frac{1}{\sqrt{N_g}} \mathbf{X}_g \Theta_g^s \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
l_2 &= \frac{1}{\sqrt{N_g}} \mathbf{P}_g \Gamma_g^s \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
l_3 &= \frac{1}{\sqrt{N_g}} \left(\mathbf{X}_g \Theta_g^s + \mathbf{P}_g \Gamma_g^c \right) \mathbf{F}' \mathbf{J}_T \left(\mathcal{P}^c - \widehat{\mathcal{P}}^c \right) \\
l_4 &= \frac{1}{\sqrt{N_g}} \mathbf{P}_g \mathbf{E}_g \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right).
\end{aligned}
$$

Also, under Assumptions 2 and 3, as N_g *increases, it holds that*

$$
l_1' l_1 \xrightarrow{p} \left(\mathbf{I}_T - \mathcal{P}^c \right) \mathbf{J}_T \mathbf{G}_g \mathcal{D}_g^{-1\prime} \mathcal{D}_g^{-1} \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \mathcal{P}^c \right),
$$

where $\mathcal{D}_g$ *is given in Lemma A.7, and that for* $i = 2, 3, 4$,

$$
l_i' l_i \xrightarrow{p} \mathbf{0}_{T \times T}.
$$

Proof From (2.7) and (2.8), we have that

$$
\begin{aligned}
\widetilde{\mathbf{R}}_g &= \widehat{\mathbf{R}}_g \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) = \mathbf{P}_g \mathbf{R}_g \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
&= \left(\mathbf{P}_g \mathbf{X}_g \Theta_g^c + \mathbf{P}_g \Gamma_g^c \right) \mathbf{F}' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) + \left(\mathbf{P}_g \mathbf{X}_g \Theta_g^s + \mathbf{P}_g \Gamma_g^s \right) \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
&+ \mathbf{P}_g \mathbf{E}_g \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
&= \mathbf{X}_g \Theta_g^s \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) + \mathbf{P}_g \Gamma_g^s \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right) \\
&+ \left(\mathbf{X}_g \Theta_g^c + \mathbf{P}_g \Gamma_g^c \right) \mathbf{F}' \mathbf{J}_T \left(\mathcal{P}^c - \widehat{\mathcal{P}}^c \right) + \mathbf{P}_g \mathbf{E}_g \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c \right),
\end{aligned}
$$

where the second equality is from $\mathbf{F}' \mathbf{J}_T \left(\mathbf{I}_T - \mathcal{P}^c \right) = \mathbf{0}_{K^c \times T}$. Hence, it holds that

$$
\frac{\widetilde{\mathbf{R}}_g}{\sqrt{N_g}} = l_1 + l_2 + l_3 + l_4, \tag{A.17}
$$

where l_1, l_2, l_3 and l_4 are defined above.

Next, we move to the limits of $l_i' l_i$ for $i = 1, \cdots, 4$. Note that

$$l_1' l_1 = \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) \mathbf{J}_T \mathbf{G}_g \mathcal{D}_g^{-1\prime} \mathcal{D}_g' \frac{\Theta_g^{s\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^s}{N_g} \mathcal{D}_g \mathcal{D}_g^{-1} \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right)$$
$$\xrightarrow{p} \left(\mathbf{I}_T - \mathcal{P}^c\right) \mathbf{J}_T \mathbf{G}_g \mathcal{D}_g^{-1\prime} \mathcal{D}_g^{-1} \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \mathcal{P}^c\right) \tag{A.18}$$

where the limit is from Assumption 3(i) and Lemmas A.4 and A.7. Also, note that

$$l_2' l_2 = \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) \mathbf{J}_T \mathbf{G}_g \mathbf{C}_2 \mathbf{G}_g' \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right),$$

where

$$\mathbf{C}_2 = \frac{\Gamma_g^{s\prime} \mathbf{X}_g}{N_g} \left(\frac{\mathbf{X}_g' \mathbf{X}_g}{N_g}\right)^{-1} \frac{\mathbf{X}_g' \Gamma_g^s}{N_g} \xrightarrow{p} \mathbf{0}_{K_g^s \times K_g^s},$$

from Assumptions 2(i) and 2(ii), which, in conjunction with Lemma A.4, gives

$$l_2' l_2 \xrightarrow{p} \mathbf{0}_{T \times T}. \tag{A.19}$$

Note that

$$l_3' l_3 = \left(\mathcal{P}^c - \widehat{\mathcal{P}}^c\right) \mathbf{J}_T \mathbf{F} \mathbf{C}_3 \mathbf{F}' \mathbf{J}_T \left(\mathcal{P}^c - \widehat{\mathcal{P}}^c\right), \tag{A.20}$$

where

$$\mathbf{C}_3 = \frac{\Theta_g^{c\prime} \mathbf{X}_g' \mathbf{X}_g \Theta_g^c}{N_g} + \frac{\Theta_g^{c\prime} \mathbf{X}_g' \Gamma_g^c}{N_g} +$$
$$+ \frac{\Gamma_g^{c\prime} \mathbf{X}_g \Theta_g^c}{N_g} + \frac{\Gamma_g^{c\prime} \mathbf{X}_g}{N_g} \left(\frac{\mathbf{X}_g' \mathbf{X}_g}{N_g}\right)^{-1} \frac{\mathbf{X}_g' \Gamma_g^c}{N_g} \xrightarrow{p} \Theta_g^{c\prime} \mathbf{V}_{X_g} \Theta_g^c \tag{A.21}$$

from Assumptions 2(i), 2(ii), and 3(i). Plugging (A.21) into (A.20) along with Lemma A.4 gives the following:

$$l_3' l_3 \xrightarrow{p} \mathbf{0}_{T \times T}. \tag{A.22}$$

Lastly, note that

$$l_4' l_4 = \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right) \mathbf{J}_T \mathbf{C}_4 \mathbf{J}_T \left(\mathbf{I}_T - \widehat{\mathcal{P}}^c\right), \tag{A.23}$$

where

$$\mathbf{C}_4 = \frac{\mathbf{E}_g'\mathbf{X}_g}{N_g}\left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\right)^{-1}\frac{\mathbf{X}_g'\mathbf{E}_g}{N_g} \xrightarrow{p} \mathbf{0}_{T\times T} \tag{A.24}$$

from Assumptions 2(i) and 2(ii). Plugging (A.24) into (A.23) along with Lemma A.4 gives the following:

$$l_4'l_4 \xrightarrow{p} \mathbf{0}_{T\times T}. \tag{A.25}$$

The expression (A.17) along with the limits (A.18), (A.19), (A.22), and (A.25), completes the proof of the lemma. □

Lemma A.9. *Under Assumptions 2 and 3, it holds that*

$$\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g} \xrightarrow{p} \left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right),$$

where $\mathcal{H}_g$ is given by Lemma A.7.

Proof From Lemma A.8, it holds that

$$\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g} = \sum_{i,j=1}^{4} l_i'l_j,$$

which gives

$$\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g} - l_1'l_1 = \sum_{\sim(i=j=1)} l_i'l_j. \tag{A.26}$$

Note that for $i = 1, 2, 3, 4$ and $j = 2, 3, 4$, it holds that

$$\|l_i'l_j\| \leq \|l_i\|\cdot\|l_j\| = \sqrt{\operatorname{tr}\left(l_i'l_i\right)\operatorname{tr}\left(l_j'l_j\right)} \xrightarrow{p} 0, \tag{A.27}$$

where the first inequality is from the submultiplicativity of Frobenius norm and the limit is from Lemma A.8. Hence, it follows that

$$\|\sum_{\sim(i=j=1)} l_i'l_j\| \leq \sum_{\sim(i=j=1)} \|l_i'l_j\| \xrightarrow{p} 0, \tag{A.28}$$

where the first inequality is from triangle inequality and the limit is from (A.27).

Lastly, from (A.26) and (A.26), it follows that

$$\|\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g} - l_1'l_1\| \xrightarrow{p} 0,$$

which in junction with Lemma A.8 confirms the claim of the lemma. This completes the proof of the lemma. □

Proof of Theorem 2.3 The following five steps complete the proof of $\widehat{\Theta}_g^s \xrightarrow{p} \Theta_g^s \mathcal{D}_g$. Consider $\widetilde{\mathbf{D}}_g$, $\widetilde{\mathbf{G}}_g$, $\widehat{\mathbf{G}}_g$ and Δ_g defined in Lemma A.6 and $\mathcal{D}_g$ and $\Delta_{\Theta_g^s}$ from Lemma A.7. Define $\mathcal{G}$ as $(\mathbf{I}_T - \mathcal{P}^c)\,\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-0.5}\Delta_{\Theta_g^s}^{-0.5}$.

Step 1. $\widehat{\mathbf{G}}_g \xrightarrow{p} \mathcal{G}$ and $\Delta_g \xrightarrow{p} T\Delta_{\Theta_g^s}$: From Lemma A.7, it holds that

$$\begin{aligned}\mathcal{G}'\mathcal{G} =& \Delta_{\Theta_g^s}^{-0.5}T^{-0.5}\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-0.5}\Delta_{\Theta_g^s}^{-0.5}\\ =& \Delta_{\Theta_g^s}^{-0.5}T^{-0.5}T\Delta_{\Theta_g^s}T^{-0.5}\Delta_{\Theta_g^s}^{-0.5} = \mathbf{I}_{K_g^s}\end{aligned} \tag{A.29}$$

and that

$$\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathcal{G} = \mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-0.5}\Delta_{\Theta_g^s}^{-0.5} = T^{0.5}\Delta_{\Theta_g^s}^{0.5},$$

which in turn gives that

$$\mathcal{G}'\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T - \mathcal{P}^c\right)\mathcal{G} = T\Delta_{\Theta_g^s}, \tag{A.30}$$

which is a diagonal matrix from Lemma A.6. Hence, from (A.29) and (A.30), each column of $\mathcal{G}$ is an eigenvector of $(\mathbf{I}_T - \mathcal{P}^c)\,\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\,(\mathbf{I}_T - \mathcal{P}^c)$, which is the limit of $\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g}$ from Lemma A.9. Because the k-th column of $\widehat{\mathbf{G}}_g$ is an eigenvector of $\frac{\widetilde{\mathbf{R}}_g'\widetilde{\mathbf{R}}_g}{N_g}$, corresponding to the k-th eigenvalue or the k-th diagonal element of Δ_g, the claim holds due to the continuity of eigen-decomposition.

Step 2. $\widetilde{\mathbf{G}}_g \xrightarrow{p} (\mathbf{I}_T - \mathcal{P}^c)\,\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}$: From Step 1, it holds that

$$\begin{aligned}\widetilde{\mathbf{G}}_g = \widehat{\mathbf{G}}_g\Delta_g^{0.5} \xrightarrow{p}& \mathcal{G}T^{0.5}\Delta_{\Theta_g^s}^{0.5}\\ &= \left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-0.5}\Delta_{\Theta_g^s}^{-0.5}T^{0.5}\Delta_{\Theta_g^s}^{0.5} = \left(\mathbf{I}_T - \mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}.\end{aligned}$$

Step 3. $\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\xrightarrow{p}\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-1}\Delta_{\Theta_g^s}^{-1}$: It holds that

$$
\begin{aligned}
&\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&\xrightarrow{p}\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}\left(\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}\right)^{-1}\\
&=\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-1}\left(\mathcal{D}_g^{-1}\frac{\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g}{T}\mathcal{D}_g^{-1\prime}\right)^{-1}\\
&=\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g\mathcal{D}_g^{-1\prime}T^{-1}\Delta_{\Theta_g^s}^{-1},
\end{aligned}
$$

where the limit is from Step 2 and the last equality is from Lemma A.7.

Step 4. $\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\xrightarrow{p}\mathbf{I}_{K_g^s}$: From Step 3 and Lemmas A.4 and A.7,

$$
\begin{aligned}
&\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&\xrightarrow{p}\mathcal{D}_g^{-1}\frac{\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\mathcal{P}^c\right)\mathbf{J}_T\mathbf{G}_g}{T}\mathcal{D}_g^{-1\prime}\Delta_{\Theta_g^s}^{-1}=\mathbf{I}_{K_g^s}.
\end{aligned}
$$

Step 5. $\widehat{\Theta}_g^s\xrightarrow{p}\Theta_g^s\mathcal{D}_g$: Note that $\widetilde{\mathbf{D}}_g$ in Lemma A.6 is expressed using (2.7) as follows:

$$
\begin{aligned}
\widetilde{\mathbf{D}}_g^s&=\widetilde{\mathbf{R}}_g\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}=\widehat{\mathbf{R}}_g\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&=\left(\mathbf{X}_g\Theta_g^s\mathcal{D}_g+\mathbf{P}_g\Gamma_g^s\mathcal{D}_g\right)\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&+\left(\mathbf{X}_g\Theta_g^c+\mathbf{P}_g\Gamma_g^c\right)\mathbf{F}'\mathbf{J}_T\left(\mathcal{P}^c-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&+\mathbf{P}_g\mathbf{E}_g\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1},
\end{aligned}
$$

where the last equality is from $\mathbf{F}'\mathbf{J}_T\left(\mathbf{I}_T-\mathcal{P}^c\right)=\mathbf{0}_{K^c\times T}$. Hence, it follows that

$$
\begin{aligned}
\widehat{\Theta}_g^s&=\left(\mathbf{X}_g'\mathbf{X}_g\right)^{-1}\left(\mathbf{X}_g'\widehat{\mathbf{D}}_g^s\right)=\left(\mathbf{X}_g'\mathbf{X}_g\right)^{-1}\left(\mathbf{X}_g'\widetilde{\mathbf{D}}_g^s\right)\\
&=\left(\Theta_g^s\mathcal{D}_g+\left(\frac{\mathbf{X}_g'\Gamma_g^s}{N_g}\right)\mathcal{D}_g\right)\mathcal{D}_g^{-1}\mathbf{G}_g'\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&+\left(\Theta_g^c+\frac{\mathbf{X}_g'\Gamma_g^c}{N_g}\right)\mathbf{F}'\mathbf{J}_T\left(\mathcal{P}^c-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\\
&+\left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\right)^{-1}\left(\frac{\mathbf{X}_g'\mathbf{E}_g}{N_g}\right)\mathbf{J}_T\left(\mathbf{I}_T-\widehat{\mathcal{P}}^c\right)\widetilde{\mathbf{G}}_g\left(\widetilde{\mathbf{G}}_g'\widetilde{\mathbf{G}}_g\right)^{-1}\xrightarrow{p}\Theta_g^s\mathcal{D}_g,
\end{aligned}
$$

where the limit is from Assumptions 2(i) and 2(ii) and Lemma A.4 and Steps 3 and 4. This completes the proof of the theorem. □

Lemma A.10. *The minimization problem in Theorem 2.4 has the following closed form solution:*

$$vec\left(\widehat{\Theta}_g^c\right) = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} - \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\left(\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\right)^{-1}\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y},$$

where $\mathbf{y} = vec\left(\mathbf{R}_g\mathbf{J}_T\right)$, $\mathbf{X} = \left(\widehat{\mathbf{F}} \otimes \mathbf{X}_g\right)$, *and* $\mathbf{Z} = \mathbf{I}_{K^c} \otimes \left(\widehat{\Theta}_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\right)$.

Proof Note that

$$\begin{aligned}\|\mathbf{R}_g\mathbf{J}_T - \left(\mathbf{X}_g\Theta_g^c\right)\widehat{\mathbf{F}}'\| &= \|\text{vec}\left(\mathbf{R}_g\mathbf{J}_T - \left(\mathbf{X}_g\Theta_g^c\right)\widehat{\mathbf{F}}'\right)\| \\ &= \|\text{vec}\left(\mathbf{R}_g\mathbf{J}_T\right) - \left(\widehat{\mathbf{F}} \otimes \mathbf{X}_g\right)\text{vec}\left(\Theta_g^c\right)\|,\end{aligned} \tag{A.31}$$

where the second equality is from the property of vectorize operator. Also, note that the constraint of $\widehat{\Theta}_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\Theta_g^c = \mathbf{0}_{K_g^s \times K^c}$ is equivalent to

$$\left(\mathbf{I}_{K^c} \otimes \left(\widehat{\Theta}_g^{s\prime}\mathbf{X}_g'\mathbf{X}_g\right)\right)\text{vec}\left(\Theta_g^c\right) = \mathbf{0}_{K_g^s \cdot K^c}. \tag{A.32}$$

Hence, from (A.31) and (A.32), the original minimization problem can be reformulated as

$$\widehat{\Theta}_g^c = \arg\min_{\mathbf{\Theta},\boldsymbol{\lambda}}\left(\mathbf{y} - \mathbf{X}\text{vec}\left(\Theta_g^c\right)\right)'\left(\mathbf{y} - \mathbf{X}\text{vec}\left(\Theta_g^c\right)\right) + \boldsymbol{\lambda}'\mathbf{Z}\text{vec}\left(\Theta_g^c\right), \tag{A.33}$$

where $\mathbf{y} = \text{vec}\left(\mathbf{R}_g\mathbf{J}_T\right)$, $\mathbf{X} = \left(\widehat{\mathbf{F}} \otimes \mathbf{X}_g\right)$, and $\mathbf{Z} = \mathbf{I}_{K^c} \otimes \left(\widehat{\Theta}_g^{s}\mathbf{X}_g'\mathbf{X}_g\right)$. Then, the first order conditions of (A.33) are

$$\begin{bmatrix} 2\mathbf{X}'\mathbf{X} & \mathbf{Z}' \\ \mathbf{Z} & 0 \end{bmatrix}\begin{bmatrix} \text{vec}\left(\widehat{\Theta}_g^c\right) \\ \widehat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 2\mathbf{X}'\mathbf{y} \\ 0 \end{bmatrix},$$

which yields

$$\begin{bmatrix} \text{vec}\left(\widehat{\Theta}_g^c\right) \\ \widehat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 2\mathbf{X}'\mathbf{X} & \mathbf{Z}' \\ \mathbf{Z} & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2\mathbf{X}'\mathbf{y} \\ 0 \end{bmatrix}.$$

Lastly, standard matrix inversion gives

$$\text{vec}\left(\widehat{\Theta}_g^c\right) = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} - \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\left(\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\right)^{-1}\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}.$$

This completes the proof of the lemma. □

Lemma A.11. *Under Assumptions 2 and 3, it holds that*

$$\begin{aligned}
\frac{\mathbf{X}'\mathbf{y}}{N_g} &\xrightarrow{p} \left(\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)\otimes\mathbf{V}_{X_g}\right) vec\left(\Theta_g^c\mathcal{S}^{-1\prime}\right) + \left(\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\otimes\left(\mathbf{V}_{X_g}\Theta_g^s\right)\right) vec\left(\mathbf{I}_{K_g^s}\right),\\
\frac{\mathbf{X}'\mathbf{X}}{N_g} &\xrightarrow{p} \left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)\otimes\mathbf{V}_{X_g},\\
\frac{\mathbf{Z}}{N_g} &\xrightarrow{p} \mathbf{I}_{K^c}\otimes\left(\mathcal{D}_g'\Theta_g^{s\prime}\mathbf{V}_{X_g}\right),
\end{aligned}$$

where $\mathbf{y}$, $\mathbf{X}$, *and* $\mathbf{Z}$ *are given in Lemma A.10.*

Proof From (2.6),

$$\begin{aligned}
\mathbf{R}_g\mathbf{J}_T &= \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\mathbf{F}'\mathbf{J}_T + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\mathbf{G}_g'\mathbf{J}_T + \mathbf{E}_g\mathbf{J}_T\\
&= \left(\mathbf{X}_g\Theta_g^c + \Gamma_g^c\right)\mathbf{F}'\mathbf{J}_T + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\mathbf{G}_g'\mathbf{J}_T + \mathbf{E}_g\mathbf{J}_T\\
&= \mathbf{X}_g\Theta_g^c\mathcal{S}^{-1\prime}\mathcal{S}'\mathbf{F}'\mathbf{J}_T + \Gamma_g^c\mathbf{F}'\mathbf{J}_T + \left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\mathbf{G}_g'\mathbf{J}_T + \mathbf{E}_g\mathbf{J}_T,
\end{aligned}$$

which yields

$$\begin{aligned}
\mathbf{y} = \text{vec}\left(\mathbf{R}_g\mathbf{J}_T\right) &= \left(\left(\mathbf{J}_T\mathbf{F}\mathcal{S}\right)\otimes\mathbf{X}_g\right)\text{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right) + \left(\left(\mathbf{J}_T\mathbf{F}\right)\otimes\Gamma_g^c\right)\text{vec}\left(\mathbf{I}_{K^c}\right)\\
&+ \left(\left(\mathbf{J}_T\mathbf{G}_g\right)\otimes\left(\mathbf{X}_g\Theta_g^s + \Gamma_g^s\right)\right)\text{vec}\left(\mathbf{I}_{K_g^s}\right) + \left(\mathbf{J}_T\otimes\mathbf{E}_g\right)\text{vec}\left(\mathbf{I}_T\right). \qquad \text{(A.34)}
\end{aligned}$$

By combining the expression of $\mathbf{y}$ in (A.34) and that of $\mathbf{X}$ in Lemma A.10, along with the properties of Kronecker product that $\left(\mathbf{A}\otimes\mathbf{B}\right)' = \left(\mathbf{A}'\otimes\mathbf{B}'\right)$ and $\left(\mathbf{A}\otimes\mathbf{B}\right)\left(\mathbf{C}\otimes\mathbf{D}\right) =$

$(\mathbf{AC}) \otimes (\mathbf{BD})$, we have that

$$\begin{aligned}
\frac{\mathbf{X}'\mathbf{y}}{N_g} &= \left(\left(\widehat{\mathbf{F}}'\mathbf{J}_T\mathbf{F}\mathcal{S} \right) \otimes \left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g} \right) \right) \text{vec}\left(\Theta_g^c \mathcal{S}^{-1\prime} \right) + \left(\left(\widehat{\mathbf{F}}'\mathbf{J}_T\mathbf{F} \right) \otimes \left(\frac{\mathbf{X}_g'\Gamma_g^c}{N_g} \right) \right) \text{vec}\left(\mathbf{I}_{K^c} \right) \\
&+ \left(\left(\widehat{\mathbf{F}}'\mathbf{J}_T\mathbf{G}_g \right) \otimes \left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g}\Theta_g^s + \frac{\mathbf{X}_g'\Gamma_g^s}{N_g} \right) \right) \text{vec}\left(\mathbf{I}_{K_g^s} \right) \\
&+ \left(\left(\widehat{\mathbf{F}}'\mathbf{J}_T \right) \otimes \left(\frac{\mathbf{X}_g'\mathbf{E}_g}{N_g} \right) \right) \text{vec}\left(\mathbf{I}_T \right) \\
&\xrightarrow{p} \left((\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}) \otimes \mathbf{V}_{X_g} \right) \text{vec}\left(\Theta_g^c \mathcal{S}^{-1\prime} \right) + \left((\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{G}_g) \otimes \left(\mathbf{V}_{X_g}\Theta_g^s \right) \right) \text{vec}\left(\mathbf{I}_{K_g^s} \right),
\end{aligned}$$

where the limit is from Assumptions 2(i) and 2(ii) and Theorem 2.2. This confirms the first claim of the lemma.

The next two limits are straightforward. From Theorem 2.2 and Assumption 2(i), it holds that

$$\frac{\mathbf{X}'\mathbf{X}}{N_g} = \left(\widehat{\mathbf{F}}'\widehat{\mathbf{F}} \right) \otimes \left(\frac{\mathbf{X}_g'\mathbf{X}_g}{N_g} \right) \xrightarrow{p} (\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}) \otimes \mathbf{V}_{X_g},$$

which confirms the second claim. Also, note that

$$\frac{\mathbf{Z}}{N_g} = \mathbf{I}_{K^c} \otimes \left(\widehat{\Theta}_g^{s\prime} \frac{\mathbf{X}_g'\mathbf{X}_g}{N_g} \right) \xrightarrow{p} \mathbf{I}_{K^c} \otimes \left(\mathcal{D}_g' \Theta_g^{s\prime} \mathbf{V}_{X_g} \right),$$

from Theorem 2.3 and Assumption 2(i). This completes the proof of the lemma. $\square$

Lemma A.12. *Under Assumptions 2 and 3, it holds that*

$$\left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{y} \xrightarrow{p} vec\left(\Theta_g^c \mathcal{S}^{-1\prime} \right) + \left(\left(\mathcal{S}'^{-1} \left(\mathbf{F}'\mathbf{J}_T\mathbf{F} \right)^{-1} \left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g \right) \right) \otimes \Theta_g^s \right) vec\left(\mathbf{I}_{K_g^s} \right),$$

and that

$$\begin{aligned}
&\left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{Z}' \left(\mathbf{Z} \left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{Z}' \right)^{-1} \mathbf{Z} \left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{y} \\
&\xrightarrow{p} \left(\left(\mathcal{S}'^{-1} \left(\mathbf{F}'\mathbf{J}_T\mathbf{F} \right)^{-1} \left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g \right) \right) \otimes \Theta_g^s \right) vec\left(\mathbf{I}_{K_g^s} \right) \qquad \text{(A.35)}
\end{aligned}$$

where $\mathbf{y}$, $\mathbf{X}$, *and* $\mathbf{Z}$ *are given in Lemma A.10.*

Proof From Lemma A.11 and the property of Kronecker product that $(\mathbf{A}\otimes\mathbf{B})^{-1} = (\mathbf{A}^{-1}\otimes\mathbf{B}^{-1})$,

$$\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1} \xrightarrow{p} \left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)^{-1}\otimes\mathbf{V}_{X_g}^{-1}, \tag{A.36}$$

which in conjunction with the limit of $\frac{\mathbf{X}'\mathbf{y}}{N_g}$ in Lemma A.11 gives

$$\begin{aligned}
\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} &= \left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{X}'\mathbf{y}}{N_g}\\
&\xrightarrow{p}\mathrm{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right)+\left(\left(\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)^{-1}\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\Theta_g^s\right)\mathrm{vec}\left(\mathbf{I}_{K_g^s}\right)\\
&=\mathrm{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right)+\left(\left(\mathcal{S}^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\Theta_g^s\right)\mathrm{vec}\left(\mathbf{I}_{K_g^s}\right).
\end{aligned} \tag{A.37}$$

This confirms the first claim of the lemma.

For the next claim, we first identify the limits of $\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{X}'\mathbf{y}}{N_g}$, $\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}$, and $\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}$ and then combine the limits. From (A.37) and the limit of $\frac{\mathbf{Z}}{N_g}$ in Lemma A.11,

$$\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{X}'\mathbf{y}}{N_g}\xrightarrow{p}p_1+p_2, \tag{A.38}$$

where

$$\begin{aligned}
p_1 &= \left(\mathbf{I}_{K^c}\otimes\left(\mathcal{D}_g'\Theta_g^{s\prime}\mathbf{V}_{X_g}\right)\right)\mathrm{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right),\\
p_2 &= \left(\left(\mathcal{S}^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\left(\mathcal{D}_g'\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\right)\right)\mathrm{vec}\left(\mathbf{I}_{K_g^s}\right)
\end{aligned} \tag{A.39}$$

Note that

$$p_1 = \mathrm{vec}\left(\left(\mathcal{D}_g'\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^c\right)\right) = \mathbf{0}_{K_g^s\cdot K^c}, \tag{A.40}$$

where the first equality is from a property of vectorize operator and the last equality is from Assumption3(ii). Also, by applying Lemma A.7 to (A.39), we have that

$$p_2 = \left(\left(\mathcal{S}^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\mathcal{D}_g^{-1}\right)\mathrm{vec}\left(\mathbf{I}_{K_g^s}\right). \tag{A.41}$$

By plugging (A.40) and (A.41) into (A.38), we have that

$$\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{X}'\mathbf{y}}{N_g}\xrightarrow{p}\left(\left(\mathcal{S}^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\mathcal{D}_g^{-1}\right)\text{vec}\left(\mathbf{I}_{K_g^s}\right). \tag{A.42}$$

Next, we move to $\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}$ and $\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}$. From the limit of $\frac{\mathbf{Z}}{N_g}$ in Lemma A.11 and (A.36), note that

$$\begin{aligned}&\frac{\mathbf{Z}}{N_g}\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}\\ &\xrightarrow{p}\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)^{-1}\otimes\left(\mathcal{D}_g'\Theta_g^{s\prime}\mathbf{V}_{X_g}\Theta_g^s\mathcal{D}_g\right)=\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)^{-1}\otimes\mathbf{I}_{K_g^s},\end{aligned} \tag{A.43}$$

where the last equality is from Lemma A.7, and that

$$\left(\frac{\mathbf{X}'\mathbf{X}}{N_g}\right)^{-1}\frac{\mathbf{Z}'}{N_g}\xrightarrow{p}\left(\mathcal{S}'\mathbf{F}'\mathbf{J}_T\mathbf{F}\mathcal{S}\right)^{-1}\otimes\mathcal{D}_g. \tag{A.44}$$

Lastly, combining (A.42), (A.43) and (A.44) yields the last claim of the lemma. This completes the proof of the lemma. □

Proof of Theorem 2.4 It suffices to show that $\text{vec}\left(\widehat{\Theta}_g^c\right)\xrightarrow{p}\text{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right)$. Lemma A.10 reformulate the solution in Theorem 2.4. Furthermore, from Lemma A.12, we have that

$$\begin{aligned}\text{vec}\left(\widehat{\Theta}_g^c\right)&=\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}-\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\left(\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{Z}'\right)^{-1}\mathbf{Z}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}\\ &\xrightarrow{p}\text{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right)+\left(\left(\mathcal{S}'^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\Theta_g^s\right)\text{vec}\left(\mathbf{I}_{K_g^s}\right)\\ &-\left(\left(\mathcal{S}'^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{J}_T\mathbf{G}_g\right)\right)\otimes\Theta_g^s\right)\text{vec}\left(\mathbf{I}_{K_g^s}\right)=\text{vec}\left(\Theta_g^c\mathcal{S}^{-1\prime}\right).\end{aligned}$$

This completes the proof of the theorem. □

The following lemma is useful for the proofs of Theorems 2.5 and 2.6.

Lemma A.13. *Under Assumptions 2 and 3, as N_1 and N_2 increase,*

$$\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}'\mathbf{R}\xrightarrow{p}\begin{bmatrix}\mathcal{S}'\left(\boldsymbol{\lambda}_1^c-\boldsymbol{\lambda}_2^c\right)\mathbf{1}_T'\\ \mathcal{S}'\left(\boldsymbol{\lambda}_2^1\mathbf{1}_T'+\mathbf{F}'\right)\end{bmatrix},$$

where $\mathbf{R}$ *is the* $(N_1+N_2)\times T$ *matrix of* $[\mathbf{R}_1' \ \mathbf{R}_1']'$ *and* $\widehat{\mathbf{B}}$ *is given in Theorem 2.5.*

Proof From (2.5), $\mathbf{R}_g$ can be rewritten as

$$\mathbf{R}_g = \mathbf{X}_g \Theta_g^c \left(\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'\right) + \mathbf{u}_g, \tag{A.45}$$

where

$$\mathbf{u}_g = \Gamma_g^c \left(\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'\right) + \left(\mathbf{X}_g \Theta_g^s + \Gamma_g^s\right)\left(\boldsymbol{\lambda}_g^s \mathbf{1}_T' + \mathbf{G}_g'\right) + \mathbf{E}_g. \tag{A.46}$$

Then, stacking (A.45) over $g = 1, 2$, we rewrite $\mathbf{R}$ as

$$\begin{aligned}
\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{X}_1 \Theta_1^c \mathcal{S}^{-1\prime} \mathcal{S}' \left(\boldsymbol{\lambda}_1^c \mathbf{1}_T' + \mathbf{F}'\right) \\ \mathbf{X}_2 \Theta_2^c \mathcal{S}^{-1\prime} \mathcal{S}' \left(\boldsymbol{\lambda}_2^c \mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{X}_1 \Theta_1^c \mathcal{S}^{-1\prime} \\ 0_{N_2 \times K^c} \end{bmatrix} \mathcal{S}' \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) \mathbf{1}_T' + \begin{bmatrix} \mathbf{X}_1 \Theta_1^c \mathcal{S}^{-1\prime} \\ \mathbf{X}_2 \Theta_2^c \mathcal{S}^{-1\prime} \end{bmatrix} \mathcal{S}' \left(\boldsymbol{\lambda}_2^c \mathbf{1}_T' + \mathbf{F}'\right) + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{X}_1 \Theta_1^c \mathcal{S}^{-1\prime} & \mathbf{X}_1 \Theta_1^c \mathcal{S}^{-1\prime} \\ 0_{N_2 \times K^c} & \mathbf{X}_2 \Theta_2^c \mathcal{S}^{-1\prime} \end{bmatrix} \begin{bmatrix} \mathcal{S}' \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) \mathbf{1}_T' \\ \mathcal{S}' \left(\boldsymbol{\lambda}_2^c \mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}.
\end{aligned} \tag{A.47}$$

Hence, using (A.47), we have that

$$\begin{aligned}
\frac{\widehat{\mathbf{B}}'\mathbf{R}}{N} = &\begin{bmatrix} \widehat{\Theta}_1^{c\prime} \frac{\mathbf{X}_1'\mathbf{X}_1}{N} \Theta_1^c \mathcal{S}^{-1\prime} & \widehat{\Theta}_1^{c\prime} \frac{\mathbf{X}_1'\mathbf{X}_1}{N} \Theta_1^c \mathcal{S}^{-1\prime} \\ \widehat{\Theta}_1^{c\prime} \frac{\mathbf{X}_1'\mathbf{X}_1}{N} \Theta_1^c \mathcal{S}^{-1\prime} & \widehat{\Theta}_1^{c\prime} \frac{\mathbf{X}_1'\mathbf{X}_1}{N} \Theta_1^c \mathcal{S}^{-1\prime} + \widehat{\Theta}_2^c \frac{\mathbf{X}_2'\mathbf{X}_2}{N} \Theta_2^c \mathcal{S}^{-1\prime} \end{bmatrix} \begin{bmatrix} \mathcal{S}' \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) \mathbf{1}_T' \\ \mathcal{S}' \left(\boldsymbol{\lambda}_2^c \mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix} \\
&+ \begin{bmatrix} \widehat{\Theta}_1^{c\prime} \frac{\mathbf{X}_1'\Gamma_1}{N} \\ \widehat{\Theta}_2^c \frac{\mathbf{X}_2'\Gamma_2}{N} \end{bmatrix}
\end{aligned} \tag{A.48}$$

From (A.46), note that

$$\begin{aligned}
\widehat{\Theta}_g^c \frac{\mathbf{X}_g'\mathbf{u}_g}{N_g} &= \widehat{\Theta}_g^c \frac{\mathbf{X}_g'\Gamma_g^c}{N_g} \left(\boldsymbol{\lambda}_g^c \mathbf{1}_T' + \mathbf{F}'\right) \\
&\quad + \left(\widehat{\Theta}_g^c \frac{\mathbf{X}_g'\mathbf{X}_g}{N_g} \Theta_g^s + \widehat{\Theta}_g^c \frac{\mathbf{X}_g'\Gamma_g^s}{N_g}\right)\left(\boldsymbol{\lambda}_g^s \mathbf{1}_T' + \mathbf{G}_g'\right) + \widehat{\Theta}_g^c \frac{\mathbf{X}_g'\mathbf{E}_g}{N_g} \\
&\xrightarrow{p} \mathcal{S}^{-1} \Theta_g^{c\prime} \mathbf{V}_{X_g} \Theta_g^s = \mathbf{0}_{K^c \times T},
\end{aligned}$$

where the limit is from Assumptions 2(ii) and Theorem 2.4 and the last equality is from Assumption 3(ii). Hence, along with Assumption 2(iii), it follows that

$$\widehat{\Theta}_1^{c\prime}\frac{\mathbf{X}_1'\mathbf{u}_1}{N} = \frac{N_1}{N}\widehat{\Theta}_1^{c\prime}\frac{\mathbf{X}_1'\mathbf{u}_1}{N_1} \xrightarrow{p} \mathbf{0}_{K^c\times T},$$
$$\widehat{\Theta}_2^{c}\frac{\mathbf{X}_2'\mathbf{u}_2}{N} = \left(1-\frac{N_1}{N}\right)\widehat{\Theta}_1^{c\prime}\frac{\mathbf{X}_2'\mathbf{u}_2}{N_2} \xrightarrow{p} \mathbf{0}_{K^c\times T},$$

which along with Assumptions 2(i) and 2(iii) gives

$$\frac{\widehat{\mathbf{B}}'\mathbf{R}}{N} \xrightarrow{p} \mathbf{V}_B \begin{bmatrix} \mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)\mathbf{1}_T' \\ \mathcal{S}'\left(\boldsymbol{\lambda}_2^c\mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix}, \tag{A.49}$$

where

$$\mathbf{V}_B = \begin{bmatrix} n_1\mathcal{S}^{-1}\Theta_1^{c\prime}\mathbf{V}_{X_1}\Theta_1^c\mathcal{S}^{-1} & n_1\mathcal{S}^{-1\prime}\Theta_1^{c\prime}\mathbf{V}_{X_1}\Theta_1^c\mathcal{S}^{-1\prime} \\ n_1\mathcal{S}^{-1}\Theta_1^{c\prime}\mathbf{V}_{X_1}\Theta_1^c\mathcal{S}^{-1\prime} & \begin{array}{c}(n_1\mathcal{S}^{-1}\Theta_1^{c\prime}\mathbf{V}_{X_1}\Theta_1^c\mathcal{S}^{-1\prime} \\ +(1-n_1)\mathcal{S}^{-1}\Theta_2^{c\prime}\mathbf{V}_{X_2}\Theta_2^c\mathcal{S}^{-1\prime})\end{array} \end{bmatrix}. \tag{A.50}$$

Also, from Assumptions 2(i) and 2(iii) and Theorem 2.4, we have that

$$\frac{\widehat{\mathbf{B}}'\widehat{\mathbf{B}}}{N} = \begin{bmatrix} \widehat{\Theta}_1^{c\prime}\frac{\mathbf{X}_1'\mathbf{X}_1}{N}\Theta_1^c\mathcal{S}^{-1\prime} & \widehat{\Theta}_1^{c}\frac{\mathbf{X}_1'\mathbf{X}_1}{N}\Theta_1^c\mathcal{S}^{-1\prime} \\ \widehat{\Theta}_1^{c\prime}\frac{\mathbf{X}_1'\mathbf{X}_1}{N}\Theta_1^c\mathcal{S}^{-1\prime} & \widehat{\Theta}_1^{c}\frac{\mathbf{X}_1'\mathbf{X}_1}{N}\Theta_1^c\mathcal{S}^{-1\prime} + \widehat{\Theta}_2^{c}\frac{\mathbf{X}_2'\mathbf{X}_2}{N}\Theta_2^c\mathcal{S}^{-1\prime} \end{bmatrix} \xrightarrow{p} \mathbf{V}_B, \tag{A.51}$$

where $\mathbf{V}_B$ is given by (A.50). Finally, combining (A.49) and (A.51), we have that

$$\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}\mathbf{R} = \left(\frac{\widehat{\mathbf{B}}'\widehat{\mathbf{B}}}{N}\right)^{-1}\frac{\widehat{\mathbf{B}}'\mathbf{R}}{N} \xrightarrow{p} \begin{bmatrix} \mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)\mathbf{1}_T' \\ \mathcal{S}'\left(\boldsymbol{\lambda}_2^1\mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix},$$

which completes the proof of the lemma. □

Proof of Theorem 2.5 From Lemma A.13, we have that

$$\widehat{\boldsymbol{\lambda}}_\Delta = [\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K^c}]\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}'\overline{\mathbf{R}} = [\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K^c}]\left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{B}}'\mathbf{R}\right)\mathbf{1}_T/T$$
$$\xrightarrow{p} [\mathbf{I}_{K^c}\ \mathbf{0}_{K^c\times K^c}]\begin{bmatrix} \mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)\mathbf{1}_T' \\ \mathcal{S}'\left(\boldsymbol{\lambda}_2^1\mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix}\mathbf{1}_T/T = \mathcal{S}'\left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right).$$

This completes the proof of the theorem. □

Proof of Theorem 2.6 From Lemma A.13 and Theorem 2.5, we have that

$$\begin{aligned}
\widehat{\mathbf{w}}\mathbf{R} &= \widehat{\boldsymbol{\lambda}}_{\Delta}' \left[\mathbf{I}_{K^c} \; \mathbf{0}_{K^c \times K^c}\right] \left(\widehat{\mathbf{B}}'\widehat{\mathbf{B}}\right)^{-1} \widehat{\mathbf{B}}'\mathbf{R} \\
&\xrightarrow{p} \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)' \mathcal{S} \left[\mathbf{I}_{K^c} \; \mathbf{0}_{K^c \times K^c}\right] \begin{bmatrix} \mathcal{S}' \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) \mathbf{1}_T' \\ \mathcal{S}' \left(\boldsymbol{\lambda}_2^1 \mathbf{1}_T' + \mathbf{F}'\right) \end{bmatrix} \\
&= \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right)' \mathcal{S}\mathcal{S}' \left(\boldsymbol{\lambda}_1^c - \boldsymbol{\lambda}_2^c\right) \mathbf{1}_T' = \delta \mathbf{1}_T'.
\end{aligned}$$

This completes the proof of the theorem. □

References

Connor, G., M. Hagmann, and O. Linton (2012). Efficient semiparametric estimation of the Fama–French model and extensions. *Econometrica 80*(2), 713–754.

Connor, G. and R. A. Korajczyk (1986). Performance measurement with the arbitrage pricing theory: A new framework for analysis. *Journal of Financial Economics 15*(3), 373–394.

Fan, J., Y. Liao, and W. Wang (2016). Projected principal component analysis in factor models. *Annals of Statistics 44*, 219–254.

Ferson, W. E. and C. R. Harvey (1999). Conditioning variables and the cross section of stock returns. *Journal of Finance 54*(4), 1325–1360.

Ghysels, E. (1998). On stable factor structures in the pricing of risk: do time-varying betas help or hurt? *Journal of Finance 53*(2), 549–573.

Harari, Y. N. (2015). *Sapiens: A Brief History of Humankind.* Harper Perennial.

Kelly, B., S. Pruitt, and Y. Su (2019). Characteristics are covariances: A unified model of risk and return. *Journal of Financial Economics 134*, 501–534.

Kim, S., R. A. Korajczyk, and A. Neuhierl (2021). Arbitrage Portfolios. *Review of Financial Studies 34*(6), 2813–2856.

Kim, S. and G. Skoulakis (2018). Ex-post risk premia estimation and asset pricing tests using large cross sections: The regression-calibration approach. *Journal of econometrics 204*(2), 159–188.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory 13*(3), 341–360.

Shanken, J. (1992). On the estimation of beta-pricing models. *The review of financial studies 5*(1), 1–33.