Modelling the Term Structure with Trends in Yields and Cycles in Excess Returns*

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First Version: October, 2023
This Version: December, 2023

Abstract

Bond yields can be decomposed into two unobservable components: the expected sequence of short-term rates and term premia. The standard literature Affine Macro Term Structure literature achieves the decomposition of yields by estimating a common factor structure for yields and excess returns. A new model in which yields are drifting, sharing a common stochastic trend driven by the drift in short-term rates and excess returns are stationary, as the compensation for risk is driven by the cycles in yields produces much better forecasts of the dynamics of US rates at all maturities and, consequently, different and stationary dynamics for the term premia.

JEL codes: E43, E52, G12.
Keywords: Affine Term Structure Models, Trends and Cycles, Term Premia.

*We are grateful to Tobias Adrian, Michael Bauer, JS Fontaine, Francesco Giavazzi, Refet Gurkaynak, Glenn Rudebusch, and Guihai Zhao for comments and suggestions and to Daniel Gros for stimulating discussions. Financial Support by the European Union-Next Generation EU, project GRINS–Growing Resilient, INclusive and Sustainable PE00000018 (CUP B43C22000760006) is gratefully acknowledged. This paper is part of the research activities of the Baffi Centre unit on Macroeconomic Trends, Cycles, and Asset Prices.

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1 Introduction

The dynamics of nominal government bond yields at different maturities plays a central role in shaping the response of the real economy to monetary and fiscal policy interventions. Yields can be decomposed into two unobservable components: the sequence of expected one-period rates and the term-premia (Campbell and Shiller, 1991; Duffee, 2002). The first component reflects the future expected path of monetary policy rates, while the second reflects both macro fundamentals, including the prospects for growth, inflation and government debt dynamics, and the investors' attitude toward risk. Policymakers are fully aware that the market-based financing conditions that matters for the control of the business cycle and inflation depend on both components of yields. Fluctuations in term premia are considered as important as the market perception of the future path of interest rates (Schnabel, 2023), and they are also used to evaluate the macroeconomic implications of monetary policy (Schnabel, 2022).

Term structure models are helpful in that they allow the identification of the two components by forecasting the expected path of interest rates and by imposing consistency with no-arbitrage restrictions for the derived term premia at different maturities. The data tell us that yields are drifting, but excess returns are cyclical.[1] In standard Affine Macro Term Structure (AMTS) models a few factors, modelled by a Vector Autoregressive Process, are the common drivers of the dynamics of both the expected path of future one-period rates and the term premia. As yields drift,

[1] Although, there is evidence that over a 700 hundred year horizon yields are stationary, (Rogoff et al., 2022), the data tell us that over a horizon of up fifty years (and especially the last fifty years) bond yields are drifting, while returns obtained by holding for one-quarter bonds at any maturity in excess of the return of the three-month rates are stationary.

factors exhibit a high level of persistence. When these factors are modelled using a Vector Autoregression (VAR), the forecast of future one-period rates gradually converges to the mean of the sample used for estimation. Standard AMTS models tend to generate term premia that are a-cyclical and parallel to the secular trend in yields. These features of the term-premia are a direct consequence of the specification strategy for the dynamics of both yields and excess-returns that adopts a common autoregressive factor structure for them.

Our contribution is to build a model consistent with the empirical evidence which tells that yields are non-stationary and driven by a common trend while excess returns are stationary. Hence, we propose a new AMTS model in which yields drift, sharing a common stochastic trend driven by the drift in short-term (monetary policy) rates and excess returns are stationary as the compensation for risk depends on the cycle in yields.

Following Favero et al. (2016), Del Negro et al. (2019), Lunsford and West (2019), and Favero et al. (2022), we decompose short-term rates in a trend component and a cycle component. The trend component is driven by the very long-run forecast of the central bank for real short-term rates and by the response of monetary policy to the very long-run forecast for inflation. The very long-run forecast for the real rates is labelled in the literature as the *natural rate of interest*. We model the natural rate as function of the equilibrium growth rate of output in the economy and the age structure of population, a time-varying determinant of household preferences. We provide statistical evidence that these variables capture successfully the trend in one-period rates and that the trend in yields at all maturities is successfully modelled by the average of the trend in one-period rates over the residual life of the different bonds. The current and future trend component of the short-term rates are constructedGiven the availability of long-run forecast for the growth rate of the economy, the age structure of population, and long-term expected inflation.

A factor model is then built for the cyclical components of yields, where the dynamics of holding period excess returns and term-premia is consistent with no-arbitrage restrictions. We keep a VAR specification for the identified factors, but, thanks to the trend-and-cycle decomposition, it is now a VAR on stationary variables. Predictions for short-term rates at any future dates are then derived by combining the predictions for the trends (not based on the VAR for factors and therefore forward-looking) and the predictions for the cyclical components (based on the VAR for factors and, therefore, backward-looking). Bonds at any maturities are then priced via pricing equations that imposes no-arbitrage restrictions. Term premia are derived as the difference of bond yields obtained when the price of risk is estimated in the affine specification and when the price of risk is restricted to zero. Bond yields are non-stationary, but their trend is the average trend of short-term rates over the maturities of the bond and term-premia are driven by the stationary state variables. However, we do not use the trend derived from the ATMS to de-trend yields. We assume that all yields have a common stochastic trend that coincides with the one period yield trend and, therefore, use it to de-trend the whole yield curve.

The specification strategy dominates standard models in term of forecasting performance for returns and yields at any maturity and leads to a very different measurement of term premia.

By shedding light on the relationships between the two components and the trends and cycles in yields, our ATSM model contributes to advancing theoretical frameworks and provides a robust foundation for more precise forecasting and effective policy simulation analysis.

The rest of the paper is organized as follows: Section 2 has a first look at the data, Section 3 places our contribution to the literature, Section 4 proposes the new specification for ATSM models, Section 5 illustrates and discusses the empirical results and Section 6 concludes.

2 A First Look at the Data

The quarterly US data from the last fourty years on the term structure of zero-coupon Government bonds show the presence of a common drift in yields to maturity which disappears for the 1-period excess holding returns for bonds at all maturities.

Figure 1 reports quarterly observations of the Treasury yield curve estimates of the Federal Reserve Board made available by Gürkaynak et al. (2007) over the sample period 1980-2023. A quick look at the shape of the figures shows that every bond yield shows the presence of a common drift. We chose the quarter to be the unit for periods. Hence, we take the three-month Treasury Bills to be our 1-period bond.

Figure 2 reports the observed 1-quarter returns of holding bonds at all maturities from 5 to 15 years in excess over the return on 1-period bond. Contrary to the previous case, no trend is evident from the data. The stationarity of one-period excess holding returns has two immediate implications. First, term premia at all maturities, i.e., the average of expected one-period excess holding returns over the residual maturity of bonds, have to be stationary as well. Second, the common drift component in the term structure is driven by the trend in the 1-period bond. Consequently, it vanishes when considering the spread on the one-period bond.

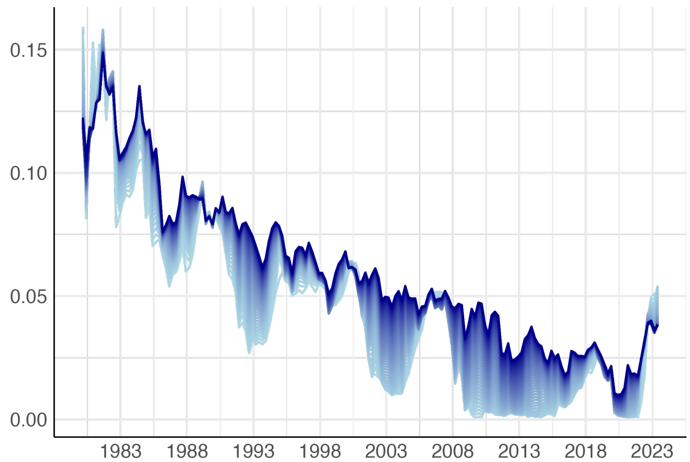


FIGURE 1. Quarterly observations on the time-series of (annualised) yields from the 3-month to the 15-year maturity. We use the same colour palette for all maturities (blue). Darkest blue indicates the highest maturity, i.e., 15 years.

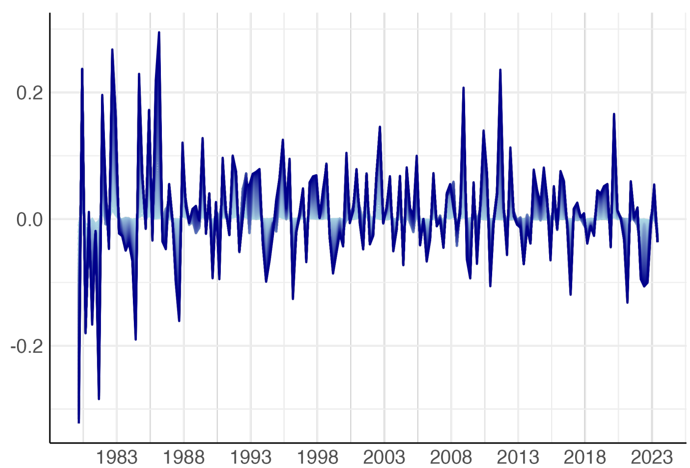


FIGURE 2. Quarterly observations on the time series of 1-quarter holding period returns for bonds at maturities between 5 and 15 years in excess of the return on three-month Treasury Bills

3 The Literature

Macro-finance models of the term structure mostly belong to the class of Affine Term Structure Models (Diebold et al., 2005). These models are originally designed for stationary processes in yields, as the yield dynamics is modelled as a vector autoregression (VAR) of a set of factors extracted from the term structure partially, like Ang and Piazzesi (2003), or totally, like Kim and Wright (2005) and Adrian et al. (2015); and VAR models are used for forecasting stationary processes. Importantly, the factor dynamics also drives the price of risk and holding period returns. The presence of a stochastic trend in yields has several negative consequences for this approach. VAR models are inappropriate for long-run forecasting of non-stationary data, biased forecast of the dynamics of short-term rates[2] do affect the measurement of term premia. The non-stationarity of factors might results in non-stationarity of term premia, which is counterfactual with respect to the empirical evidence of stationarity of holding period (excess) returns.

Several papers have documented the existence of a slow-moving component common to the entire term structure (see, for example, Bakshi and Chen, 1994 and Fama, 2006). An important and growing literature has modeled Treasury yields using shifting endpoints (Kozicki and Tinsley, 2001), near-cointegration (Jardet et al., 2013) or long memory (Golinski and Zaffaroni, 2016), vector autoregressive models (VAR) with common trends (Del Negro et al., 2019), slow-moving averages of inflation (Cieslak and Povala, 2015) and consumption (Jørgensen, 2018), or an (unobserved) stochastic trend common across Treasury yields (Bauer and Rudebusch, 2020).

[2]It has also been recognised that OLS estimates of near-unit roots are notoriously biased downward, thus overestimating the amount of mean reversion in yields.

Interestingly, Bauer and Rudebusch (2020), in their model that allows for a trend in yields and returns, note that the loading of returns on the unobserved common stochastic trend is an order of magnitude smaller than the loading of prices and they also report that predictive regressions of returns on de-trended yields and trend proxies lead to coefficients on the trend that are not significantly different from zero. Bauer et al. (2014) observe that, as a consequence of the very high persistence in yields, term premia implied by maximum likelihood estimates of affine term structure models are misleading because of small-sample bias. They show that ATSM models, such as that estimated by Wright (2011), tend to produce cyclical risk premium estimates, often just parallel to the secular trend in interest rates, while bias corrected term-premia show strong (counter-)cyclical behaviour. Christensen and Rudebusch (2012) address the problem of non-stationarity of yields by imposing a unit root in the factor capturing the level of the term structure, showing that this restriction produces a clear improvement in the forecasting performance of the model. Their evidence illustrates that "standing still" (i.e. using random walk forecast) is better than "moving in the wrong direction" (using VAR based forecast that affected by the small-sample bias produce mean-reversion towards the wrong mean); in this paper we consider the alternative of "moving in the right direction" (by identifying the drivers of the stochastic trends in yields and using predictions on these variables for forecasting). Campbell and Shiller (1987) have proposed a stationary representation of spreads and changes in short-term rates, based on cointegration between short-term rates and yields at any maturity, but their approach has never found its way in Affine Term Structure models. The existence of a $(1, -1)$ cointegrating vector between long-term and short term rates, i.e. the stationarity of the spread, provides statistical evidence against the non-stationarity of the term premia and it does not justify a pattern of term premia that reflects the secular trend in interest rates. Piazzesi et al. (2015) use survey data on interest rate forecasts to construct subjective bond risk premia to find that subjective premia are less volatile and not very cyclical. They explain this evidence by pointing out that survey forecasts of interest rates are made as if both the level and the slope of the yield curve are more persistent than under common statistical models. Zhao (2020) and Feunou and Fontaine (2023) propose structural models of trends and cycles in the term structure capable of explaining several features of the data. However, these models rely on statistical trend-cycle decompositions and do not relate the trend component of the yield curve to observable slow-moving variables, such as the demographic structure of the population and potential output growth, whose properties can be exploited for forecasting purposes.

4 An Affine Term Structure model with Trend and Cycle in Monetary Policy Rates

Affine models of the term structure of interest rates are a popular way of determining the term premia, that are derived as the difference between observed yields and the model-based expectation of the future path of short rates[3]. The affine models typically use state variables (latent factors) to model the shocks that drive the economy. The key assumptions are: First, the pricing kernel is exponentially affine in the state variables, whose dynamics is described by a VAR. Second, market prices of risk are

[3]In fact, the models also include a correction for convexity, which is empirically small and, ironically, of second-order importance.

affine in the state variables. Finally, the innovations to state variables and one-period holding excess returns are jointly normal-distributed.

Using these assumptions, together with no-arbitrage restrictions, delivers generating processes for continuously compounded excess returns and continuously compounded yields at any maturity that are a function of the state variables. Yields can be decomposed into a term premium, a convexity correction, and a part reflecting expectations for the one-period rate over the residual life of the bond. In the light of the evidence reported in the introduction, this specification strategy suffers from a clear shortcoming: the state variables have to capture the drift in the data, and a VAR model is not the most appropriate specification for long-run projections of the relevant variables if they are highly persistent. Indeed, long-run projections are needed because pricing a long-dated bond with quarterly data will require to project the three-month rates over an horizon equal to the maturity of the bond.

To deal with this problem, we propose to specify an Affine Term Structure model with two sets of states variables: the trending ones and the stationary ones. The trending variables will be related to slow moving components in the structure of the economy and will not be predicted by a VAR. The VAR specification will then be limited to the stationary state variables.

4.1 Detrending the Term Structure to model excess returns

The identification of the two sets of state variables is implemented starting from the specification of the one-period nominal risk free rate $r_t^{(1)}$. The risk free rate can be decomposed in a trend and a cycle. The trend, i.e. long-run risk free rate, is made of two components: the natural rate of interest, r_t^*, and a component that reflects long-term inflation expectations.

Laubach and Williams (2003) show that in the standard Ramsey model households intertemporal optimization delivers a positive relationship between the natural rate of interest and both the growth rate of output in the economy and household preferences. This motivates the inclusion of (log) growth rate of potential output, Δy_t^{pot}, as a variable explaining the trend. However, Jordà and Taylor (2019) and Mian et al. (2021) illustrate that fluctuations in output growth (*per capita*) of the economy cannot fully explain the drift in natural rate, therefore, other time-varying determinants of the rate of time preference of the agents in the economy should be considered. On the one hand, we follow Favero et al. (2016), Lunsford and West (2019), and Favero et al. (2022), and consider the age structure of the population as the driver of changing preferences, in particular MY_t, the ratio of middle-aged (40-49) to young (20-29) population. On the other hand, Gürkaynak et al. (2005) convincingly argue that private agents views of long-run inflations are subject to fluctuations. In line with this evidence we use the survey-based measure of long-run inflation expectations, π_t^{LR}, also considered in the Fed's FRB/US model[4] as the proxy for long-run inflation expectations. This is a reasonable proxy under the assumption that the central bank is credible. The yield's cyclical part can thus be identified with the residual after regressing the short rate on those three variables, Δy_t^{pot}, MY_t, and π_t^{LR}.

Once the trend and the cycle in the one-period rate are identified, the trend and the cycle for yields at all maturities can be consistently derived. In fact, by using the relationship between prices and yields to maturities and by applying no-arbitrage to

[4]Available at `https://www.federalreserve.gov/econres/us-models-package.htm`.

zero-coupon bond we have:

$$
\begin{aligned}
p_t^{(n)} &= -nr_t^{(n)}, && (1)\\
rx_{t+1}^{(n-1)} &= p_{t+1}^{(n-1)} - p_t^{(n)} - r_t^{(1)}, && (2)\\
\mathbb{E}_t\left[rx_{t+1}^{(n-1)}\right] &= \phi_{t,t+1}^{n}, && (3)\\
r_t^{(1)} &= r_t^{*,(1)} + u_t^{(1)} && (4)\\
r_t^{*,(1)} &= \gamma_1 MY_t + \gamma_2 \Delta y_t^{pot} + \gamma_3 \pi_t^{LR} && (5)\\
r_t^{(n)} &= r_t^{(1)} + \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\mathbb{E}_t\left[\Delta r_{t+i}^{(1)}\right] + \frac{1}{n}\sum_{i=0}^{n-1}\phi_{t+i,t+i+1}^{n} && (6)\\
r_t^{(n)} &= r_t^{*,(1)} + u_t^{(n)} && (7)\\
u_t^{(n)} &= u_t^{(1)} + \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\mathbb{E}_t\left[\Delta r_{t+i}^{(1)}\right] + \frac{1}{n}\sum_{i=0}^{n-1}\phi_{t+i,t+i+1}^{n} && (8)
\end{aligned}
$$

The model is naturally interpreted within a cointegration approach (Engle and Granger, 1987) to the stochastic drift in rates: if demographics, productivity and the inflation target of the central bank successfully capture the trend in nominal rates, then $u_t^{(1)}$ should be stationary. Stationarity of $u_t^{(1)}$, paired with stationarity of the term premia[5], implies that $u_t^{(n)}$ are stationary. Note also that, in this framework, the stochastic trends in yields at all maturities are all driven by the trend in one period rates.

Long-run forecast for MY_{t+i}, Δy_{t+1}^{pot}, π_{t+i}^{LR} are readily available in that demographics and potential output long-term forecast can be respectively downloaded from the Bureau of Census and the Fred database, while credibility of the central bank implies that long forecast for inflation cannot diverge from the CB target. Therefore, no VAR

[5]The term premium at time t and maturity n is given by $\frac{1}{n}\sum_{i=0}^{n-1}\phi_{t+i,t+i+1}^{n}$, which is the average of the expected one-period risk-premia over the residual maturity of the bond

is needed to obtain $r^{*,(1)}_{t+i}$, as these forecasts can be derived directly by using (4) with the appropriate scenario for the exogenous variables MY_{t+i}, Δy^{pot}_{t+1}, π^{LR}_{t+i}.

Once the stochastic trend has been removed from yields (and, potentially, excess returns), the general procedure consists of extrating the K factors from principal components to the N cyclical components of the yield curve $u^{(j)}_t$, for $j = 1, \dots, n$, which we stack into a $T \times N$ matrix, $\mathbf{U}$. We denote these K factors as $X_t \in \mathbb{R}^K$, and they are the first K principal components of $\mathbf{U}$. This procedure ensures the stationarity of X_t to specify a VAR, i.e.,

$$X_{t+1} = \mu + \Phi X_t + v_{t+1} \tag{9}$$

$$v_{t+1} | \left(X_s\right)^t_{s=0} \sim \mathcal{N}(0, \Sigma), \tag{10}$$

where $\mu \in \mathbb{R}^K$, $\Phi \in \mathbb{R}^{K \times K}$ and $\Sigma \in \mathbb{R}^{K \times K}$. On the other hand, the variables in X_t determine the market price of risk, λ_t, in the following affine form:

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t), \tag{11}$$

The assumption of no-arbitrage implies that there exists a pricing kernel, M_t, such that:

$$P^{(n)}_t = \mathbb{E}_t \left(M_{t+1} P^{(n-1)}_{t+1} \right), \tag{12}$$

for every $n > 0$ and $t \geq 0$, and where $P^{(n)}_t = \exp\left[-n r^{(n)}_t\right]$ is the price of a zero coupon bond with maturity n. We are strictly following Adrian et al. (2015). Hence, we assume that the pricing kernel is exponentially affine, i.e.,

$$m_{t+1} = -r_t^{(1)} - \frac{1}{2}\lambda_t^{\mathrm{T}}\lambda_t - \lambda_t^{\mathrm{T}}\Sigma^{-1/2}v_{t+1}, \tag{13}$$

where $r_t^1 = -\log\left(P_t^{(1)}\right) = -p_t^{(1)}$ is the continuously compounded risk-free rate, and $m_t = \log M_t$. The excess log returns are given by:

$$xr_{t+1}^{(n-1)} = p_{t+1}^{(n-1)} - p_t^{(n)} - r_t^{(1)},, \tag{14}$$

where $p_t^{(n)} = \log P_t^{(n)}$. After some derivations using (13) and (12) (see Appendix A.1), we arrive to

$$\mathbb{E}_t\left(xr_{t+1}^{(n-1)}\right) = \mathrm{cov}_t\left[xr_{t+1}^{(n-1)}, v_{t+1}'\Sigma^{-1/2}\lambda_t\right] - \frac{1}{2}\mathbb{V}_t\left(xr_{t+1}^{(n-1)}\right), \tag{15}$$

In the same fashion as Adrian et al. (2013), we can define $\beta_t^{(n-1)'}$ as

$$\beta_t^{(n-1)} := \Sigma^{-1}\mathrm{cov}_t\left(xr_{t+1}^{(n-1)}, v_{t+1}\right) \in \mathbb{R}^K. \tag{16}$$

By substituting from (16) into (15) and using (11), we have:

$$\mathbb{E}_t\left[xr_{t+1}^{(n-1}\right] = \lambda_t \cdot \beta_t^{(n-1)} - \frac{1}{2}\mathbb{V}_t\left[xr_{t+1}^{(n)}\right], \tag{17}$$

The unexpected excess return can be decomposed in a component that is correlated with v_{t+1}, and whose correlation vector coincides with $\beta_t^{(n-1)}$, and another component which is conditionally orthogonal to v_t, and which can be interpreted as the return pricing error:

$$xr_{t+1}^{(n-1)} - \mathbb{E}_t\left(xr_{t+1}^{(n-1)}\right) = \beta_t^{(n-1)} \cdot v_{t+1} + e_{t+1}^{(n-1)}, \tag{18}$$

Under the assumption that the return pricing error are i.i.d. with variance σ^2 and that β_t is constant, the generating process for log excess returns becomes:

$$\begin{aligned} xr_{t+1}^{(n-1)} = (\lambda_0 + \lambda_1 X_t)^{\mathrm{T}} \beta^{(n-1)} - \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) \\ + v_{t+1}^{\mathrm{T}} \beta^{(n-1)} + \varepsilon_{t+1}^{(n-1)}, \end{aligned} \tag{19}$$

and so it's clear now that the (log) excess returns can be decomposed into the expected return (first term), a convexity correction (second term), and a return innovation. This expression also allows us to see that the time-varying component of expected excess returns is stationary and driven by the dynamics of the stationary state variables. We can thus stack (19) across N maturities and T time-periods to obtain the following matrix-form representation:

$$\mathbf{xr} = \left(\lambda_0 \mathbb{1}_{T\times 1}^{\mathrm{T}} + \lambda_1 \mathbf{X}_-^{\mathrm{T}}\right)^{\mathrm{T}} \mathbf{B} - \frac{1}{2}\left(\mathbf{B}^* \mathrm{vec}\left(\Sigma\right) + \sigma^2 \mathbb{1}_{K\times 1}\right) \mathbb{1}_{T\times 1}^{\mathrm{T}} + \mathbf{V}^{\mathrm{T}}\mathbf{B} + \mathbf{E} \tag{20}$$

where $\mathbb{1}_{l\times m}$ is a matrix of ones for each $l, m \in \mathbb{N}$, and

1. $\mathbf{xr} \in \mathbb{R}^{T\times N}$.
2. $\lambda_0 \in \mathbb{R}^K$, $\lambda_1 \in \mathbb{R}^{K\times K}$,
3. $\mathbf{X}_- = [X_1 \mid X_2 \mid \cdots \mid X_{T-1}]^{\mathrm{T}} \in \mathbb{R}^{T\times K}$,
4. $\mathbf{B} \in \mathbb{R}^{K\times N}$,
5. $\mathbf{B}^* = [\mathrm{vec}\left(B_1 B_1^{\mathrm{T}}\right) \mid \cdots \mid \mathrm{vec}\left(B_n B_n^{\mathrm{T}}\right)]^{\mathrm{T}} \in \mathbb{R}^{K^2\times N}$,
6. $\mathbf{V} \in \mathbb{R}^{T\times K}$ and $\mathbf{E} \in \mathbb{R}^{T\times N}$.

4.2 Parameters' Estimation

We proceed with the parameter estimation by extending the 3-step procedure proposed by Adrian et al. (2013) to a 4-step procedure. All details can be found in the Appendix A.2, but everything can be summarised to adapting the classical model to the de-trended yield curve.

1. Construct the cyclical components of yields at all maturities by estimating a (cointegrating) regression of the one-period rate as a function of predictable slow-moving variables and use the available predictions on the drivers of the trend in one-year yields to model the trend for yields at all maturities by taking the appropriate average of the expected trends in the one-period rate as described in (**??**).

2. Construct the pricing factors, $\mathbf{X}$, from principal component analysis (PCA) of the cyclical components of yields derived in the first step, $\mathbf{U}$. Estimate the equation (9) using OLS, decomposing the pricing factors into predictable components and factor innovations $\hat{V}$.

3. Regress excess returns on a constant, lagged pricing factors and contemporaneous pricing factor innovations according to

$$\mathbf{xr} = a\mathbb{1}_{T\times K}\mathbb{1}_{K\times N} + \hat{\mathbf{V}}b + \mathbf{X}_{-}c + \mathbf{E} \tag{21}$$

4. We show in the Appendix A.2 that

$$a = \left(\lambda_0 \mathbb{1}_{T\times 1}^{\mathrm{T}}\right)^{\mathrm{T}} \mathbf{B} - \frac{1}{2}\left(\mathbf{B}^{*}\mathrm{vec}\left(\Sigma\right) + \sigma^2 \mathbb{1}_{K\times 1}\right)\mathbb{1}_{T}^{\mathrm{T}} \tag{22}$$

$$c = \lambda_1^{\mathrm{T}}\mathbf{B} \tag{23}$$

From these, market price of risk's estimates are given by

$$\hat{\lambda}_0 = \left(\hat{\mathbf{B}}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\hat{\mathbf{B}}\left[\hat{a}^{\mathrm{T}} + \frac{1}{2}\mathbb{1}_{T\times 1}\left(\mathbf{B}^{*}\mathrm{vec}\left(\Sigma\right) + \sigma^2\mathbb{1}_{N\times 1}\right)^{\mathrm{T}}\right], \tag{24}$$

$$\hat{\lambda}_1 = \left(\hat{\mathbf{B}}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\hat{\mathbf{B}}\hat{c}^{\mathrm{T}}. \tag{25}$$

4.3 Modelling Trending Yields

Bond prices at any maturity can be obtained by recursive forward substitution of prices in (15), keeping in mind that the (log) price of all bonds at maturity is zero, i.e., $p_{t+n}^{(0)} = 0$. The cyclical component of the one-period bond r_t^1, i.e., $u_t^{(1)} \coloneqq r_t^{(1)} - r_t^{*,(1)}$, can be expressed as a linear function of the underlying factors, i.e.,

$$r_t^{(1)} = r_t^{*,(1)} + \delta_0 + \delta_1 \cdot X_t + \varepsilon_t^{(1)}, \tag{26}$$

$$p_t^{(1)} = -r_t^{(1)}, \quad p_t^{*,(1)} = -r_t^{*,(1)},$$

where parameters $\hat{\delta}_0$ and $\hat{\delta}_1$ can be estimated by projecting the cycle $u_t^{(1)}$ on the stationary factors X_t, and $\cdot$ denotes the inner product.

On the other hand, No-A2rbitrage implies that bond prices depend linearly on a trend component and on a stationary component[6]:

$$p_t^{(n)} = p_t^{*,(n)} + A_n + X_t^{\mathrm{T}} B_n + \varepsilon_t^{(n)}, \tag{27}$$

where $p_t^{n,*}$ captures the trend component of bond prices. The model also implies

[6]See Appendix A.3 for more details

cross-equation restrictions on the parameters A_n, B_n and on the trend $p_t^{n,*}$.

$$A_n = A_{n-1} + (\mu - \lambda_0)^{\mathrm{T}} B_{n-1} + \frac{1}{2}\left(B_{n-1}^{\mathrm{T}} \Sigma B_{n-1} + \sigma^2\right) - \delta_0 \tag{28}$$

$$B_n = (\Phi - \lambda_1)^{\mathrm{T}} B_{n-1} - \delta_1 \tag{29}$$

$$p_t^{(n),*} = p_{t+1}^{*,(n-1)} - r_t^{*,(1)} \tag{30}$$

In this specification, trends affect yields but excess returns are driven exclusively by stationary variables. The main innovation in our proposal is that the vector X_t is extracted from the de-trended term structure and therefore the drivers of the excess-returns are the factors extracted from the cyclical components of the yield curve. Note that our specification imposes on the dynamics of de-trended bond prices exactly the same restrictions that a standard model imposes on the dynamics of bond prices. Hence, the comparison of the output of our model with that of a comparable ATSM model is immediate. In the case of a standard ATSM, the same VAR structure that we use for factors extracted from the cyclical components of yields is adopted directly for factors extracted from yields, without de-trending them. In this specification,

$$r_t^{(1)} = \delta_0 + \delta_1 \cdot X_t^{ACM} + \epsilon_t^{(1)}, \tag{31}$$

$$p_t^{(n)} = C_n + \left(X_t^{ACM}\right)^{\mathrm{T}} D_n + \epsilon_t^{(n)}, \tag{32}$$

where the recursive restrictions apply to C_n, and D_n. Basically, everything is the same but the trendy terms are drifting prices and yields. Hence, in this specification yields (trendy) and excess returns (stationary) are driven by the same set of state variables, X_t^{ACM} (Adrian et al., 2013).

4.4 Model Simulation, Forecasting and Term Premia

After the estimation is completed, we have the following model:

$$
\begin{aligned}
r_t^{(1)} &= r_t^{*,(1)} + u_t^{(1)} && (33)\\
r_t^{*,(1)} &= \gamma_1 MY_t + \gamma_2 \Delta y_t^{pot} + \gamma_3 \pi_t^{LR} && (34)\\
p_t^{(n)} &= p_t^{*,(n)} + A_n + B_n^{\mathrm{T}} X_t + \varepsilon_t^{(n)}, && (35)\\
X_t &= \mu + \Phi X_{t-1} + v_{t+1} && (36)\\
r_t^{(n)} &= -\frac{1}{n} p_t^{(n)} && (37)
\end{aligned}
$$

in which the factors X_t are extracted from the cyclical components of yields, u_t^n, after the completion of the first stage of estimation. The model fit can be readily assessed, by comparing actual data with fitted data from the model, model forecasts are also naturally constructed using the factor structure. Finally, model simulation in two scenarios, a baseline with all parameters set are their fitted values and an alternative one in which the market price of risk is set to zero, i.e. $\lambda_0 = \lambda_1 = 0$, allows to compute term premia as the differences between the model implied yields and the risk neutral yields.

The performance of our model in terms of fit, forecast and the properties of the derived term premia can be compared with that of a standard ATSM model:

$$
\begin{aligned}
p_t^{(n)} &= C_n + \left(X_t^{ACM}\right)^{\mathrm{T}} D_n + \varepsilon_t^{(n)}, && (38)\\
X_t^{ACM} &= \mu + \Phi X_{t-1}^{ACM} + v_{t+1}, && (39)
\end{aligned}
$$

in which estimation is implemented in three steps and the factors X_t^{ACM} are extracted directly from the yield curve (i.e., not detrended).

5 Empirical Results

Estimation and simulation[7] is performed by using the zero coupon yields provided by the FED[8] (Gürkaynak et al., 2007), data on MY_t, the ratio of middle-aged (40-49) to young (20-29) obtained from the Bureau of Census, the survey-based measure of long-run inflation expectations, used in the Fed's FRB/US model[9] and the measure of potential Gross Domestic Product available from the FRED database.[10] Quarterly data over the period 1980:1-2023:2 are considered. In this section, we shall report evidence based on the comparison between the simulation of our model estimated in four steps, which we label as FF, and a standard ATS model estimated in three steps, which we label as ACM.

5.1 Detrending Yields

The trend in the one-period (three-month) rate is captured by projecting it on the proxy for the age structure of the population, potential output growth and the survey-based measure of long-run inflation expectations. The results, reported in Table 1, show that the estimated model produces stationary residuals, as witnessed by the reported value of the Augmented-Dickey Fuller (ADF) test, with estimated coefficients on the drivers of the drift on short-term rates in line with previous studies Favero and

[7]A full replication package in R is available from the authors' website

[8]https://www.federalreserve.gov/econres/feds/the-us-treasury-yield-curve-1961-to-the-present.htm

[9]https://www.federalreserve.gov/econres/us-models-package.htm.

[10]https://fred.stlouisfed.org/series/GDPPOT.

Fernandez-Fuertes (2023), Bauer and Rudebusch (2020), with a negative coefficients on MY capturing the effects of the age structure of the population on the supply of savings, and positive, and slightly larger than one, coefficients on potential output growth and long-run inflation expectations.

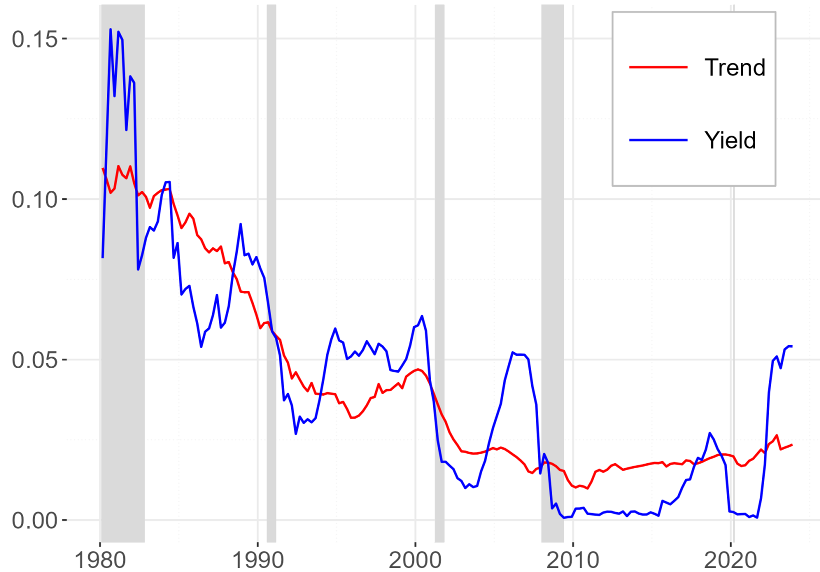
Table 1. Modelling the Trend in three-month yields

	Dependent variable:
	$r_t^{(1)}$
MY_t	-0.037^{***}
	(0.004)
Δy_t^{pot}	1.418^{***}
	(0.192)
π_t^{LR}	1.315^{***}
	(0.090)
Observations	174
Adjusted R^2	0.907
ADF test on residuals	-4.66^{***}
Residual Std. Error	0.017 (df = 171)
F Statistic	567.984^{***} (df = 3; 171)
Note:	*p<0.1; **p<0.05; ***p<0.01

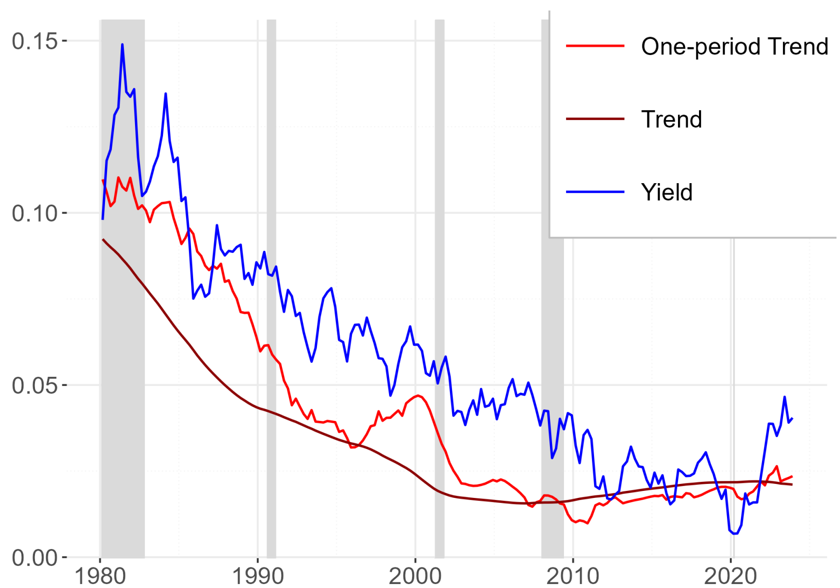
Given the trend component on one-period rates we derive the trend components for yields at any maturity as specified in Section 4. Figure 3 illustrates our results for the 3-month and the 10-year yields. Note that the cyclical components of yields contain information on the term-premia, therefore we expect them to fluctuate around a level different from zero and different across different maturities.

Figure 3. Trend Components

(a) Three month yield time series against its trend.



(b) Ten year yield time series against its trend.



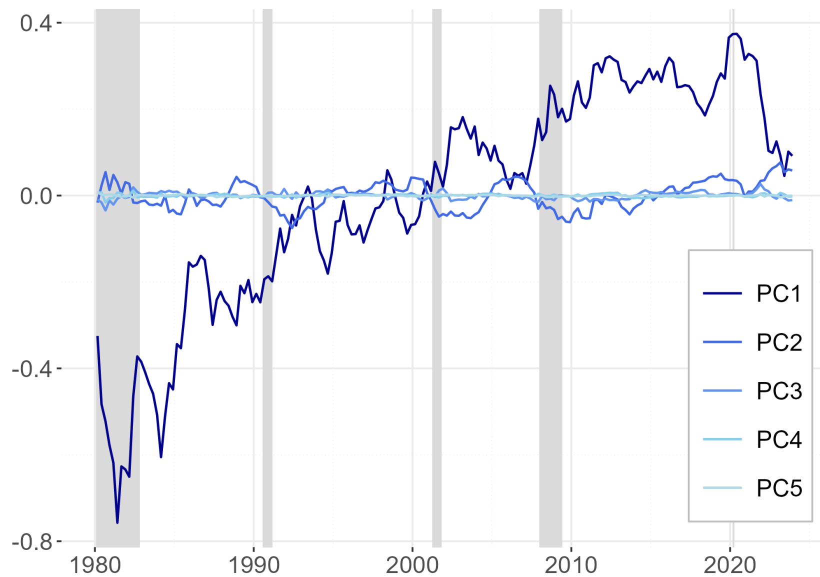
5.2 The VAR in factors

The second step in the approach we follow is the extraction of principal components from detrended yields and the specification of a VAR to model their dynamics.

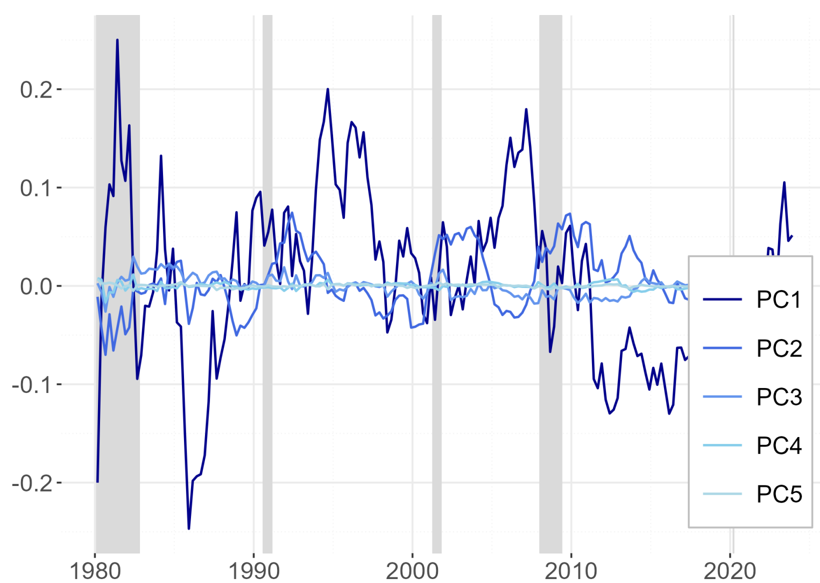
Following the specification choice path of Adrian et al. (2013), we consider as factors the first five principal components of a term structure of sixty cyclical components of yields with maturities from three-month (one quarter) to 15-year (sixty quarters),so we have , $N = 60$. Figure 4 illustrates the time series of these factors and compares them with those of the equivalent factors extracted from the term structure of yields with maturities from three-month to 15-year. The graphical evidence clearly hints at the presence of a drift in at least one of the factors estimated in the standard ACM approach, while the FF framework seems successful in removing it.

Figure 4. Principal Components

(a) ACM



(b) FF



Indeed, if we look at the two alternative VAR specifications for the five factors , we see that there exists a near unit-root in the VAR associated to the ACM model in which there's no correction for potential trends, which is not present in the VAR specification for the five PC extracted from detrended yields in the FF model.

TABLE 2. Roots of the characteristic polynomial for the VAR specification of PC extracted from yields (Panel A) and of the VAR specification for PC extracted from the detrended yields (Panel B).

Panel A: VAR on PCs from yields (Adrian et al., 2013)				
PC1	PC2	PC3	PC4	PC5
0.9721	0.9053	0.6890	0.5127	0.3467

Panel B: VAR on PCs from yields' cyclical component				
PC1	PC2	PC3	PC4	PC5
0.89473	0.89473	0.7625	0.4935	0.3490

As we see in Table 2, the highest eigenvalue of the coefficient matrix of the VAR model, i.e. equals $0.9721 \approx 1$. However, this almost-unit root is eliminated in our model, in which the highest eigenvalue is just 0.89473.

The empirical findings presented herein validate the assertions established in prior research (Campbell and Shiller, 1987) regarding the stationarity of term premia. By subtracting the average predicted trend of one-period rates from yields across all maturities, we effectively eliminate the drift from the entire yield curve.

Note that the observed stationarity of term premia emerges from our empirical investigation rather than being a predetermined assumption. In cases where term premia exhibit non-stationarity, subtracting the expected trend in one-period rates from yields fails to eliminate the underlying drift in yields. This underscores the robustness of our empirical approach in identifying the dynamic properties of term premia and their impact on yield behaviour.

5.3 Excess Returns regressions

We report in Table 3 the results of regressing excess returns on a constant, lagged pricing factors, X_t, and contemporaneous pricing factor innovations, v_{t+1}, for the standard factor specification and our factor specification in the spirit of equation (19). In particular, we consider the R^2 from the "predictive" specification in which contemporaneous pricing factor innovations are not included and the full specification and compare them with the version in which contemporaneous pricing factor innovations are included in both models, ACM and FF. It's worth highlighting that the predictive version of the model, which incorporates factors extracted from the cyclical components of yields, outperforms the standard model. Indeed, R^2 is below 0.20 across all maturities and around 0.1 in almost all maturities in the standard ACM model, whilst it's higher than 0.10 in all maturities in our model. However, when we consider the full specification, the standard model achieves a nearly perfect fit, with R^2 near one in every maturity, surpassing the alternative FF model's performance.

TABLE 3. This table reports the R^2 of regresing excess returns on, either only the lagged pricing factors, X_t, or on lagged pricing factors together with contemporaneous pricing innovations, v_{t+1}.

Panel A: ACM model

	1	2	3	4	5	6	7	8
X_t	0.17	0.13	0.10	0.09	0.09	0.09	0.10	0.11
X_t and v_{t+1}	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Panel BB: FF model

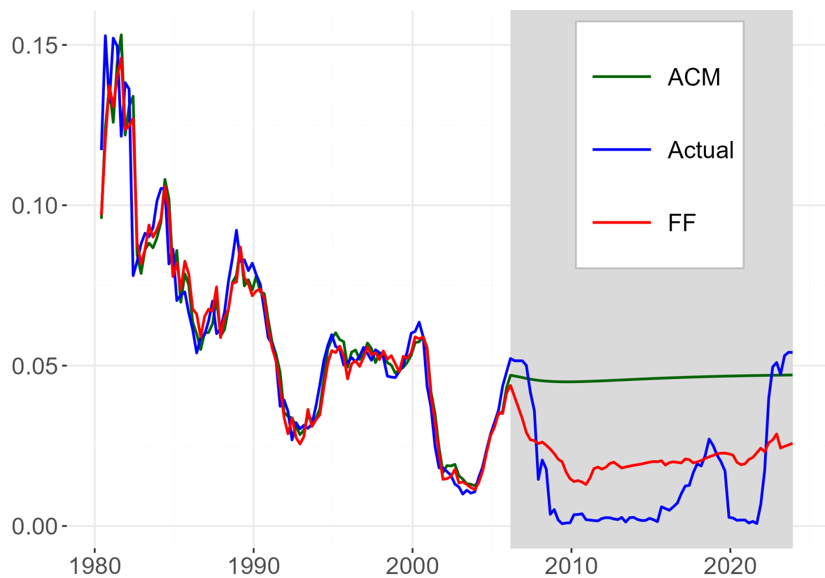
	1	2	3	4	5	6	7	8
X_t	0.15	0.13	0.13	0.13	0.13	0.13	0.13	0.13
X_t and v_{t+1}	0.94	0.93	0.92	0.91	0.90	0.89	0.87	0.86

5.4 Fit and Forecast Performance

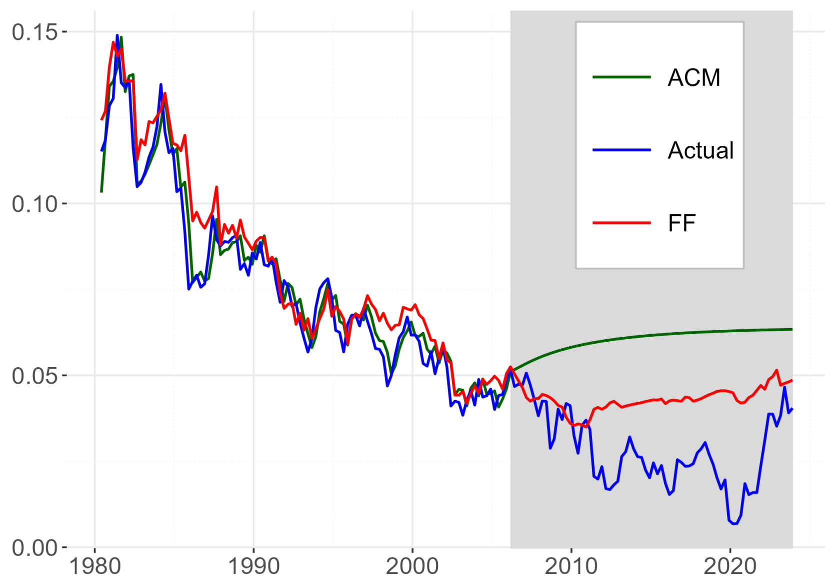
To illustrate the fit and the forecasting performance of the two alternative specifications, we report in Figure 5 the results of a within-sample model simulation up to 2005:Q4, where current values of the factors are used to predict yields, and of out-sample model simulation from 2006:Q1 onward, where n-step ahead forecasts of the factors (with n going from 1-quarter to 70-quarters) are used to predict yields.

Figure 5. This graph reports the fitted (1980Q1:2005Q4) and the forecasted (2006Q1:2023Q2) time series of 1Y and 10Y yields given by the standard ACM model (green) and our model (red) against the actual values (blue). The shaded area indicates our selected out-of-sample period.

(a) 3-month yield.



(b) Ten year yield.



Results for 3-month and 10-year yields are reported. The within-sample performance of our models is slightly inferior to that of the standard model. This is to be expected in that the trend components of yields are not as effective in fitting the term structure as the five factors extracted from yields. However, the FF model clearly dominates out-of-sample, showing the capability of tracking well the long-term dynamics of yields, especially at long-horizon. Indeed, the standard model behaves very poorly out-of-sample, as it is evident by simply looking at the picture: the forecasted path is basically a straight line whose level is much higher than the average of the realised one. The presence of a unit root together with the very low R^2 in the regressions without innovations, v_{t+1}, compared to the near one R^2 when including them may explain these phenomena (Panel A, first row of Table 3). The FF model instead exploits cointegration and the predictability of the slow-moving components driving the trend in yields for long-term forecasting purposes.

5.5 Term Premia

As it seems clear from Figure 6, the term premia derived by the two alternative models appear to be quite distinct: the 10-year term premium suggested by the conventional model exhibits a noticeable trend, in contrast to the model that uses the trend-cycle decomposition of yields, where a cyclicality in term premia emerges. The a-cyclicality of term premia estimated by standard ATS models and their parallelism to the secular trend in long-term interest rates has been already noted by Bauer et al. (2014) in commenting on the estimates provided by Wright (2011). Bauer et al. (2014) attribute the acyclicality to small sample bias caused by the very high persistence in the VAR model for factors; they show that biased-adjusted estimates produce instead countercyclical term premia. In fact, adjusting for small sample-bias produces estimates that are much closer to the unit root, preventing the sequence of predicted one-period rate to converge to a biased estimate of their level. Christensen and Rudebusch (2012) build a model in which the (three: level, slope, curvature) unobservable latent factors indeed follow a mean-reverting Ornstein-Uhlenbeck process. However, because of the reasons already exposed, they are forced to break this mean-reversion property by killing the drift term of the first factor, letting it being a standard Brownian Motion. In discrete time jargon, they are just forcing their model to have a unit root (at least in the first factor). This issue has been common across the classic ATSM models, as it's also evident from Figure 4. Our approach is different. We strongly claim that it's not necessary to rely on unit roots to solve the small-sample bias problem. Since the one-period rate's trend is captured by the long-term drivers, factors are then extracted from the deviations of yields from their drift explained by productivity demographics and long-term inflation forecasts. Our VAR for factors is much less persistent and the parameters' estimates do not require a small sample adjustment. As a result, the sequence of predicted one-period rates features much smaller forecast errors than the equivalent in standard ATS models and also our estimates of the term premia show some counter-cyclical behaviour visible in Figure 6, where the NBER recessions highlighted by shaded areas. This evidence is in line with the empirical and theoretical research that has found support for countercyclical risk premia, including, among many others, Campbell and Shiller (1987), Cochrane and Piazzesi (2005),Campbell and Cochrane (1999), and Wachter (2006). It is also easy to mathematically see that, under this view, the term premia is decontaminated from trendy terms. The term premium is defined by

$$TP_t^{(n)} = r_t^{(n)} - \sum_{k=1}^{n} \mathbb{E}_t \left[r_{t+k}^{(1)} \right]. \tag{40}$$

If we just decompose it in $r_t^{(n)} = r_t^{*,(n)} + u_t^{(n)}$, we get

$$TP_t^{(n)} = u_t^{(n)} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_t \left[r_{t+i}^{*,(1)} \right] + r_t^{*(n)} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_t \left[r_{t+i}^{*,(1)} \right] \tag{41}$$

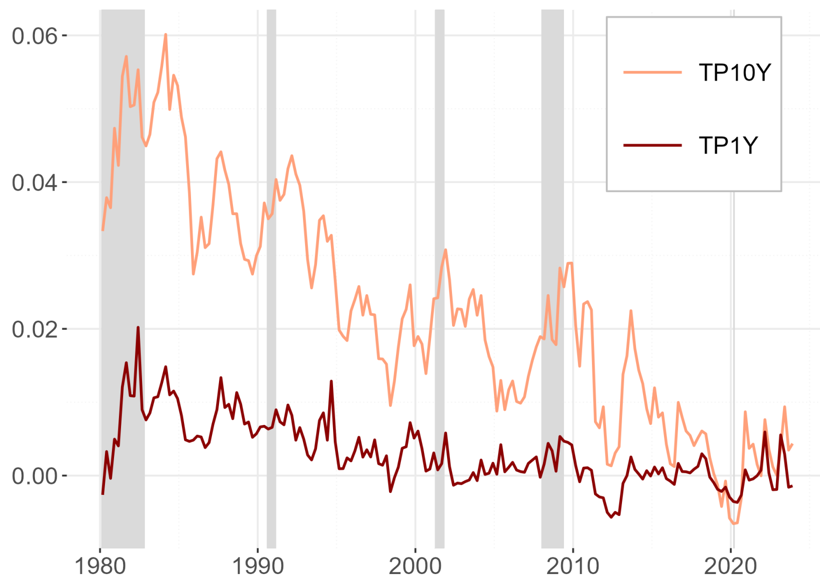
From (**??**), we can thus write it as

$$TP_t^{(n)} = u_t^{(n)} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_t \left[u_{t+i}^{*,(1)} \right] \tag{42}$$

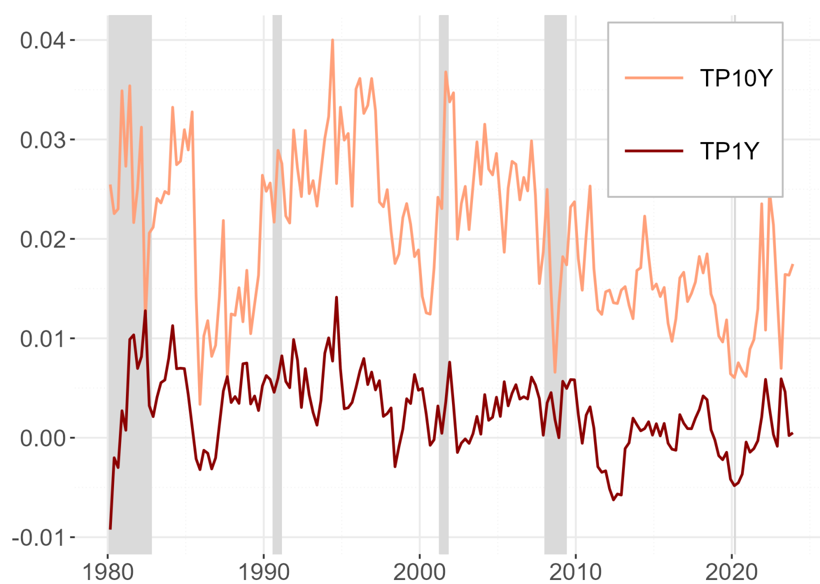
Therefore, $TP^{(n)}$ has no trend.

FIGURE 6. This figure reports the 1Y (red) and 10Y (orange) term premia in the two models. Shaded areas coincide with recession periods.

(A) ACM.



(B) FF.



6 Conclusions

Yields to maturity are (co-)drifting and holding period excess returns are (co-)cycling. Standard Affine Term Structure model do not separate trends and cycles in the data, but use factors extracted from yields to maturity to explain holding period excess returns as well as yields to maturity. As a consequence, the empirical model has a rather disappointing performance in predicting short-term rates and generates trending risk premia. This trend is steeper at longer horizons. As risk premia are not observable, term structure models should be evaluated by their performance in predicting the future path of short-term rates. Risk premia are very strongly dependent on this path. We propose a novel way to improve on the standard approach by applying the no-arbitrage restrictions to a model in which the factor structure adopted to explain holding period excess returns is extracted from de-trended yields. The trend in yields is a common trend driven by the drift in short-term rates. The drift in short-term rates in turn is not predicted by a VAR but it is related to long-term forecast for slow-moving variables such as the demographic structure of the population, potential output growth and long-term inflation forecast. A VAR structure is then adopted to model the dynamics of the stationary cyclical components. Our proposed model outperforms the standard approach in forecasting short-term rates and produces stationary risk premia, very different from those produced by the standard approach.

A Appendix

A.1 Derivations

As Adrian et al. (2013), we assume that the systematic risk is represented by a stochastic vector, $(X_t)_{t\geq 0}$, that follows a stationary vector autoregression

$$X_t = \mu + \Phi X_{t-1} + v_t \tag{A.1}$$

with initial condition X_0 and whose residual terms, $(v_t)_{t\geq 0}$ follow a Gaussian distribution with variance-covariance matrix, Σ, i.e,.

$$v_t \big| \left(X_s\right)_{0\leq s\leq t} \sim \mathcal{N}\left(0, \Sigma\right). \tag{A.2}$$

Let's denote the zero coupon treasury bond price with maturity n at time t by $P_t^{(n)}$. We take the following assumptions:

Assumption 1. No-arbitrage condition holds (Dybvig and Ross, 1989), i.e.,

$$P_t^{(n)} = \mathbb{E}_t\left[M_{t+1}P_{t+1}^{n-1}\right]. \tag{A.3}$$

Assumption 2. The pricing kernel, $m_{t+1} \coloneqq \log M_{t+1}$, is exponentially affine

$$m_{t+1} = -r_t^{(1)} - \frac{1}{2}||\lambda_t||^2 - \lambda_t^{\intercal}\Sigma^{-\frac{1}{2}}v_{t+1}, \tag{A.4}$$

where $r_t^{(1)} \coloneqq -p_t^{(1)}$ is the continuously compounded risk-free rate, and $\lambda_t \in \mathbb{R}^K$.

Assumption 3. Market prices of risk are affine

$$\lambda_t = \Sigma^{-\frac{1}{2}} \left(\lambda_0 + \lambda_1 X_t \right), \tag{A.5}$$

where $\lambda_0 \in \mathbb{R}^K$ and $\lambda_1 \in \mathbb{R}^{K \times K}$.

Assumption 4. $\left(xr_t^{(n-1)}, v_t \right)_{t \geq 0}$ are jointly normally distributed for $n \geq 2$.

Thanks to all these assumptions, we can continue our modelling by recalling the definition of the excess holding return of a bond maturing in n periods, i.e.,

$$xr_{t+1}^{(n-1)} \coloneqq p_{t+1}^{(n-1)} - p_t^{(n)} - r_t^{(1)}, \tag{A.6}$$

where $n-1$ indicates the $n-1$ periods remaining since time $t+1$ with respect to which the return is computed. Now, (A.3) can be rewritten as

$$\begin{aligned}
1 &= \mathbb{E}_t \left[\exp \left\{ m_{t+1} + p_{t+1}^{(n-1)} - p_t^{(1)} \right\} \right] \\
&= \mathbb{E}_t \left[\exp \left\{ -r_t^{(1)} - \frac{1}{2} ||\lambda_t||^2 - \lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1} + xr_{t+1}^{(n)} + r_t^{(1)} \right\} \right] \\
&= \mathbb{E}_t \left[\exp \left\{ xr_{t+1}^{(n)} - \frac{1}{2} ||\lambda_t||^2 - \lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1} \right\} \right] \\
&= \exp \left\{ \mathbb{E}_t \left[\xi_{t+1} \right] + \frac{1}{2} \mathbb{V} \left[\xi_{t+1} \right] \right\},
\end{aligned} \tag{A.7}$$

where $\xi_{t+1} \coloneqq xr_{t+1}^{(n)} - \frac{1}{2} ||\lambda_t||^2 - \lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1}$, and

$$\begin{aligned}
\mathbb{E}_t \left[\xi_{t+1}^{(n-1)} \right] &= \mathbb{E}_t \left[xr_{t+1}^{(n-1)} \right] - \frac{1}{2} ||\lambda_t||^2 \\
\mathbb{V}_t \left[\xi_{t+1}^{(n-1)} \right] &= \mathbb{V}_t \left[xr_{t+1}^{(n-1)} - \lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1} \right] \\
&= \mathbb{V}_t \left[xr_{t+1}^{(n-1)} \right] + \mathbb{V}_t \left[\lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1} \right] - 2 \mathrm{cov} \left(xr_{t+1}^{(n-1)}, \lambda_t^{\mathrm{T}} \Sigma^{-\frac{1}{2}} v_{t+1} \right)
\end{aligned} \tag{A.8}$$

$$= \mathbb{V}_t\left[xr_{t+1}^{(n-1)}\right] + \lambda_t^{\mathrm{T}}\Sigma^{-\frac{1}{2}}\mathbb{V}_t\left[v_{t+1}\right]\Sigma^{-\frac{1}{2}}\lambda_t - 2\lambda_t^{\mathrm{T}}\Sigma^{-\frac{1}{2}}\mathrm{cov}_t\left(xr_{t+1}^{(n-1)}, v_{t+1}\right)$$

$$= \mathbb{V}_t\left[xr_{t+1}^{(n-1)}\right] + ||\lambda_t||^2 - 2\lambda_t^{\mathrm{T}}\Sigma^{\frac{1}{2}}\beta_t^{(n-1)}. \tag{A.9}$$

where

$$\beta_t^{(n-1)} := \Sigma^{-1}\mathrm{cov}_t\left(xr_{t+1}^{(n-1)}, v_{t+1}\right) \in \mathbb{R}^K. \tag{A.10}$$

Therefore, no-arbitrage condition (A.3) is equivalent to

$$0 = \mathbb{E}_t\left[xr_{t+1}^{(n-1}\right] + \frac{1}{2}\mathbb{V}_t\left[xr_{t+1}^{(n)}\right] - \lambda_t^{\mathrm{T}}\Sigma^{\frac{1}{2}}\beta_t^{(n-1)}, \tag{A.11}$$

which gives us the following expression for the expected returns:

$$E_t\left[xr_{t+1}^{(n-1)}\right] = \lambda_t^{\mathrm{T}}\Sigma^{\frac{1}{2}}\beta_t^{(n-1)} - \frac{1}{2}\mathbb{V}_t\left[xr_{t+1}^{(n)}\right]. \tag{A.12}$$

Assumption 5. $\beta_t^{(n)} = \beta^{(n)}$ for every $t \geq 0$.

If we were to decompose the unexpected excess return, $xr_{t+1}^{(n-1)} - \mathbb{E}_t\left[xr_{t+1}^{(n-1)}\right]$ into a component that is correlated with v_{t+1} and another component which is conditionally orthogonal, $\varepsilon_{t+1}^{(n-1)}$ (return pricing error), we could simply write the following OLS-wise form

$$xr_{t+1}^{(n-1)} - \mathbb{E}_t\left[xr_{t+1}^{(n-1)}\right] = v_{t+1}^{\mathrm{T}}\gamma^{(n-1)} + \varepsilon_{t+1}^{(n-1)}. \tag{A.13}$$

and try to figure out who the $\gamma^{(n-1)}$ is. To do so, notice that

$$\beta_t^{(n-1)} = \Sigma^{-1}\left(\mathbb{E}\left[xr_{t+1}^{(n-1)}v_{t+1}\right] - \mathbb{E}\left[xr_{t+1}^{(n-1)}\right]\mathbb{E}_t\left[v_{t+1}\right]\right) = \Sigma^{-1}\mathbb{E}\left[xr_{t+1}^{(n-1)}v_{t+1}\right]$$

and

$$\gamma^{(n-1)} = \left(\mathbb{E}\left[v_{t+1}^{\mathrm{T}} v_{t+1}\right]\right)^{-1} \mathbb{E}\left[v_{t+1} xr_{t+1}^{(n-1)}\right] = \Sigma^{-1} \mathbb{E}\left[xr_{t+1}^{(n-1)} v_{t+1}\right],$$

because $\mathbb{E}\left[v_{t+1}^{\mathrm{T}} v_{t+1}\right] = \Sigma$. Therefore, $\gamma^{(n)} = \beta^{(n)}$ for every $n \geq 0$. With this identity in our hands,

$$\begin{aligned}
\mathbb{V}\left[xr_{t+1}^{(n-1)}\right] &= \mathbb{E}_t\left[\left(xr_{t+1}^{(n-1)} - \mathbb{E}_t\left[xr_{t+1}^{(n-1)}\right]\right)^2\right] \\
&= \mathbb{E}_t\left[\left(v_{t+1}^{\mathrm{T}} \beta^{(n-1)} + \varepsilon_{t+1}^{n-1}\right)^2\right] \\
&= \mathbb{E}_t\left[\left(v_{t+1}^{\mathrm{T}} \beta^{(n-1)}\right)^2 + 2 v_{t+1}^{\mathrm{T}} \beta^{(n-1)} \varepsilon_{t+1}^{(n-1)} + \left(\varepsilon_{t+1}^{(n-1)}\right)^2\right] \\
&= \left(\beta^{(n-1)}\right)^{\mathrm{T}} \mathbb{E}_t\left[v_{t+1} v_{t+1}^{\mathrm{T}}\right] \beta^{(n-1)} + \sigma^2 \\
&= \left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2,
\end{aligned}$$

Finally,

$$\begin{aligned}
xr_{t+1}^{(n-1)} = (\lambda_0 + \lambda_1 X_t)^{\mathrm{T}} \beta^{(n-1)} - \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) \\
+ v_{t+1}^{\mathrm{T}} \beta^{(n-1)} + \varepsilon_{t+1}^{(n-1)}.
\end{aligned} \tag{A.14}$$

A.2 Estimation

We can then rewrite (A.14) as

$$xr_{t+1}^{(n-1)} = (\lambda_0 + \lambda_1 X_t)^{\mathrm{T}} B_{n-1} - \frac{1}{2}\left(B_{n-1}^{\mathrm{T}} \Sigma B_{n-1} + \sigma^2\right) + v_{t+1}^{\mathrm{T}} B_n + e_{t+1}^{(n-1)} \tag{A.15}$$

and therefore have a vectorial form:

$$\mathbf{xr} = \left(\lambda_0 \mathbb{1}_{T\times 1}^{\mathrm{T}} + \lambda_1 \mathbf{X}_{-}^{\mathrm{T}}\right)^{\mathrm{T}} \mathbf{B} - \frac{1}{2}\left(\mathbf{B}^* \mathrm{vec}\left(\Sigma\right) + \sigma^2 \mathbb{1}_{K\times 1}\right) \mathbb{1}_T^{\mathrm{T}} + \mathbf{V}^{\mathrm{T}} \mathbf{B} + \mathbf{E} \tag{A.16}$$

where

1. $\mathbf{xr} \in \mathbb{R}^{T\times N}$.
2. $\lambda_0 \in \mathbb{R}^K$, $\lambda_1 \in \mathbb{R}^{K\times K}$,
3. $\mathbf{X}_- = [X_1 \mid X_2 \mid \cdots \mid X_{T-1}]^\mathrm{T} \in \mathbb{R}^{T\times K}$,
4. $\mathbf{B} \in \mathbb{R}^{K\times N}$,
5. $\mathbf{B}^* = [\mathrm{vec}\,(B_1 B_1^\mathrm{T}) \mid \cdots \mid \mathrm{vec}\,(B_n B_n^\mathrm{T})]^\mathrm{T} \in \mathbb{R}^{K^2\times N}$,
6. $\mathbf{V} \in \mathbb{R}^{T\times K}$ and $\mathbf{E} \in \mathbb{R}^{T\times N}$.

So we take (A.16) as our reference point in the estimation process that we do in three steps following Adrian et al. (2013) procedure:

1. Construct the pricing factors, $(X_t)_{t=1}^T$ and estimate the VAR coefficients $\mu \in \mathbb{R}^K$ and $\Phi \in \mathbb{R}^K$ in (A.1) using OLS. Then take $(\hat{v}_t)_{t=1}^T$ from $\hat{v}_t \coloneqq X_t - \hat{X}_t \in \mathbb{R}^K$, where $\hat{X}_t = \mu + \Phi X_{t-1}$ for every $t = 1, \ldots, T$. Stack the time series $(v_t)_{t=1}^T$ into the matrix $\hat{\mathbf{V}} \in \mathbb{R}^{T\times K}$. The variance-covariance matrix is thus

$$\hat{\Sigma} = \frac{\hat{\mathbf{V}}^\mathrm{T}\hat{\mathbf{V}}}{T} \tag{A.17}$$

2. Perform the regression according to (A.16), i.e.,

$$\mathbf{xr} = a\mathbb{1}_{T\times K}\mathbb{1}_{K\times N} + \hat{\mathbf{V}}b + \mathbf{X}_- c + \mathbf{E} \tag{A.18}$$

where $a \in \mathbb{R}$, $b, c \in \mathbb{R}^{K\times N}$. Collect everything into single matrices

$$\mathbf{Z} = \left[\mathbb{1}_{T\times 1} \mid \hat{\mathbf{V}} \mid \mathbf{X}_-\right] \in \mathbb{R}^{T\times(2K+1)} \tag{A.19}$$

$$d = [a\mathbb{1}_{K\times 1} \mid b \mid c]^\mathrm{T} \in \mathbb{R}^{(2K+1)\times N} \tag{A.20}$$

so we can write $\mathbf{xr} = \mathbf{Z}d + \mathbf{E}$ and therefore

$$\hat{d} = \left(\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\right)^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{xr}. \tag{A.21}$$

Then, collect the residuals from this regression into the matrix

$$\hat{\mathbf{E}} = \mathbf{xr} - \mathbf{Z}\hat{d} \in \mathbb{R}^{T\times N}. \tag{A.22}$$

and estimate

$$\hat{\sigma}^2 = \frac{\mathrm{tr}\left(\hat{\mathbf{E}}^{\mathrm{T}}\hat{\mathbf{E}}\right)}{NT}. \tag{A.23}$$

Finally, we construct $\hat{\mathbf{B}}^*$ from $\hat{b}$.

3. Estimate the price of risk parameters, λ_0 and λ_1 via cross-sectional regression. Recall from (A.16) that

$$a = \left(\lambda_0 \mathbb{1}_{T\times 1}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{B} - \frac{1}{2}\left(\mathbf{B}^*\mathrm{vec}\left(\Sigma\right) + \sigma^2\mathbb{1}_{K\times 1}\right)\mathbb{1}_T^{\mathrm{T}} \tag{A.24}$$

$$c = \lambda_1^{\mathrm{T}}\mathbf{B} \tag{A.25}$$

If we transpose them, we can estimate λ_0 and λ_1 via OLS, i.e.,

$$\hat{\lambda}_0 = \left(\hat{\mathbf{B}}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\hat{\mathbf{B}}\left[\hat{a}^{\mathrm{T}} + \frac{1}{2}\mathbb{1}_{T\times 1}\left(\mathbf{B}^*\mathrm{vec}\left(\Sigma\right) + \sigma^2\mathbb{1}_{N\times 1}\right)^{\mathrm{T}}\right] \tag{A.26}$$

$$\hat{\lambda}_1 = \left(\hat{\mathbf{B}}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\hat{\mathbf{B}}\hat{c}^{\mathrm{T}} \tag{A.27}$$

A.3 Recursion for the Term Structure

Consider the generating process for log excess returns in our model:

$$xr_{t+1}^{(n-1)} = (\lambda_0 + \lambda_1 X_t)^{\mathrm{T}} \beta^{(n-1)} - \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) + v_{t+1}^{\mathrm{T}} \beta^{(n-1)} + \varepsilon_{t+1}^{(n-1)}. \tag{A.28}$$

We need now to find two sequences of coefficients, $(A_n)_{n=1}^N$ and $(B_n)_{n=1}^N$, that allow us to express bond prices as exponentially affine in the vector of state variables, X_t, plus a trend term, $p_t^{*,(n)}$, i.e.,

$$p_t^{(n)} = p_t^{*,(n)} + A_n + X_t^{\mathrm{T}} B_n + e_t^{(n)}, \tag{A.29}$$

where $p_t^{(n)} \coloneqq \log P_t^{(n)}$. Notice that

$$p_t^{(1)} = -r_t^{(1)} = -r_t^{*,(1)} - \delta_0 - X_t^{\mathrm{T}} \delta_1, \tag{A.30}$$

motivating that $A_1 = -\delta_0$, $B_1 = -\delta_1$, and $p_t^{1,*} = -r_t^{*,(1)}$. For any $n > 1$,

$$\begin{aligned}
xr_{t+1}^{(n-1)} &= p_{t+1}^{*,(n-1)} + A_{n-1} + X_{t+1}^{\mathrm{T}} B_{n-1} + e_{t+1}^{(n-1)} \\
&- p_t^{*,(n)} - A_n - X_t^{\mathrm{T}} B_n - e_t^{(n)} \\
&+ p_t^{*,(1)} + A_1 + X_t^{\mathrm{T}} B_1 + e_t^{(1)} \\
&= p_{t+1}^{*,(n-1)} + A_{n-1} + (\mu + \Phi X_t + v_{t+1})^{\mathrm{T}} B_{n-1} + e_{t+1}^{(n-1)} \\
&- p_t^{*,(n)} - A_n - X_t^{\mathrm{T}} B_n - e_t^{(n)} \\
&+ p_t^{*,(1)} + A_1 + X_t^{\mathrm{T}} B_1 + e_t^{(1)} \qquad \text{(A.31)} \\
&= xr_{t+1}^{*,(n-1)} + (A_{n-1} - A_n + A_1 + \mu^{\mathrm{T}} B_{n-1})
\end{aligned}$$

$$+ X_t^{\mathrm{T}} \left(\Phi^{\mathrm{T}} B_{n-1} - B_n + B_1\right) + \left(e_{t+1}^{n-1} - e_t^{(n)} + e_t^{(1)}\right) + v_{t+1}^{\mathrm{T}} B_{n-1}.$$

Hence, the following must hold

$$\begin{aligned}
& xr_{t+1}^{*,(n-1)} + \left(A_{n-1} - A_n + A_1 + \mu^{\mathrm{T}} B_{n-1}\right) \\
& + X_t^{\mathrm{T}} \left(\Phi^{\mathrm{T}} B_{n-1} - B_n + B_1\right) + \left(e_{t+1}^{n-1} - e_t^{(n)} + e_t^{(1)}\right) \\
& = (\lambda_0 + \lambda_1) X_t^{\mathrm{T}} \beta^{(n-1)} - \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) + v_{t+1}\beta^{(n-1)} + \varepsilon_{t+1}^{(n-1)}
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_{n-1} - A_n + A_1 + \mu^{\mathrm{T}} B_{n-1} &= \lambda_0^{\mathrm{T}} \beta^{(n-1)} - \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) \\
\Phi^{\mathrm{T}} B_{n-1} - B_n + B_1 &= \lambda_1^{\mathrm{T}} \beta^{(n-1)} \\
u_{t+1}^{n-1} - u_t^{(n)} + u_t^{(1)} + v_{t+1}^{\mathrm{T}} B_{n-1} &= \varepsilon_{t+1}^{(n-1)} \\
xr_{t+1}^{*,(n-1)} &= 0 \\
v_{t+1}^{\mathrm{T}} \beta^{(n-1)} &= v_{t+1}^{\mathrm{T}} B_{n-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
A_n &= A_{n-1} + \mu^{\mathrm{T}} B_{n-1} - \lambda_0^{\mathrm{T}} \beta^{(n-1)} + \frac{1}{2}\left(\left(\beta^{(n-1)}\right)^{\mathrm{T}} \Sigma \beta^{(n-1)} + \sigma^2\right) + A_1 \\
B_n &= \Phi^{\mathrm{T}} B_{n-1} + B_1 - \lambda_1^{\mathrm{T}} \beta^{(n-1)} \\
p_t^{*,(n)} &= p_{t+1}^{*,(n-1)} - r_t^{*,(1)} \\
\beta^{(n)} &= B_n
\end{aligned}$$

The last equation simplifies everything even more:

$$A_n = A_{n-1} + (\mu - \lambda_0)^{\mathrm{T}} B_{n-1} + \frac{1}{2}\left(B_{n-1}^{\mathrm{T}} \Sigma B_{n-1} + \sigma^2\right) - \delta_0 \tag{A.32}$$

$$B_n = (\Phi - \lambda_1)^{\mathrm{T}} B_{n-1} - \delta_1 \tag{A.33}$$

$$p_t^{(n),*} = p_{t+1}^{(n-1),*} - r_t^{*,(1)} \tag{A.34}$$

Equation (A.34) for the price stochastic trend implies that

$$r_t^{*,(n)} = \frac{1}{n}\sum_{i=0}^{n-1} r_{t+i}^{*,(1)}. \tag{A.35}$$

On the other hand, these equations are fully deterministic, meaning that one can iterate all the equations back to get expressions that depend only on the initial values, A_1 and B_1. First,

$$\begin{aligned} B_n &= (\Phi - \lambda_1)^{\mathrm{T}} \left((\Phi - \lambda_1)^{\mathrm{T}} B_{n-2} - \delta_1\right) - \delta_1 \\ &= \cdots \\ &= \left[(\Phi - \lambda_1)^{\mathrm{T}}\right]^{n-1} B_1 - \sum_{j=1}^{n-2} \left[(\Phi - \lambda_1)^{\mathrm{T}}\right]^j \delta_1. \\ &= -\sum_{j=1}^{n-1} \left[(\Phi - \lambda_1)^{\mathrm{T}}\right]^j \delta_1 \end{aligned} \tag{A.36}$$

Second,

$$\begin{aligned} A_n &= A_{n-2} + (\mu - \lambda_0)^{\mathrm{T}} (B_{n-1} + B_{n-2}) + \frac{1}{2}\left(B_{n-1}^{\mathrm{T}} \Sigma B_{n-1} + B_{n-2}^{\mathrm{T}} \Sigma B_{n-2}\right) + 2\left(\frac{1}{2}\sigma^2 - \delta_0\right) \\ &= A_{n-2} + (\mu - \lambda_0)^{\mathrm{T}} (B_{n-1} + B_{n-2}) \end{aligned}$$

$$+\frac{1}{2}\left([B_{n-1}+B_{n-2}]^{\mathrm{T}}\Sigma\,[B_{n-1}+B_{n-2}]\right)+2\left(\frac{1}{2}\sigma^2-\delta_0\right)$$

$$=A_1+(\Phi-\lambda_1)^{\mathrm{T}}\sum_{j=1}^{n-1}B_{n-j}+\frac{1}{2}\left(\sum_{j=1}^{n-1}B_{n-j}\right)^{\mathrm{T}}\Sigma\left(\sum_{j=1}^{n-1}B_{n-j}\right)+(n-1)\left(\frac{1}{2}\sigma^2-\delta_0\right)$$

It's not difficult to see that

$$\sum_{j=1}^{n-1}B_{n-j}=\sum_{j=1}^{n-1}\sum_{k=1}^{n-j}\left[(\Phi-\lambda_1)^{\mathrm{T}}\right]^j\delta_1=\sum_{j=1}^{n-1}(n-j)\left[(\Phi-\lambda_1)^{\mathrm{T}}\right]^j\delta_1. \tag{A.37}$$

That allows us to write

$$\begin{aligned}A_n&=(\Phi-\lambda_1)^{\mathrm{T}}\sum_{j=1}^{n-1}(n-j)\left[(\Phi-\lambda_1)^{\mathrm{T}}\right]^j\\&+\frac{1}{2}\left(\sum_{j=1}^{n-1}(n-j)\,(\Phi-\lambda_1)^j\right)\Sigma\left(\sum_{j=1}^{n-1}(n-j)\left[(\Phi-\lambda_1)^{\mathrm{T}}\right]^j\right)\\&+n\left(\frac{1}{2}\sigma^2-\delta_0\right).\end{aligned} \tag{A.38}$$

A.4 Recursion for Term Premia

Remember that

$$TP_t^{(n)}=u_t^{(n)}-\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_t\left[u_{t+i}^{(1)}\right], \tag{A.39}$$

where $u_t^{(n)}=r_t^{(n)}-r_t^{*,(n)}$. The affine model implies that

$$u_t^{(n)}=-n\left(A_n+X_t^{\mathrm{T}}B_n+e_t^{(n)}\right). \tag{A.40}$$

In particular, for $n=1$,

$$u_t^{(1)}=-A_1-X_t^{\mathrm{T}}B_1-e_t^{(1)}. \tag{A.41}$$

Hence,

$$\mathbb{E}_t\left[u_{t+i}^{(1)}\right] = -A_1 - \mathbb{E}_t\left[X_{t+i}^{\mathrm{T}}\right] B_1. \tag{A.42}$$

Now, since $X_{t+i} = \mu + \Phi X_{t+i-1} + v_{t+i}$, then, we can iterate backwards to get

$$\begin{aligned} X_{t+i} &= \mu + \Phi X_{t+i-1} + v_{t+i} \\ &= \mu + \Phi\left(\mu + \Phi X_{t+i-2} + v_{t+i-1}\right) + v_{t+i} \\ &= (1+\Phi)\mu + \Phi^2 X_{t+i-2} + \Phi v_{t+i-1} + v_{t+i} \\ &= \cdots \\ &= \left(\sum_{j=0}^{i-1} \Phi^j\right)\mu + \Phi^i X_t + \sum_{j=0}^{i-1} \Phi^j v_{t+i-j}. \end{aligned} \tag{A.43}$$

Since $\mathbb{E}_t\left[v_s\right] = 0$ for every $s > t$, then

$$\mathbb{E}_t\left[X_{t+i}\right] = \widetilde{\Phi}_i \mu + \Phi^i X_t, \tag{A.44}$$

where

$$\widetilde{\Phi}_i = \left(\sum_{j=0}^{i-1} \Phi^j\right). \tag{A.45}$$

Hence,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_t\left[u_t^{(1)}\right] &= -A_1 - \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{\Phi}_i\mu + \Phi^i X_t\right)^{\mathrm{T}} B_1 \\ &= -A_1 - \frac{1}{n}B_1^{\mathrm{T}}\left(\sum_{i=1}^{n}\widetilde{\Phi}_i\right)\mu - \frac{1}{n}B_1^{\mathrm{T}}\left(\sum_{i=1}^{n}\Phi^i\right)X_t \\ &= -A_1 - \frac{1}{n}B_1^{\mathrm{T}}\left(\sum_{i=1}^{n}\widetilde{\Phi}_i\right)\mu - \frac{1}{n}B_1^{\mathrm{T}}\widetilde{\Phi}_n X_t \\ &= \Xi_n + \Psi_n X_t\end{aligned} \tag{A.46}$$

where

$$\Xi_n = -\frac{1}{n}A_1 - \frac{1}{n}B_1^{\mathrm{T}}\left(\sum_{i=1}^{n}\widetilde{\Phi}_i\right)\mu \tag{A.47}$$

$$\Psi_n = -\frac{1}{n}B_1^{\mathrm{T}}\widetilde{\Phi}_i \tag{A.48}$$

Hence,

$$TP_t^{(n)} = u_t^{(n)} + \Xi_n + \Psi_n X_t \tag{A.49}$$

References

Adrian, Tobias, Richard K Crump, and Emanuel Moench (2013) "Pricing the term structure with linear regressions," *Journal of Financial Economics*, 110 (1), 110–138.

——— (2015) "Regression-based estimation of dynamic asset pricing models," *Journal of Financial Economics*, 118 (2), 211–244.

Ang, Andrew and Monika Piazzesi (2003) "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables," *Journal of Monetary economics*, 50 (4), 745–787.

Bakshi, Gurdip S and Zhiwu Chen (1994) "Baby boom, population aging, and capital markets," *Journal of business*, 165–202.

Bauer, Michael D. and Glenn D. Rudebusch (2020) "Interest Rates under Falling Stars," *American Economic Review*, 110 (5), 1316–54.

Bauer, Michael D, Glenn D Rudebusch, and Jing Cynthia Wu (2014) "Term premia and inflation uncertainty: Empirical evidence from an international panel dataset: Comment," *American Economic Review*, 104 (1), 323–337.

Campbell, John Y and John H Cochrane (1999) "By force of habit: A consumption-based explanation of aggregate stock market behavior," *Journal of political Economy*, 107 (2), 205–251.

Campbell, John Y and Robert J Shiller (1987) "Cointegration and tests of present value models," *Journal of political economy*, 95 (5), 1062–1088.

——— (1991) "Yield spreads and interest rate movements: A bird's eye view," *The Review of Economic Studies*, 58 (3), 495–514.

Christensen, Jens HE and Glenn D Rudebusch (2012) "The response of interest rates to US and UK quantitative easing," *The Economic Journal*, 122 (564), F385–F414.

Cieslak, Anna and Pavol Povala (2015) "Expected returns in Treasury bonds," *The Review of Financial Studies*, 28 (10), 2859–2901.

Cochrane, John H and Monika Piazzesi (2005) "Bond risk premia," *American economic review*, 95 (1), 138–160.

Del Negro, Marco, Domenico Giannone, Marc P Giannoni, and Andrea Tambalotti (2019) "Global trends in interest rates," *Journal of International Economics*, 118, 248–262.

Diebold, Francis X, Monika Piazzesi, and Glenn D Rudebusch (2005) "Modeling bond yields in finance and macroeconomics," *American Economic Review*, 95 (2), 415–420.

Duffee, Gregory R (2002) "Term premia and interest rate forecasts in affine models," *The Journal of Finance*, 57 (1), 405–443.

Dybvig, Philip H and Stephen A Ross (1989) "Arbitrage," in *Finance*, 57–71: Springer.

Engle, Robert F and Clive WJ Granger (1987) "Co-integration and error correction: Representation, estimation, and testing," *Econometrica*, 55 (2), 251–276.

Fama, Eugene F (2006) "The behavior of interest rates," *The Review of Financial Studies*, 19 (2), 359–379.

Favero, Carlo A and Ruben Fernandez-Fuertes (2023) "Monetary Policy in the COVID Era and Beyond: The Fed vs the ECB," *Available at SSRN 4557795*.

Favero, Carlo A, Arie E Gozluklu, and Haoxi Yang (2016) "Demographics and the behavior of interest rates," *IMF Economic Review*, 64 (4), 732–776.

Favero, Carlo A, Alessandro Melone, and Andrea Tamoni (2022) "Monetary policy and bond prices with drifting equilibrium rates," *Journal of Financial and Quantitative Analysis*, 1–26.

Feunou, Bruno and Jean-Sébastien Fontaine (2023) "Secular economic changes and bond yields," *The Review of Economics and Statistics*, 105 (2), 408–424.

Golinski, Adam and Paolo Zaffaroni (2016) "Long memory affine term structure models," *Journal of Econometrics*, 191 (1), 33–56, https://EconPapers.repec.org/RePEc:eee:econom:v:191:y:2016:i:1:p:33-56.

Gürkaynak, Refet S, Brian Sack, and Eric Swanson (2005) "The sensitivity of long-term interest rates to economic news: Evidence and implications for macroeconomic models," *American economic review*, 95 (1), 425–436.

Gürkaynak, Refet S, Brian Sack, and Jonathan H Wright (2007) "The US Treasury yield curve: 1961 to the present," *Journal of monetary Economics*, 54 (8), 2291–2304, https://www.federalreserve.gov/econres/feds/the-us-treasury-yield-curve-1961-to-the-present.htm, Accessed on 2023/10.

Jardet, Caroline, Alain Monfort, and Fulvio Pegoraro (2013) "No-arbitrage Near-Cointegrated VAR(p) term structure models, term premia and GDP growth," *Journal of Banking & Finance*, 37 (2), 389–402, 10.1016/j.jbankfin.2012.0.

Jordà, Òscar and Alan M Taylor (2019) "Riders on the Storm,"Technical report, National Bureau of Economic Research.

Jørgensen, Kasper (2018) "How Learning from Macroeconomic Experiences Shapes the Yield Curve," Finance and Economics Discussion Series 2018, Board of Governors of the Federal Reserve System, `https://ideas.repec.org/p/fip/fedgfe/2015-77.html`.

Kim, Don H and Jonathan H Wright (2005) "An arbitrage-free three-factor term structure model and the recent behavior of long-term yields and distant-horizon forward rates."

Kozicki, Sharon and P. A. Tinsley (2001) "Shifting endpoints in the term structure of interest rates," *Journal of Monetary Economics*, 47 (3), 613–652.

Laubach, Thomas and John C Williams (2003) "Measuring the natural rate of interest," *Review of Economics and Statistics*, 85 (4), 1063–1070.

Lunsford, Kurt G and Kenneth D West (2019) "Some evidence on secular drivers of US safe real rates," *American Economic Journal: Macroeconomics*, 11 (4), 113–39.

Mian, Atif R, Ludwig Straub, and Amir Sufi (2021) "What Explains the Decline in r*? Rising Income Inequality Versus Demographic Shifts," *Becker Friedman Institute for Economics Working Paper* (2021-104).

Piazzesi, Monika, J.Salomao, and Martin Schneider (2015) "Trend and cycle in bond premia," `https://web.stanford.edu/~piazzesi/trendcycle.pdf`, Accessed on 2023/10.

Rogoff, Kenneth S, Barbara Rossi, and Paul Schmelzing (2022) "Long-run trends in long-maturity real rates 1311-2021,"Technical report, National Bureau of Economic Research.

Schnabel, Isabel (2022) "United in diversity-Challenges for monetary policy in a currency union," `https://www.ecb.europa.eu/press/key/date/2022/html/ecb.sp220614~67eda62c44.en.html`, Accessed on 2023/08.

——— (2023) "The Last Mile. Keynote speech at the annual Homer Jones Memorial Lecture," `https://www.ecb.europa.eu/press/key/date/2023/html/ecb.sp231102_1~4bb07ebef7.en.html`, Accessed on 2023/11.

Wachter, Jessica A (2006) "A consumption-based model of the term structure of interest rates," *Journal of Financial economics*, 79 (2), 365–399.

Wright, Jonathan H (2011) "Term premia and inflation uncertainty: Empirical evidence from an international panel dataset," *American Economic Review*, 101 (4), 1514–1534.

Zhao, Guihai (2020) "Learning, equilibrium trend, cycle, and spread in bond yields,"Technical report, Bank of Canada Staff Working Paper.