Equilibrium expectation errors and asset pricing anomalies

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Abstract

We introduce a financial market model populated by imperfectly informed mean-variance investors. The set-up is identical to the Capital Asset Pricing Model (CAPM), but with the perfect information assumption fully relaxed - that is, heterogeneous and biased beliefs on the asset means, variances and correlations. The model shows that not only exposures to the market return matter in equilibrium, but also exposures to the expectation errors which have a persistent three-term structure. The pricing relationship is hence a four-dimensional space but can be conveniently expressed as a two-dimensional plane, the Security Market Plane (SMP). We provide an empirical procedure similar to two-pass regressions that allows to test for the out-of-sample pricing relevance of any arbitrary mean-variance beliefs, for a given set of observed prices. We use the procedure to assess whether the beliefs implied by the Institutional Brokers' Estimate System (I/B/E/S) can explain well-known asset pricing anomalies such as the value, size and idiosyncratic volatility premia.

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1 Introduction

The Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) states that the market return should be the only priced risk factor. Any cash-flow can be priced with respect to the risk-free rate and its covariance with the market. It is intuitively attractive because it is an equilibrium single factor model and it conveniently summarizes the relationship between expected return and risk. However, it is valid under strict assumptions and has been heavily criticized in the literature. In particular, the assumption of perfect information which implies that the representative investor represented by the market has full knowledge of the asset true moments. The assumption automatically implies that the market is mean-variance efficient, leading to the pure one-factor pricing relationship represented by the Security Market Line (SML). As an equilibrium model, the CAPM is based on the work of Markowitz (1952) on Modern Portfolio Theory (MPT). In that set-up, investors choose mean-variance efficient portfolios. While appealing in theory, a proper optimization requires knowledge of the mean vector and the variance-covariance matrix of the asset returns. In practice, the asset return moments are unknown and must be estimated using available information, likely with errors. The portfolios identified using the MPT are known to be extremely sensitive to the moment estimates and often result in unreasonably large weights. The optimality of the MPT solution is only as valid as the moment estimates used for the computations. The historical expected return vector and variance-covariance matrix are known to be backward looking and hence not representative of future returns.

Crucial to the CAPM is the perfect information assumption, which implies both homogeneous and correct beliefs for all investors. Using intuition, the homogeneity assumption can be easily invalidated: in practice, there are many different actors co-existing within one market. The divergences in investor beliefs can come from different information sets or different ways to interpret the same information. Moreover, investors have differentiated access to funds and financial constraints, which can arguably preclude investors to fully act on their beliefs. However, heterogeneity in itself does not directly challenge the CAPM model results. Indeed, Levy, Levy and Benita (2006) relax the homogeneity assumption and show that, although the two-fund separation theorem does not hold, the CAPM pricing relationship still holds under *unbiased* heterogeneous beliefs. They deduce that a violation of the homogeneity assumption in itself can not explain the lack of empirical support to the SML relationship. However, their results do not generalize to the case of *biased* heterogeneous beliefs. In other words, the perfect information assumption is only partly relaxed since the market correctly aggregates information. In contrast, Williams (1977) studies heterogeneous and biased beliefs with respect to the asset means, using Brownian motion processes. Since security prices are distributed log-normally in continuous time, investors estimate the unknown variances and covariances with complete accuracy. He shows that biased heterogeneity in the asset means can explain momentary deviations to the CAPM pricing relationship. However, as time passes and information unfolds, the biased beliefs should become unbiased and the SML should converge to its usual pure one-factor form. For linear risk tolerance utility functions in a dynamic set-up, Jouini and Napp (2006) show that a heterogeneous belief equilibrium is equivalent to a homogeneous or consensus belief equilibrium plus an aggregation bias which takes the form of a discount rate, proportional to the belief dispersion. They show that if investors are pessimistic, the bias results in an increase in the market price of risk with respect to the standard case. However, as it is the case in Williams (1977), there can be no heterogeneity with respect to volatilities and correlations since the model is in continuous time. In a companion paper but in discrete time, Jouini and Napp (2006) study the impacts of heterogeneous beliefs for general utility functions on the market price of risk and risk-free rate. Since the model is in discrete time, there can also be disagreements on the assets' volatilities, leading to richer distortion effects. They focus on the beliefs aggregation and the effects of pessimism or doubt, without formally distinguishing the impacts each type of expectation errors. The impact of disagreement and errors in the risk dimension is likely to be important and complex, especially in high-dimension. In the static CAPM set-up with mean-variance preferences, Chiarella, He and Dieci (2006) develop an aggregation procedure that allows for investors' heterogeneity in the mean, variance and correlation beliefs. It relies on the rewriting of the sum of the individual optimal weights into a representative belief, which is a risk tolerance weighted sum of the beliefs. Contrary to the approach developed by Jouini and Napp (2006), there is no bias due to the aggregation. Chiarella et al. do not construct the homogeneous belief from the heterogeneous belief for general utility functions, but rather mechanically find the representative belief for mean-variance preferences and in a static set-up. Investors are allowed to have different views not only about the asset expected returns, but also about the risk related to these returns. Indeed, in discrete time, there is no reason to assume that heterogeneity is restricted to the mean dimension. Investor views about expected returns are likely to be associated with personal views about the risk inherent to these same returns. As highlighted by Levy, Levy and Benita (2006), the full relaxation of the assumption implies not only heterogeneous beliefs, but also biased beliefs. In addition, it should allow for biased beliefs about the components of portfolio risk - that is, asset variances and covariances.

The linear SML relationship has found little empirical support in the empirical asset pricing literature, as early as Black, Jensen and Scholes (1972). They find a positive linear relationship between market beta and expected returns, but also intercepts (alphas) significantly different from zero, contradicting the results of the CAPM. There seems to be a consistent related structure behind the alphas, as low (high) beta portfolios tend to have positive (negative) alphas. They suggest that expected returns are better explained by a two-factor model than the single-factor model represented by the CAPM. The expected return variation left unexplained by the market return is considered as anomalous if it is found to have a common cross-sectional structure. Asset pricing anomalies usually relate to firm characteristics, like the well-documented value and size anomalies. The size premium investigated by amongst others by Banz (1981) describes the fact that stocks with low market capitalization (small stocks) tend to earn positive abnormal returns with respect to the CAPM. It is the opposite for stocks with high market capitalization (big stocks), which tend to under-perform. Similarly, the value premium is the observation that stocks with high book-to-market ratio (value stocks) tend to earn positive abnormal returns whereas stocks with low book-to-market ratio (growth stocks) tend to earn negative abnormal returns. It has long been observed in empirical return data, as early as Graham and Dodd (1934). Another well-documented anomaly is the low volatility or low beta anomaly in the context of the CAPM. Stocks with low beta tend to have low risk as expected, but earn significantly higher returns than expected by the theory. It dates from as early as Black (1972) and Black, Jensen and Scholes (1972), but the anomaly is more recently investigated by Baker, Bradley and Wurgler (2011) who give a behavioural explanation to the anomaly. Based on the value and size effects observed in the data, Fama-French (1993) build a three-factor model by adding the value and size factors to the market factor. The linear pricing relationship stemming from the CAPM is used as the basis, but it is augmented by two exogenous characteristic-based portfolio factor returns. They are constructed as long-short portfolios that invest into the available asset returns according to binary rules in the underlying characteristics (book-to-market and market capitalization). They capture the spread in expected returns generated by the characteristic in the cross-section. Fama-French (1996) show that the three-factor model explain many anomalies identified in previous work. Since the three-factor model is based on empirical performance ex-post, there is no clear factor identification method. That is, many characteristic-based factors can be added to the pricing relationship if they are found to be empirically meaningful. In addition, there are degrees of freedom in the construction of the characteristic-based factors themselves since they are portfolios invested in the underlying assets. The Arbitrage Pricing Theory (APT) from Ross (1976) is the foundation for the linear factor structure in returns. It is based on no-arbitrage arguments rather than equilibrium mean-variance preferences, as it is the case for the CAPM. Stock returns are modelled as linear functions of arbitrary systematic risk factors and since the factors explain common variations in returns, they should earn positive risk premia. From this perspective, the market return is not special and only one amongst many possible factors. The APT and the Fama-French model identify additional sources of risk premia in the cross-section, aside from the market excess return explained by mean-variance theory. The APT factor selection method is not specified, aside that they should relate to common sources of risk. There has been hundreds of factor candidates in the literature, such that it has been termed as the "factor zoo" by Cochrane (2011). The asset pricing empirical research has then focused on the selection of the correct factor structure. Amongst others, the two-pass testing procedure from Feng, Giglio and Xiu (2020), which allows to systematically discriminate across models while taking omitted factors into account.

We propose a belief-based approach of the CAPM in which mean-variance investors do not know the true asset return underlying process. The perfect information assumption is fully relaxed in the sense that the consensus or representative belief is allowed to differ from the true return moments. The focus is not on the aggregation of the individual beliefs into the representative belief, but rather on the resulting bias or error between the representative belief and the true return distribution moments. More specifically, it is on the expectation errors on the risk dimension, i.e. the variances and correlations. The theoretical model stays agnostic on the sources of belief heterogeneity and biases. In other words, it is valid for arbitrary mean-variance investor beliefs implying arbitrary expectation errors. The representative market beliefs correspond to a perceived probability measure $\mathbb{P}$, which is different than the real or objective measure $\mathbb{O}$. Under the perspective of the representative investor, the market is efficient and the CAPM is valid. However, since the market is really inefficient, the CAPM that holds ex-ante under the subjective measure does not hold ex-post. The representative distribution and expected betas are unobservable. In that sense, the intuition behind the change of measure in the model is similar to the difference between the investor and the empiricist's CAPM explained in Andrei, Cujean and Wilson (2021). To explain the difference, they focus on the variation in the individual investors' betas, whereas we focus on the difference between the representative investor's beliefs and the true moments. In the presence of expectation errors, the SML slope can be positive under the representative measure $\mathbb{P}$ and flat under the objective measure $\mathbb{O}$. Two-fund separation holds with respect to the representative belief, but the market return is located within the mean-variance efficient frontier and the Capital Market Line (CML) is not an efficient frontier ex-post. The market weights are decomposed into four portfolios. The first one is defined by the true asset moments and corresponds to the efficient or informed part of the market portfolio. The other three error portfolios are defined by the expectation errors on both moments and the combination of both: Mean Errors (ME), Precision Errors (PE) and combined Mean and Precision errors (MPE). The belief deviations are not observable under the representative measure, that is from the point-of-view of the market. However, they are observable under the objective measure, with knowledge of the true moments. The precision matrix is the inverse of the variance-covariance matrix and summarizes all relevant information with respect to risk. The market weights decomposition is the starting point of all results in the model, such that all the additional error terms respect the three-term structure. The SML has additional systematic and idiosyncratic expectation error terms. There is a slope adjustment to the SML and the total equity premium (required excess return on exposure to the market portfolio) is not equal to the market excess return. The adjustment relates to the expectation errors made on the market return: it is positive when there is over-optimism (mean over-estimation or volatility under-estimation) and negative when there is over-pessimism (mean under-estimation or volatility over-estimation). It consists in an additional premium to consider when pricing any covariance with the market return. Simultaneously, there are idiosyncratic asset alphas which are consequences of the expectation errors made on each of the individual asset returns: they are positive when there is over-pessimism and negative when there is over-optimism. The market itself has an alpha with respect to the true error-adjusted SML, which is equal to the negative of the slope adjustment in the market premium formulation.

Using covariances, the alphas can be written as exposures to expectation error portfolios. As it is the case for the market factor, the asset exposures to each of the three error portfolios are measured by their corresponding betas: covariance with the given error portfolio return divided by its variance. This formulation shows that, when the perfect information is relaxed, the true pricing relationship is a four-dimensional space rather than a line. That is the case even if there is only one true external source of systematic risk. Due to the expectation errors, which are by definition not taken into account in the representative investor's utility function, four portfolio returns (instead of one) explain the cross-section of equilibrium expected returns ex-post. Hence, what appears as unexplained performance on the one-dimensional line is actually explained when observing the four-dimensional space. The three error portfolios can be conveniently regrouped into one error portfolio, summarizing all types of expectation errors. In this case, the four-factor pricing relationship becomes a two-dimensional plane; the Security Market Plane (SMP). The error-adjusted SML with idiosyncratic and systematic alphas consists then in a cut of the asset pricing plane at an error portfolio beta equal to zero. Equilibrium expected returns are then functions of two exposures, the market beta and the overall error portfolio beta. That is, two portfolios only explain the full cross-section of returns. This result is consistent with the ones of Andrei, Cujean and Fournier (2019) who show that adding a long-short portfolio to the market portfolio allows to always explain the cross-section of returns. In their paper, the additional portfolio is identified empirically using the distance between the market and the tangency portfolios. The expectation errors bring an equilibrium explanation to this deviation. The error portfolio stemming from the approach conducted in this paper corresponds to the negative of the Low-Minus-High portfolio and indeed does not proxy for fundamental risk. Regrouping all three portfolios into one error portfolio reduces the number of factors and makes the relationship easier to represent graphically, with the disadvantage of losing the information with respect to each type of errors. As previously stated, the most interesting effects are the one related to the expectation errors on the second market moment, combined with the error on the means. Biases with respect to risk, especially with respect to the correlation beliefs, have not been extensively studied in the literature although they are likely to be important in a high-dimensional context.

We show that the linear regression is a too simple model to test for the error-adjusted version of the CAPM. When the assumption is relaxed, the true pricing relationship is really four-dimensional. The expectation error risk is not a true risk factor since it is not related to any external source of risk. By definition, it is unaccounted for in the representative investor's optimal portfolio problem ex-ante. Nevertheless, it has cross-sectional power in explaining expected returns. Through the error pricing terms, the model can theoretically and jointly explain several characteristic-based cross-sectional anomalies. To derive the error-adjusted results, only the standard CAPM model is needed. Williams (1977) can be seen as a special case of the error-adjusted or imperfectly informed version of the CAPM in continuous time when there are only errors in the asset means. Kozak, Nagel and Santosh (2018) argue that factor models are closely related to investors' beliefs. The Fama-French portfolios are economically founded, but it is difficult to theoretically explain why they still exist in the data. The analysis conducted in this paper suggests that their existence relates to persistent biases in the way the market forms its expectations. It makes a structural link between beliefs and returns, showing that a multi-factor structure can be inferred from the data even in a single factor set-up due to expectation errors. Andrei, Cujean and Fournier (2019) argue that the existence of anomalies with respect to the market return is not evidence that the theory is wrong. Indeed, we show that anomalies are theoretically consistent with the CAPM once the perfect information is relaxed. That is, empirical tests based on the standard one-factor SML are tests of the perfect information assumption rather than the CAPM theory itself.

The theoretical framework is valid regardless of the objective and representative distributions on the mean-variance space. It describes the relationship between belief biases and observable equilibrium returns without taking a stand on where the errors come from. If the objective distribution can be estimated using samples of asset returns, the representative measure $\mathbb{P}$ is unobservable. By definition, it is the measure that generates the market weights. In theory, the asset moments under $\mathbb{P}$ could be inferred from the observed market weights. However, there are more unknowns than equations and hence an infinity of potential solutions. In turn, it implies that exogenous beliefs will (almost) never match the strong market weights condition. Rather than aiming for the correct beliefs, we provide an empirical method that uses the model structure and allows to test for any arbitrary beliefs on the mean-variance space. More precisely, it tests if the biases implied by the exogenous beliefs help explaining the cross-section of asset returns out-of-sample. In particular, the focus is on well-known asset pricing anomalies, like the value, size or idiosyncratic volatility. The procedure is similar to Fama-MacBeth (1973) two-pass regressions, except that it does not use linear regressions in the first step. In the empirical tests, the price targets from the Institutional Brokers' Estimate System (I/B/E/S) are used to estimate the representative belief but other beliefs formation system can be used. Given estimates for the objective return distribution moments, different beliefs implies different expectation errors and hence different error portfolios with different out-of-sample explaining power. Expectation errors in the means and especially variances and correlations can be tested within the framework of the model. Although it can be used on any asset returns, the approach requires the estimation of the variance-covariance matrix and hence imposes a condition on the number of assets considered. To that purpose, the available asset returns are regrouped into portfolios formed according to given characteristics, similarly to the 25 Fama-French portfolios formed on size and book-to-market. The market weights are used to weight the individual assets into each of the portfolios, such that they always sum to the market portfolio. Using portfolio returns rather than asset returns allows to have balanced samples and reduce idiosyncratic noise.

Explaining the anomalies corresponds then to explaining the cross-section of characteristic-sorted portfolio returns.

The paper is organized as follows. In section 2, the model is described, leading to the main propositions. In section 3, the risk-return relationship stemming from the model and the links with asset pricing puzzles are studied in more details. Section 4 describes the main pricing result of the paper, which is that the error-adjusted SML can be expressed as a four-factor beta relationship. Section 5 focuses on the empirical estimation of the model. Finally, section 6 concludes.

2 The model

The model is static, although the same method can likely be applied (with additional complications) to dynamic models. The objective is to capture the effect of imperfect information on both the return and risk dimensions. On the mean-variance space, it corresponds to expectation errors on the means, variances and covariances of the asset returns. The set-up is the same as the CAPM, but with the perfect information assumption relaxed. It is flexible and allows for a high-dimensional analysis of the asset returns. Under the true probability measure, expectation errors are observable. Using a belief-based decomposition of the market weights, it results in propositions relating the representative measure to the real one. The latter is the only measure under which the relationships are generated and observed ex-post.

2.1 Assets

The market is composed of N risky assets and one risk-free asset. The risky assets' gross returns multivariate distribution is denoted $\mathbf{F}$, with $N \times 1$-dimensional vector mean $\mu^{\mathbb{O}}$ and $N \times N$-dimensional variance-covariance matrix $\mathbf{\Sigma}^{\mathbb{O}}$:

$$\mathbf{R} \sim \mathbf{F}(\mu^{\mathbb{O}}, \mathbf{\Sigma}^{\mathbb{O}})$$

The rates of return from 0 to 1 are denoted $\mathbf{R}$ and R_f for respectively the risky assets and the risk-free asset. The asset returns can follow any multivariate distribution as long as they have finite means and variance-covariance.

2.2 Investors

There are J classes of investors who take positions in the assets at 0 such as to optimize their portfolio return at maturity 1. They have mean-variance preferences but are imperfectly informed and heterogeneous. The mean-variance set-up allows for flexibility as it does not require any assumption on the terminal wealth distribution. Investors only consider the mean and variance of their portfolio returns regardless of the true underlying distribution. The market is competitive and investors act as price takers.

2.2.1 The objective function

Since beliefs are heterogeneous across investor classes, the return moments are computed according to investor j's beliefs. Portfolio weights at time 0 and portfolio return at time 1 for investor j are denoted respectively by π^j and R_1^j. The problem is written as:

$$\begin{aligned} \underset{\pi^j}{\text{maximize}} \quad & E^j\left[R_1^j\right] - \theta_j Var^j\left[R_1^j\right] \\ \text{subject to} \quad & R_1^j = \pi^{j\prime}\mathbf{R} + (1 - \mathbb{1}'\pi^j)R_f \qquad \forall j = 1,..,J \end{aligned}$$

The portfolio return mean and variance from investor j's perspective depend on his beliefs about the expected asset return vector μ^j and return variance-covariance matrix $\mathbf{\Sigma}^j$. They correspond to a perceived multivariate return probability measure specific to investor j, denoted $\mathbb{P}^j$. The model stays agnostic on the sources of the belief heterogeneity and biases (behavioural biases, incorrect estimation, financial constraints,...). Since more informed agents have beliefs closer to the true moments, their portfolio positions should be more mean-variance efficient than the ones of uninformed agents, whose beliefs are further away from the true ones. Investors can be too optimistic or pessimistic on the asset expected returns, but also on the asset dispersions (risks) and co-dispersions. It can generate many belief combinations and attitudes with respect to the assets. For example, an agent might under-estimate an asset expected return, over-estimate its standard deviation and under-estimate its correlations with the rest of the market. Each class of investors j is allowed to have a different relative risk aversion coefficient θ_j. The framework is able to nest different strategies and investor types through different subjective moments and risk-aversion coefficients.

2.3 Equilibrium

The individual investors solve for their optimal portfolios, and the sum of their positions defines the time zero asset prices. The individual beliefs are aggregated into a representative belief which defines the multivariate probability measure corresponding to the market's perspective about the next period asset returns. The analysis is done on the representative investor which mechanically results from an aggregation of the individual investors. The market clears when the aggregate investment in the risky assets is equal to total market wealth, implying that the risk-free asset is in zero net aggregate supply. The market corresponds to a representative investor and a corresponding probability measure on the mean-variance space. It summarizes the aggregate views about the expected returns, the expected risks and the (linear) dependences in the individual asset returns. The representative distribution implies aggregate holdings, which are portfolio weights and prices when multiplied by the current total market wealth. When studied on the portfolio return space, the market is the return implied by the aggregate market weights. In the theoretical part, the market has several meanings as it is referred to as a probability measure, an investor and a portfolio.

2.3.1 The beliefs aggregation procedure

The solution of the one-period problem is the well-known mean-variance optimal portfolio:

$$\pi^j = \frac{\mathbf{\Omega}^j}{\theta_j}\left[\mu^j - \mathbb{1}R_f\right]$$
$$\pi_0^j = 1 - \mathbf{1}'\pi^j$$

The aggregation procedure is from Chiarella, He and Dieci (2006) and relies on writing the market weights as the individual optimal weights. The asset return representative moments correspond to a multivariate market probability measure, denoted $\mathbb{P}$. The solution is written as:

$$\pi_M = \frac{\mathbf{\Omega}^{\mathbb{P}}}{\theta_M}\left[\mu^{\mathbb{P}} - \mathbb{1}R_f\right]$$

$$\text{where: } \theta_M = \frac{1}{\sum_{j=1}^{J} \frac{1}{\theta_j}} \qquad \mathbf{\Omega}^{\mathbb{P}} = \theta_M \sum_{j=1}^{J} \frac{1}{\theta_j} \mathbf{\Omega}^j \qquad \mu^{\mathbb{P}} = (\Omega^{\mathbb{P}})^{-1} \theta_M \sum_{j=1}^{J} \frac{1}{\theta_j} \mathbf{\Omega}^j \mu^j$$

The representative beliefs are weighted sums of the individual beliefs. The weights depend on the investors' risk aversion coefficient, but also on the beliefs themselves. Extreme views, especially about the risk, imply a high contribution to the market representative measure. All the investors use the same portfolio policy, so the market weights depend on the overall wealth but not on the investors' individual wealth. The market weights formula above is actually a pricing formula since multiplying both sides by the market wealth gives the observed stock prices.

2.3.2 Market-clearing and the endogenous risk-free rate

The market clears when the aggregate investment in the risky assets equates the total market wealth, with the risk-free asset in zero aggregate net supply. The equilibrium risk-free rate can be endogenously determined:

$$r_f = \frac{\mathbb{1}' \mathbf{\Omega}^{\mathbb{P}} \mu^{\mathbb{P}} - \theta_M}{\mathbb{1}' \mathbf{\Omega}^{\mathbb{P}} \mathbb{1}} = R_f - 1$$

It is equal to the Lagrange multiplier corresponding to the full investment constraint $\sum \pi_M = 1$. The investors can lend and borrow to each other at the equilibrium risk-free rate but on the aggregate the borrowing amount is equal to the lending amount. The risk-free rate depends on the representative belief about the risky assets, the risk-aversion coefficient and the market wealth. The dependence on the two latter parameters is simple and intuitive: higher are the market wealth and the absolute risk aversion coefficient, lower is the risk-free rate. They decrease the investments in the risky assets, pushing the rate lower to increase the risk premia and prevent the market from investing in the risk-free asset. The effects of the represent beliefs are more complex and different beliefs result in a wide range of possible risk-free rates. In general, more attractive is the risky market (higher means, lower volatilities, better diversification opportunities), and higher is the risk-free rate. The asset impacts are differentiated in the cross-section depending on the perceived asset profile (low or high mean, low or high risk, highly correlated or not,...) considered.

2.4 From the subjective to the objective measure

In the perfect information case, the market knows the correct asset moments. Therefore, its aggregate holdings are efficient and the market lies on the MVE frontier. The market is the best attainable portfolio given the aggregate risk-aversion, with all the idiosyncratic risk effectively diversified. It can then be used as a factor to price all the assets exactly. When there are expectation errors, the market is no longer mean-variance efficient and the mechanism breaks down. Errors on the variances and correlations imply that idiosyncratic volatilities are not efficiently diversified. The usual equilibrium relationships only hold under the representative measure, but not under the objective measure which is observed ex-post. The objective of this section is to nest this duality and reconstruct the well-known relationships under $\mathbb{O}$ in order to capture the effects of imperfect information. The analysis is purely descriptive and hence agnostic about the origins of the expectation errors, as well as their repartition across the available assets. The framework developed in this section holds for any real distribution (projected on the mean-variance space) and any belief formation system.

2.4.1 The equilibrium relationships

The following propositions link the outputs of interest under $\mathbb{P}$ to the real ones under $\mathbb{O}$ by switching from the representative measure to the real measure. Hence, expectation errors which are unobservable under the representative measure become observable. $\mu^{\mathbb{O}}$, $\mathbf{\Sigma}^{\mathbb{O}}$ and $\mathbf{\Omega}^{\mathbb{O}} = \left(\mathbf{\Sigma}^{\mathbb{O}}\right)^{-1}$ denote respectively the real asset mean return vector, the real variance-covariance matrix and the real precision matrix. $\mu^{\mathbb{D}} = \mu^{\mathbb{P}} - \mu^{\mathbb{O}}$, $\mathbf{\Sigma}^{\mathbb{D}} = \mathbf{\Sigma}^{\mathbb{P}} - \mathbf{\Sigma}^{\mathbb{O}}$ and $\mathbf{\Omega}^{\mathbb{D}} = \mathbf{\Omega}^{\mathbb{P}} - \mathbf{\Omega}^{\mathbb{O}}$ denote the beliefs deviations from $\mathbb{P}$ to $\mathbb{O}$ for respectively the expected returns, the variance-covariance matrix and the precision matrix. $\mu_i^{\mathbb{D}}$ is positive in the case of an over-estimation of the mean return of asset i, and negative in the case of an under-estimation. When variances or covariances are over-estimated, $\mathbf{\Sigma}^{\mathbb{D}}$ has mostly positive elements. In the opposite, it has mostly negative elements when diversification opportunities are over-estimated or when variances are under-estimated. The decompositions often rely on the precision matrix bias $\mathbf{\Omega}^{\mathbb{D}}$, which is high-dimensional and complicated to analyze.

Proposition 1. *Market weights*

$$\begin{aligned}
&\textit{Under } \mathbb{P}: \\
\pi_M &= \frac{\mathbf{\Omega}^{\mathbb{P}}}{\theta_M}(\mu^{\mathbb{P}} - \mathbb{1}R_f) \\
&\textit{Under } \mathbb{O}: \\
\pi_M &= \frac{1}{\theta_M}\left[\mathbf{\Omega}^{\mathbb{O}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Omega}^{\mathbb{O}}\mu^{\mathbb{D}} + \mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}\right] \\
&= \pi_* + \pi_{PE} + \pi_{ME} + \pi_{MPE}
\end{aligned}$$

The market weights decomposition is the foundation for all the results in the model as it makes the link between expectation errors and prices. The market portfolio is a sum of four portfolios. The first one π_* relates to the correct asset moments, or the fundamental part of the market portfolio. The other ones are due to the beliefs errors, or the non-fundamental part of the market portfolio: Precision-Errors (PE), Mean-Errors (ME) and Mean-Precision-Errors (MPE). The existence of the three additional expectation errors portfolios characterizes market sub-optimality on the portfolio space. For simplicity, they can be considered altogether as one global error portfolio, which is a sum of the three different error portfolios. The more share they have in the market portfolio, the less efficient is the market return and the more important is the mispricing. In general, in the presence of errors, the portfolio corresponding to the correct moments will not sum to one, implying that a part of the market wealth is sub optimally invested. It in turn implies that the sum of the global error portfolio weights does not sum to zero. Expectation errors simultaneously impact the risk-free rate, which adjusts itself such as to respect the full investment constraint.

Proposition 2. *Covariances and market variance*

$$\begin{aligned}
&\textit{Under } \mathbb{P}: \\
\mathbf{\Sigma}^{\mathbb{P}}\pi_M &= \frac{(\mu^{\mathbb{P}} - \mathbb{1}R_f)}{\theta_M} = cov\left[\mathbf{R}, R_M\right]^{\mathbb{P}} \\
\pi_M^{'}\mathbf{\Sigma}^{\mathbb{P}}\pi_M &= \frac{(\mu_M^{\mathbb{P}} - R_f)}{\theta_M} = var\left[R_M\right]^{\mathbb{P}}
\end{aligned}$$

$$
\begin{aligned}
&\textit{Under } \mathbb{O}: \\
\mathbf{\Sigma}^{\mathbb{O}}\pi_M &= \frac{1}{\theta_M}\left[(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mu^{\mathbb{D}} + \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}\right] \\
&= cov\left[\mathbf{R}, R_*\right]^{\mathbb{O}} + cov\left[\mathbf{R}, R_{PE}\right]^{\mathbb{O}} + cov\left[\mathbf{R}, R_{ME}\right]^{\mathbb{O}} + cov\left[\mathbf{R}, R_{MPE}\right]^{\mathbb{O}} = cov\left[\mathbf{R}, R_M\right]^{\mathbb{O}} \\
\pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\pi_M &= \frac{1}{\theta_M}\left[(\mu_M^{\mathbb{O}} - R_f) + \pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mu_M^{\mathbb{D}} + \pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}\right] \\
&= cov\left[R_M, R_*\right]^{\mathbb{O}} + cov\left[R_M, R_{PE}\right]^{\mathbb{O}} + cov\left[R_M, R_{ME}\right]^{\mathbb{O}} + cov\left[R_M, R_{MPE}\right]^{\mathbb{O}} = var\left[R_M\right]^{\mathbb{O}}
\end{aligned}
$$

Only the first term in the expressions is actually related to the real market moments. The other ones are related to the expectations errors. Multiplying the different portfolios by the variance-covariance matrix gives the asset covariances with the different portfolio returns. The asset covariances with the market is equal to the sum of the covariances with each of the four portfolios, and they are proportional to some asset excess returns. In other words, scaled asset covariances are on the asset expected return space. Only asset covariances with the efficient part of the market portfolio are rewarded in true asset expected excess return. The error terms are rewarded in expected excess returns, but not the ones of the asset themselves, rather to a transformation of the true asset excess returns. The two terms that relate to the precision matrix expectation errors depend on $\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}$, whose components do not cancel out: it is a term that represents deviations from the usual diagonal matrix of ones under perfect information $\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{O}} = \mathbb{1}$. Multiplying by the market weights again gives the market variance, which is the sum of the market return covariance with the different portfolios. The covariance of the market with its efficient part is proportional to the market excess return. As it is the case for individual assets, only the informed part of the market portfolio is actually rewarded by mean market portfolio excess returns. The other covariance terms are not directly equal to mean excess returns, but since they are on the excess return space they must correspond to some portfolio expected excess return.

Proposition 3. *Asset betas*

$$
\begin{aligned}
&\textit{Under } \mathbb{P}: \\
\beta^{\mathbb{P}} &= \frac{\mathbf{\Sigma}^{\mathbb{P}}\pi_M}{\pi_M^{'}\mathbf{\Sigma}^{\mathbb{P}}\pi_M} = \frac{(\mu^{\mathbb{P}} - \mathbb{1}R_f)}{(\mu_M^{\mathbb{P}} - R_f)} = \frac{cov\left[\mathbf{R}, R_M\right]^{\mathbb{P}}}{var\left[R_M\right]^{\mathbb{P}}} \\
&\textit{Under } \mathbb{O}: \\
\beta^{\mathbb{O}} &= \frac{(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mu^{\mathbb{D}} + \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}}{(\mu_M^{\mathbb{O}} - R_f) + \pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mu_M^{\mathbb{D}} + \pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}} = \frac{cov\left[\mathbf{R}, R_M\right]^{\mathbb{O}}}{var\left[R_M\right]^{\mathbb{O}}}
\end{aligned}
$$

Errors in the means imply different betas than under perfect information, but it does not lead to any beta expectation errors. In other words, if there are only errors on the asset means, the asset betas are inefficient since based on incorrect return moments but they correspond to the ones perceived by the market. From the beta relationship above, the SML under imperfect information can be derived.

Proposition 4. *The SML*

$$
\begin{aligned}
&\textit{Under } \mathbb{P}: \\
\mu^{\mathbb{P}} - R_f &= \beta^{\mathbb{P}}(\mu_M^{\mathbb{P}} - R_f)
\end{aligned}
$$

Under $\mathbb{O}$:

$$\mu^{\mathbb{O}} - R_f = -\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) - \mu^{\mathbb{D}} - \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}} + \beta^{\mathbb{O}}(\mu_M^{\mathbb{O}} - R_f) + \beta^{\mathbb{O}}\mu_M^{\mathbb{D}}$$
$$+ \beta^{\mathbb{O}}\pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \beta^{\mathbb{O}}\pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}$$

The relationship that holds under $\mathbb{P}$ does not hold when departing from the perfect information assumption. Only the relationship under the objective measure $\mathbb{O}$ is observed ex-post. The expectation errors terms create noise around the standard CAPM relationship. There are idiosyncratic asset alphas and systematic terms that affect the slope of the SML. Both the idiosyncratic and systematic pricing terms have the similar common three-component structure, which relate to the types of expectation errors: mean errors, precision errors and mean-precision errors. The error-adjusted SML is studied in more details in Section 3.2.

3 The risk-return relationships

Expectation errors imply that the two main risk-return relationships of the standard CAPM i.e. the Capital Market Line (CML) and the Security Market Line (SML), do not hold. Since the market is mean-variance inefficient, the CML is not an efficient frontier and the SML has additional terms that are consequences of the expectation errors. In this section, the error-adjusted risk-return relationships are studied. Exogenous beliefs biases are considered separately in order to better understand how each type of errors (means, volatilities, correlations) impacts the two equilibrium risk-return relationship in the model. Errors on the variance dimension i.e. on the asset variances and covariances have different and more complex impacts than the ones on the mean dimension. Considering each type of error separately implies that only one error portfolio (out of three) is active at the same time. In general, all types of errors can be made at the same time and the additional joint mean-precision error term is non-zero. However, it is necessary to first understand the individual effects before studying the joint ones. In the illustrations, the asset returns' distribution is simulated as a multivariate normal. A sample of $N = 130$ asset monthly returns over 21 years from the CRSP database is used to compute the sample moments, which are then used as the objective moments in the theoretical analysis (that is, $\mu^{\mathbb{O}}$ and $\mathbf{\Sigma}^{\mathbb{O}}$). Exogenous beliefs deviations from the objective moments are then considered in the representative distribution (that is, $\mu^{\mathbb{P}}$ and $\mathbf{\Sigma}^{\mathbb{P}}$). The expectation errors are simulated randomly or equally across the assets, depending on the scenario considered. In practice, the errors are not random and some types of assets are likely to be related to some types of errors.

3.1 The Capital Market Line (CML)

The market price of risk corresponds to the reward for risk-taking on efficient portfolios i.e. the slope of the mean-variance efficient frontier. If risk is measured by volatility and without the risk-free asset, the Mean-Variance Efficient Frontier (MVEF) is a parabola on the $\left[\sigma^{\mathbb{O}}; \mu^{\mathbb{O}}\right]$ graph. In the presence of the risk-free asset and under the perfect information assumption, the efficient frontier becomes the Capital Market Line (CML), connecting the risk-free return to the efficient market return and leading to the two-fund decomposition. When there are expectation errors, the market return is mean-variance inefficient. It is on the MVEF which corresponds to the representative beliefs and characterizes the market weights. However, the latter frontier does not exist since it is based on expectation errors and holds only under the representative measure. Under the objective one, the market return is suboptimal and located within the MVEF. Hence, the error-adjusted CML links the risk-free rate to an inefficient portfolio return. The equations for the CML under both measures are given in the following proposition.

Proposition 5. *The CML*

$$\begin{aligned}&\textit{Under } \mathbb{P}:\\ \mu_*^{\mathbb{P}} &= R_f + \frac{(\mu_M^{\mathbb{P}} - R_f)}{\sigma_M^{\mathbb{P}}}\sigma_*^{\mathbb{P}} = R_f + \Lambda^{\mathbb{P}}\sigma_*^{\mathbb{P}}\\ &\textit{Under } \mathbb{O}:\\ \mu^{\mathbb{O}} &= R_f + \frac{(\mu_M^{\mathbb{O}} - R_f)}{\sigma_M^{\mathbb{O}}}\sigma^{\mathbb{O}} = R_f + \Lambda^{\mathbb{O}}\sigma^{\mathbb{O}}\end{aligned}$$

The star is used for the moments under the representative measure to show that the relationship applies to efficient returns. Under the objective measure, the market is inefficient and the relationship is not a frontier. It does not apply to mean-variance efficient returns. The relationship between the market price of volatility under the representative measure $\Lambda^{\mathbb{P}}$ and the one under the objective measure $\Lambda^{\mathbb{O}}$ is the following:

$$\Lambda^{\mathbb{O}} = \frac{\mu_M^{\mathbb{O}} - R_f}{\sigma_M^{\mathbb{O}}} = \Lambda^{\mathbb{P}}\left(1 + \frac{\sigma_M^{\mathbb{D}}}{\sigma_M^{\mathbb{O}}}\right) - \frac{\mu_M^{\mathbb{D}}}{\sigma_M^{\mathbb{O}}}$$

The values $\mu_M^{\mathbb{D}}$ and $\sigma_M^{\mathbb{D}}$ can both be positive or negative depending on the belief biases. For example, a negative $\sigma_M^{\mathbb{D}}$ means that market volatility is over-estimated and a negative $\mu_M^{\mathbb{D}}$ that the market mean return is under-estimated. Different beliefs and different investment sets can lead to a wide range of risk-free rates and prices of risk, corresponding to various market environments.

3.1.1 Exogenous beliefs analysis

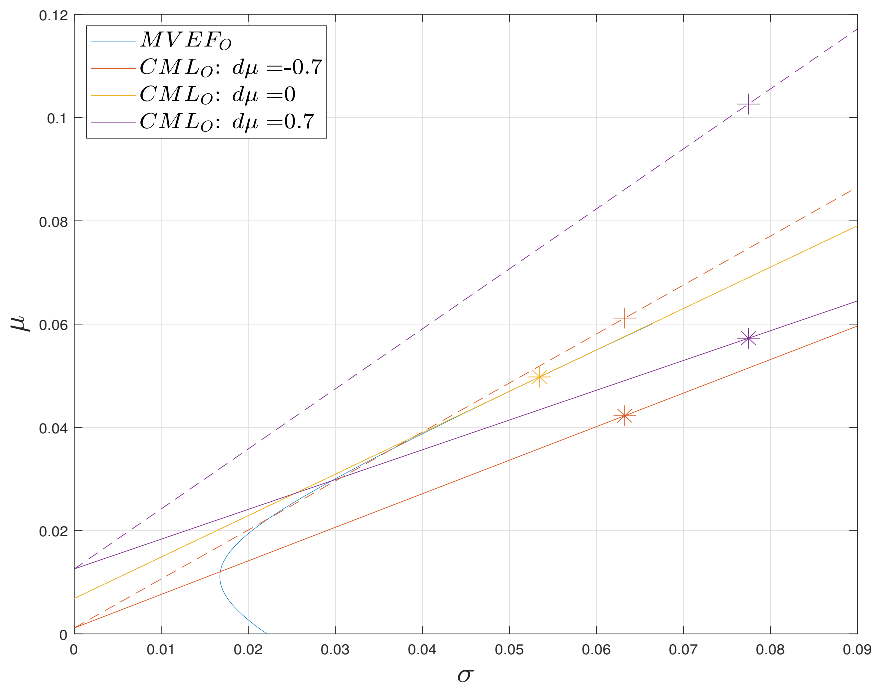
In this section, three belief scenarios are presented for each moment (on the mean-variance space) of the multivariate return distribution. The perfect information case that leads to the regular CAPM is used as a benchmark. It is represented by the yellow color in the $[\sigma^{\mathbb{O}}; \mu^{\mathbb{O}}]$ graphs. The over-estimation scenario, which is optimistic in the case of the means and pessimistic in the case of the variances/covariances, is represented by the purple color. Finally, the under-estimation scenario corresponds to the red color in the graphs. For each scenario, the true CML, which corresponds to the true market price of risk, is represented by a solid line. The dashed line is the CML perceived by the market when it creates its positions. It is imaginary in the case of expectation errors. The moments corresponding to both measures are represented on the same scale. The dashed CML is tangent to an unobservable mean-variance efficient frontier which is not presented in the graphs for clarity. In the benchmark case, the true and the perceived CML's are indistinguishable. The perceived market return, which is represented by a cross in the graphs, is on the CML under $\mathbb{P}$, whereas the true market return, represented by a asterisk in the graphs, is on the true CML under $\mathbb{O}$. Their positions on the mean-volatility graph can be contrasted with the true risky Mean-Variance Efficient Frontier (MVEF), which corresponds to the blue parabola in the graphs.

Errors in the asset means In the case of only mean errors, volatility is correctly estimated by the market i.e. $\sigma_M^{\mathbb{O}} = \sigma_M^{\mathbb{P}}$. The true market price of risk is given by:

$$\Lambda^{\mathbb{O}} = \frac{\mu_M^{\mathbb{O}} - R_f}{\sigma_M^{\mathbb{O}}} = \frac{\mu_M^{\mathbb{O}} - R_f}{\sigma_M^{\mathbb{P}}} = \Lambda^{\mathbb{P}} - \frac{\mu_M^{\mathbb{D}}}{\sigma_M^{\mathbb{O}}} = \Lambda^{\mathbb{P}} - \frac{\mu_M^{\mathbb{D}}}{\sigma_M^{\mathbb{P}}}$$

The market return under the objective measure is translated vertically from the one under the representative measure on the $\left[\sigma^{\mathbb{O}}; \mu^{\mathbb{O}}\right]$ graph. The translation is positive if the means are over-estimated, and negative if the means are under-estimated. In Figure 1, the effects of mean expectation errors on the CML are illustrated.

Figure 1: The effects of mean expectation errors on the CML



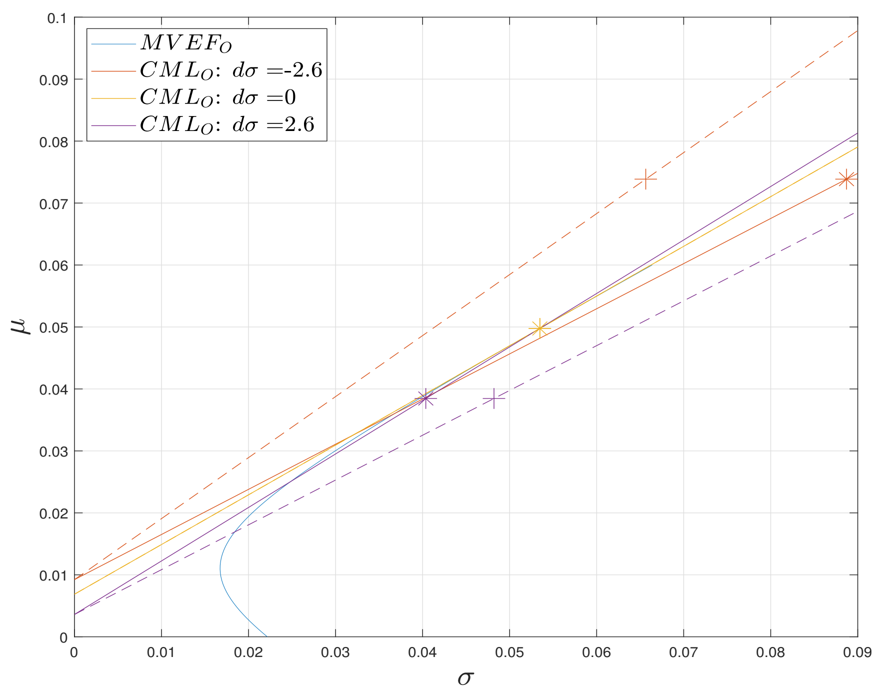
In the perfect information case, the CML under $\mathbb{P}$ is the same as the one under $\mathbb{O}$. The true market return is mean-variance efficient and hence located on the true risky MVEF. In the cases of mean under and over-estimation, the CML under $\mathbb{P}$ (dashed line) is different than the one under $\mathbb{O}$ (solid line). Although the slope is different, they start from the same intercept since the risk-free rate is the same under both measures. The true market return is different than the perceived market return, which is located on the imaginary CML under $\mathbb{P}$. Under-estimated asset means can lead to a market whose mean is over-estimated because the unprofitable assets (from the market point-of-view) are sold short to finance more investments in profitable assets. Since it is based on incorrect beliefs, the CML under $\mathbb{P}$ is different than the one under $\mathbb{O}$. In Figure 1, each asset is given a random share of the overall mean expectation error (equal to 70%).

Errors in the asset volatilities In the case of volatility errors, the true market price of risk is given by:

$$\Lambda^{\mathbb{O}} = \frac{\mu_M^{\mathbb{O}} - R_f}{\sigma_M^{\mathbb{O}}} = \frac{\mu_M^{\mathbb{P}} - R_f}{\sigma_M^{\mathbb{O}}} = \Lambda^{\mathbb{P}} \left(1 + \frac{\sigma_M^{\mathbb{D}}}{\sigma_M^{\mathbb{O}}}\right)$$

The true market return is translated horizontally from the perceived market return. If market volatility is over-estimated, corresponding to a pessimistic scenario, the market return under the objective measure is located on the left of the market return under the representative measure. In the opposite, if market volatility is under-estimated, the market return under the objective measure is located on the right of the market return under the representative measure. The effect is represented in Figure 2, using an overall volatility error of 260% allocated equally across assets.

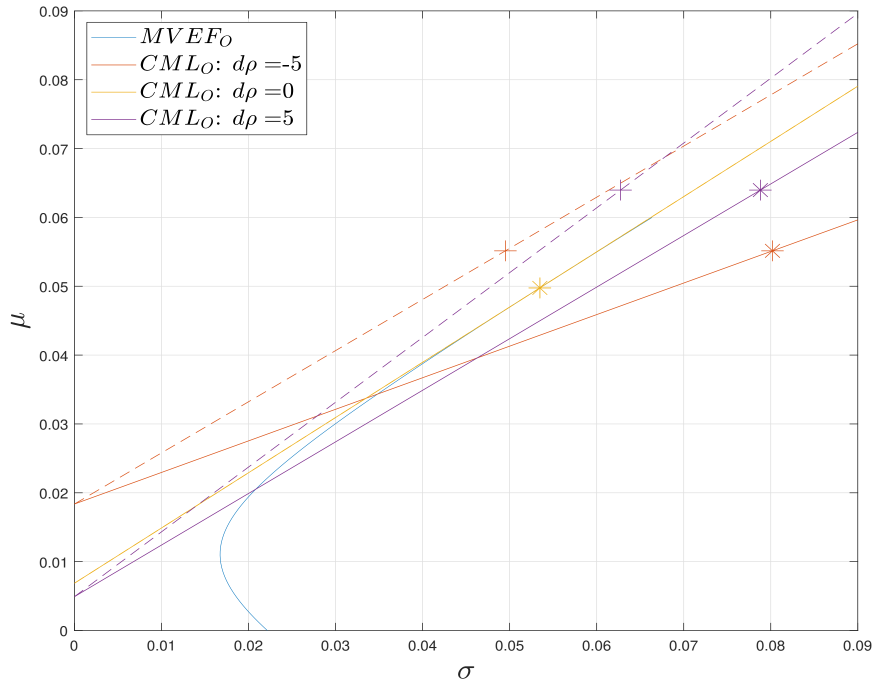
Figure 2: The effects of volatility expectation errors on the CML



Although the market return moments are different across the two volatility error scenarios, the slopes of the CML's under $\mathbb{O}$ are close to the perfectly informed one. Market risk is under-estimated when volatilities are under-estimated and over-estimated when volatilities are over-estimated. Due to the structure of the precision matrix, volatility expectation errors are more complicated to analyze than mean expectation errors. Contrary to mean belief deviations, the effect of volatility belief deviations is asymmetric and non-linear on the error portfolio weights.

Errors in the asset correlations In general, overall negative correlation belief deviations should correspond to an optimistic scenario since diversification opportunities are over-estimated, whereas overall positive correlation belief deviations should correspond to a pessimistic scenario. However, since higher correlations imply more potential hedging trades amongst assets, it can also result in an under-estimation of the market volatility. An illustration of correlation expectation errors can be seen in Figure 3. The overall correlation error of $+/-5$ is equally allocated across all (off-diagonal) correlation coefficients.

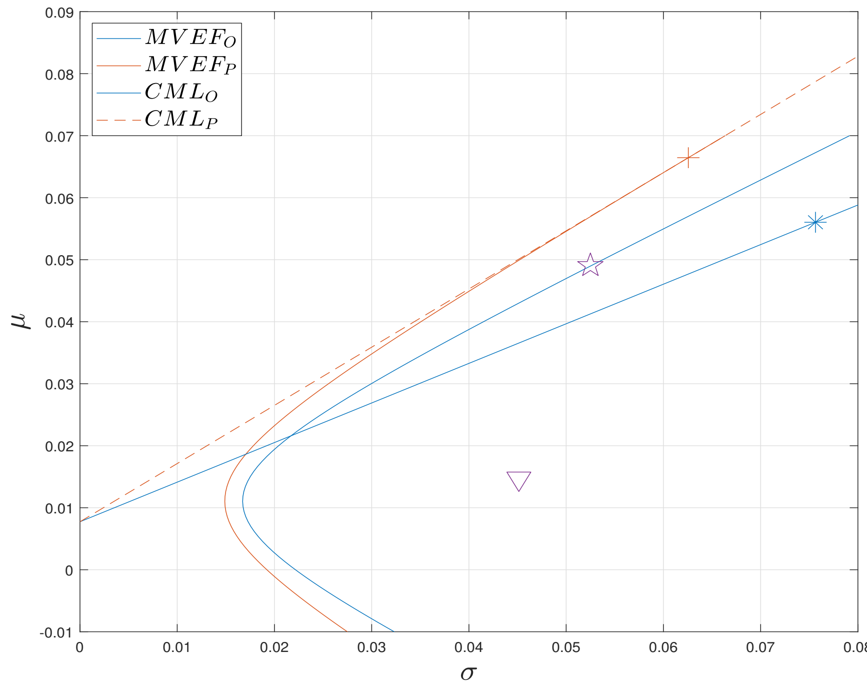
Figure 3: The effects of correlation expectation errors on the CML



In both cases of overall positive and negative correlation belief deviations, market volatility is under-estimated. The expectation error on the market return is more important in the under-estimation case. The increase in perceived diversification opportunities leads to a too aggressive market. Under the representative measure, the market return is perceived to be low risk and therefore the market is willing to accept a low price of risk. However, since market risk is under-estimated, the price of risk is even lower under the objective measure.

A sample example In this paragraph, an example of expectation errors on both mean and variance dimensions is studied. In addition to the sum of the individual effects, there is an additional joint effect stemming from the Mean-Precision-Error portfolio. To generate Figure 4, a sample is drawn from the multivariate normal distribution and the sample moments are computed to give the representative beliefs.

Figure 4: A sample example of expectation errors and the CML



The market return under the objective measure, represented by the asterisk, is translated both horizontally and vertically from the one under the representative measure, represented by the cross. From the location of the market return under both measures, it can deduced that market risk is under-estimated and its mean is over-estimated. Hence, the market invests on the basis of a highly fictional market price of risk. The imaginary MVEF which corresponds to the market beliefs is presented in solid red, tangent to the dash-lined CML under $\mathbb{O}$. Regardless of the errors, the true market return can be decomposed as the sum of an efficient or fundamental part, represented by a star in the graph, and an inefficient or error-based part, represented by a downward-pointing triangle. Trading the efficient part of the market portfolio allows to retrieve an optimal portfolio return. The error-based part of the market portfolio is highly mean-variance inefficient and located within the MVEF. Short-selling the error portfolio to invest aggressively in the star portfolio should be a highly profitable strategy.

3.2 The Security Market Line (SML)

With the perfect information relaxed, the market excess return does not carry sufficient information to correctly price all the assets in the model. There are idiosyncratic asset alphas and a slope adjustment to the SML with respect to the market excess return:

$$\mu^{\mathbb{O}} - \mathbb{1}R_f = \alpha^{\mathbb{O}} + \beta^{\mathbb{O}} \left[(\mu_M^{\mathbb{O}} - R_f) + \gamma_M^{\mathbb{O}} \right]$$

Where:

$$\alpha^{\mathbb{O}} = -\mathbf{\Sigma}^{\mathbb{O}} \mathbf{\Omega}^{\mathbb{D}} (\mu^{\mathbb{O}} - \mathbb{1}R_f) - \mu^{\mathbb{D}} - \mathbf{\Sigma}^{\mathbb{O}} \mathbf{\Omega}^{\mathbb{D}} \mu^{\mathbb{D}}$$

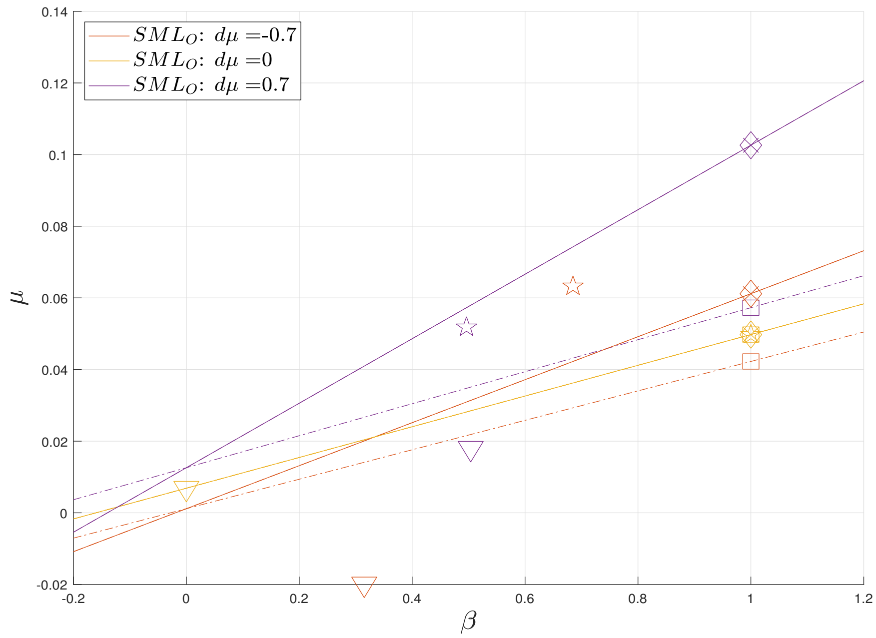
$$\gamma^{\mathbb{O}}_M = \mu^{\mathbb{D}}_M + \pi^{'}_M \boldsymbol{\Sigma}^{\mathbb{O}} \boldsymbol{\Omega}^{\mathbb{D}} (\mu^{\mathbb{O}} - \mathbb{1} R_f) + \pi^{'}_M \boldsymbol{\Sigma}^{\mathbb{O}} \boldsymbol{\Omega}^{\mathbb{D}} \mu^{\mathbb{D}}$$

$\alpha^{\mathbb{O}}$ are alphas with respect to the correct SML i.e. the one including the slope adjustment $\gamma^{\mathbb{O}}_M$. They have the three-term structure stemming from the market weights decomposition: mean-error, precision-error and mean-precision error. Errors on the asset risk are more complex than the ones on the mean return dimension. Indeed, the precision-error and mean-precision-error terms depend on the product of the true variance-covariance with the precision deviation matrices $\boldsymbol{\Sigma}^{\mathbb{O}} \boldsymbol{\Omega}^{\mathbb{D}}$. The resulting matrix correspond to the belief deviations on the assets' risks, multiplied by the true asset risks. For a given asset, risk under-estimation or mean over-estimation by the market i.e. over-optimism results in a negative alpha, while risk over-estimation or mean under-estimation i.e. over-pessimism results in a positive alpha. Since individual asset returns have alphas, the market return also has an alpha with respect to the true SML. The market alpha is equal to the market-weighted average of the ones of the individual assets. The systematic adjustment $\gamma^{\mathbb{O}}_M$ is the negative of the market alpha and is directly related to the expectation errors made on the market return. It has the three-term structure: market mean-error, variance-error and mean-variance-error. The mean-error systematic term $\mu^{\mathbb{D}}_M$ corresponds to a market portfolio of the $N \times 1$ individual mean belief deviation vector. For errors on the variance dimension, it is less straightforward. As it is the case for the individual assets, a negative alpha for the market (positive $\gamma^{\mathbb{O}}_M$) implies over-optimism: risk under-estimation or mean over-estimation. In the opposite, a positive market alpha (negative $\gamma^{\mathbb{O}}_M$) implies over-pessimism: risk over-estimation or mean under-estimation. The slope adjustment is an additional premium required on any covariance with the market, compensating for its mean-variance inefficiency. Hence, the total equity risk premium is equal to the market true excess return plus the adjustment term $\gamma^{\mathbb{O}}_M$. For the latter to be positive and the total equity premium to be higher than the market excess return, the market mean return and/or its risk must be respectively over-estimated and under-estimated. The relationship automatically holds for the market itself despite impacting the whole cross-section of assets. Since it is composed of alphas, the market return is itself not on the SML under $\mathbb{O}$. Note that there are no constraints on the alpha values. If the risk and/or mean returns of all assets are under-estimated, all assets alphas are negative and the slope adjustment term in the SML $\gamma^{\mathbb{O}}_M$ is positive. In the opposite, if all asset means and risks are respectively under and over-estimated, all asset alphas are positive. Different types of expectation errors affect the alphas differently, and it is studied in more details in section 3.2.1.

3.2.1 Exogenous beliefs analysis

In this section, the effects of exogenous belief deviations are studied on the SML under the objective measure $\mathbb{O}$. It allows to better understand the shapes of the additional terms in the true equilibrium asset pricing relationship, and their links with the belief biases. In the following $\left[\beta^{\mathbb{O}}; \mu^{\mathbb{O}}\right]$ graphs, the over and under-estimation scenarios are represented by the purple and red colors respectively. The moments under both the representative and the objective measures are shown on the same graph. The true SML under $\mathbb{O}$ is a solid line, and the position of the market on it is represented by a diamond marker. It does not correspond to the market return, but rather to the required return on an asset with a beta equal to one (without alphas). Although it is only a part of the assets' risk premia, the line corresponding to the market's true Sharpe ratio is represented by the dashed-dotted line and the market return by a square marker. The SML under $\mathbb{P}$ is a dashed line, whose slope corresponds to the perceived market price of risk. The perceived market return is on it and corresponds to a cross marker on the graphs. In addition, the positions of the efficient and inefficient parts of the market portfolio (under the true objective measure $\mathbb{O}$) are represented by the star and downward triangle markers respectively. The benchmark perfect information case is represented by the yellow color. In this case, all the markers are on the same point and all the lines (SML under $\mathbb{P}$, SML under $\mathbb{O}$ and market excess return) are indistinguishable.

Figure 5: The effects of mean expectation errors on the SML



Errors in the means In the case of only errors in the asset means, the SML is given by:

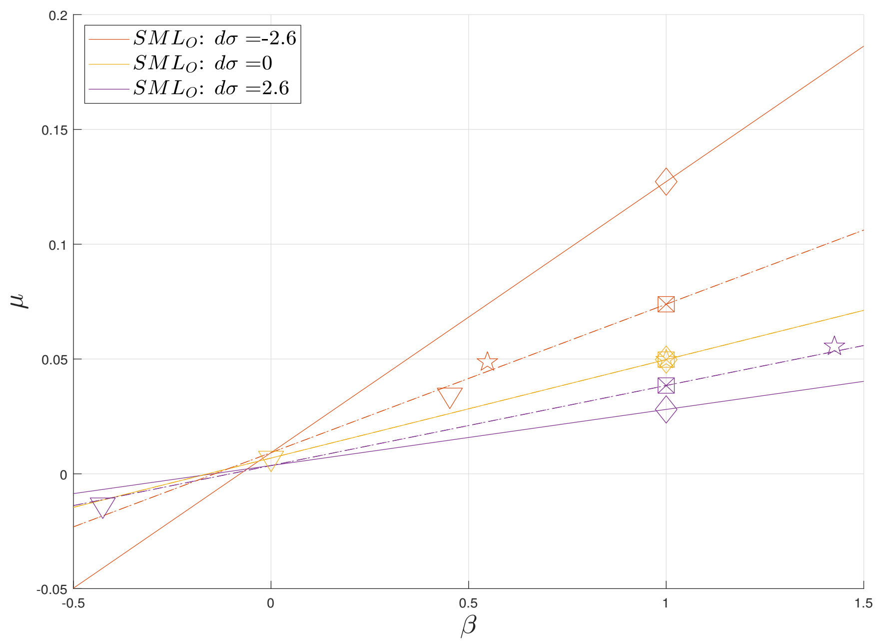
$$\mu^{\mathbb{O}} - \mathbb{1}R_f = -\mu^{\mathbb{D}} + \beta^{\mathbb{O}}(\mu_M^{\mathbb{O}} - R_f) + \beta^{\mathbb{O}}\mu_M^{\mathbb{D}}$$

The market estimates the different asset betas correctly and there are no discrepancies between the SML under $\mathbb{P}$ and the one under $\mathbb{O}$. However, their slopes are different than the market excess return due to the systematic mean error term. A mean over-estimation for an asset i tends to imply a negative alpha for asset i and for the market (assuming it is positively invested in asset i). The mean-error systematic term is then positive to compensate for the negative market alpha and the equity risk premium is higher than the market excess return. Due to their simple structure, mean errors only affect assets individually. That is, a mean over-estimated by 1% for asset i correspond to a negative asset alpha equal to 1% for asset i. Assets whose means are correctly estimated have alphas equal to zero. In Figure 5, an illustration of the effects of mean errors on the SML is presented. The optimistic scenario corresponding to the mean over-estimation case (in purple) leads to the highest market price of risk, illustrated by the highest slope of the SML under $\mathbb{O}$. The position of the market on the correct SML (diamond marker) is the same as the perceived market excess return under $\mathbb{P}$ (cross marker). However, the true market excess return is about 4% lower, the difference being equal to the negative market alpha. The mean under-estimation scenario (in red) also leads to an over-estimation of the market expected return due to short positions in assets whose means are sufficiently under-estimated.

Errors in the volatilities In the case of only errors in the asset volatilities, the SML is given by:

$$\mu^{\mathbb{O}} - \mathbb{1}R_f = -\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \beta^{\mathbb{O}}(\mu_M^{\mathbb{O}} - R_f) + \beta^{\mathbb{O}}\left[\pi_M^{'}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f)\right]$$

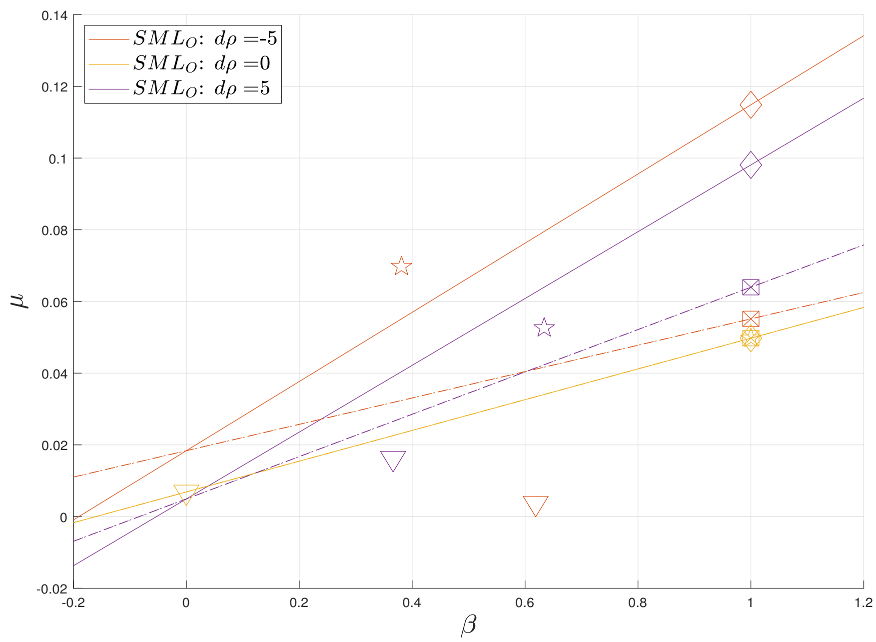
Figure 6: The effects of volatility expectation errors on the SML



Since the market estimates the means correctly, the slope of the SML under $\mathbb{P}$ is equal to the market excess return. However, it is different than the slope of the SML under $\mathbb{O}$. Due to the structure of the variance-based terms, an error on only one of the asset volatility results in non-zero alphas for all the assets in the market. Moreover, a mis-estimation equal to $|x_i|$% does not result in an alpha equal to $|x_i|$% for asset i, as it was the case for mean errors. Figure 6 shows an illustration of the impact volatility expectation errors on the SML. When volatility is under-estimated, the market is optimistic and takes more risk than it should. The asset alphas are then negative in general, especially for the mis-estimated asset, and the systematic variance-error term is positive. In the opposite, when there is a volatility over-estimation, the market is pessimistic and the asset alphas are negative in general, especially for the mis-estimated asset. The market variance-error term is hence negative, implying a flatter SML under $\mathbb{O}$ than the market excess return.

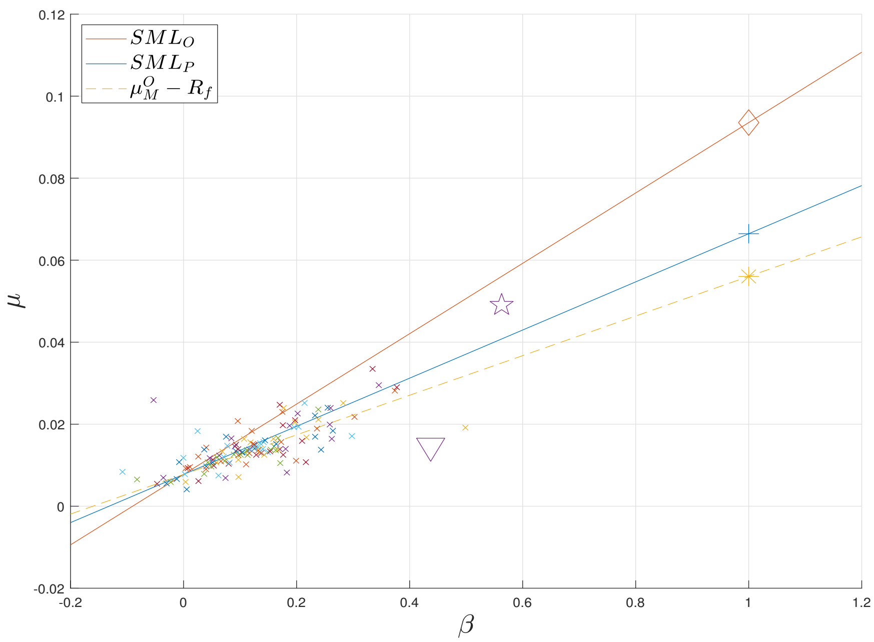
Errors in the correlations As for volatility errors, correlation errors result in a different SML under $\mathbb{O}$ than under $\mathbb{P}$. All the asset alphas are different than zero, even if only one correlation pair is mis-estimated. Both under and over-estimation scenarios can be seen as optimistic scenarios because of better diversification opportunities and more hedging trades, respectively. The impacts of correlation expectation errors are illustrated in Figure 7. Indeed, both error scenarios result in an under-estimation of the market price of risk. Asset alphas tend to be negative, implying a negative alpha for the market and a positive systematic slope adjustment term in the SML. The under-estimation scenario leads to the largest slope adjustment, implying a more incorrect risk assessment. It also leads to the most important difference between the systematic risk-return of the error and efficient part of the market portfolio, the latter having a positive alpha for a low beta.

Figure 7: The effects of correlation expectation errors on the SML



A sample example A sample example of the SML is shown in Figure 8 with joint errors on means, variances and covariances. In this case, the SML under $\mathbb{O}$ (solid red line) is different than the SML under $\mathbb{P}$ (solid blue line), and both are different than the true market excess return line (dashed yellow line). The risk premium is under-estimated by the market and the SML under $\mathbb{P}$ has a lower slope than the SML under $\mathbb{O}$. Since the market excess return (which is only a part of the total risk premium) is under-estimated by the market, the true market excess return (yellow asterisk) is lower than the perceived market excess return (blue cross). The efficient part of the market portfolio (purple star) allows to have an excess return similar to the market for a systematic exposure cut by more than 40%. Nevertheless, it still has a negative alpha with respect to the true SML. The error part of the market portfolio, represented by the downward-facing purple triangle, has an excess return approximately equal to the asset with the lowest excess return. Its systematic risk exposure is also higher than (almost) all assets. The cross markers represent each of the assets in the market portfolio. As expected, they mostly have negative alphas with respect to the true SML under $\mathbb{O}$. Assets with positive alphas tend to have low betas and the asset with the most negative alpha is the one with the highest beta. In this case, the expected return of the asset in question is over-estimated by about 2%, leading to a more negative alpha for the asset, a higher market weight and hence a higher beta.

Figure 8: A sample example of expectation errors on the SML



4 The SML revisted: the four-factor model

The asset alphas do not consist in unexplained performance because they are explained by expectation errors. However, they still consist in deviations from the error-adjusted SML. It suggests that at least one dimension is missing from the pricing line. From Proposition 2, it can be seen that returns are proportional to covariances and variances. More precisely, the true asset excess returns correspond to (scaled) covariances with the efficient part of the market portfolio whereas the belief-based error pricing terms are proportional to covariances with the three error portfolios:

$$\begin{aligned}(\mu^{\mathbb{O}} - R_f) &= cov(\mathbf{R}, R_*)\theta_M \\ \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) &= cov(\mathbf{R}, R_{PE})\theta_M \\ \mu^{\mathbb{D}} &= cov(\mathbf{R}, R_{ME})\theta_M \\ \mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}} &= cov(\mathbf{R}, R_{MPE})\theta_M\end{aligned}$$

Since the relationships hold for individual assets, they must also hold for any portfolios. From market variance, the market excess return is proportional to its covariance with the star portfolio and the three systematic error pricing terms are proportional to the (scalar) market covariances with the different error portfolios. It leads to a simple rewriting of the SML pricing relationship:

$$\frac{(\mu^{\mathbb{O}} - R_f)}{\theta_M} = -cov(\mathbf{R}, R_{ME}) - cov(\mathbf{R}, R_{PE}) - cov(\mathbf{R}, R_{MPE}) + cov(\mathbf{R}, R_M)$$

Under perfect information, the market portfolio is equal to the star portfolio. Hence, covariances with the market portfolio are directly proportional to the asset excess returns. Market variance is then also proportional to the market excess return, which is the definition of the equity premium in standard models. In the presence of expectation errors, the market variance still corresponds to the equity premium but the latter is not equal to the market excess return because it also includes the slope adjustment. The covariance formulation shows that the exposures to the error portfolios matter in the cross-section, resulting in a four-dimensional pricing relationship. Contrary to the market premium, the signs of the error premia are negative. That is, positive exposures to the error portfolios are rewarded in negative excess returns. The three error covariances can then be written as beta exposures, leading to a four-dimensional beta relationship named the Security Market Space (SMS). The error exposures correspond to a rewriting of the idiosyncratic asset alphas. Since they have a defined structure in the cross-section, they can be expressed as systematic premia.

Proposition 6. *The four-dimensional Security Market Space*

$$\begin{aligned}(\mu^{\mathbb{O}} - R_f) &= -\beta^{\mathbb{O}}_{ME} var\left[R_{ME}\right]\theta_M - \beta^{\mathbb{O}}_{PE} var\left[R_{PE}\right]\theta_M - \beta^{\mathbb{O}}_{MPE} var\left[R_{MPE}\right]\theta_M + \beta^{\mathbb{O}}_{M} var\left[R_M\right]\theta_M \\ &= -\beta^{\mathbb{O}}_{ME}\mu^{\mathbb{D}}_{ME} - \beta^{\mathbb{O}}_{PE}\pi'_{PE}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) - \beta^{\mathbb{O}}_{MPE}\pi'_{MPE}\mathbf{\Sigma}^{\mathbb{O}}\mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}} + \beta^{\mathbb{O}}_{M}\left((\mu^{\mathbb{O}}_M - R_f) + \gamma^{\mathbb{O}}_M\right)\end{aligned}$$

The total equity premium is composed of four pricing terms, rather than one. The premium that rewards market exposure is not equal to the market excess return because there is a slope adjustment that accounts for the expectation errors made on the market return. Although the market excess return can be negative, the sum of the excess return and the slope adjustment can not be negative because it is proportional to the market variance (assuming implicitly positive risk-aversion). Although the error premia are not usual portfolio excess returns since they depend on the belief deviations, they are on the portfolio excess return space and can be expressed as such. As it is the case for the market factor, the error premia must each be positive because they correspond to the given portfolio variance, scaled by the risk-aversion. Mechanically, it is the case because the same belief deviations which are in the premia are also in the error portfolio weights. The presence of expectation errors can generate significant deviations from the usual one-factor relationship. Although they are expressed as factors, exposures to the expectation error portfolios are rewarded negatively in equilibrium, contrary to the market factor. That is, depending on the expectation errors made on the asset and the corresponding beta values, the exposures to the error portfolios can offset the positive exposure to the market factor. However, the presence of the three additional factors is fully consistent with the assumptions of the market model. That is, deviations from the one-factor relationship do not consist in a rejection of the CAPM as an equilibrium asset pricing model. Since the SMS is four-dimensional, it can not be represented graphically. However, all three error portfolios can be regrouped into one overall error portfolio, which includes all three types of errors. In this case, only two exposures matter in the cross-section: the market beta and the expectation error beta. The four-dimensional space becomes a two-dimensional plane and the Security Market Plane (SMP) equation is given in Proposition 7.

Proposition 7. *The two-dimensional Security Market Plane*

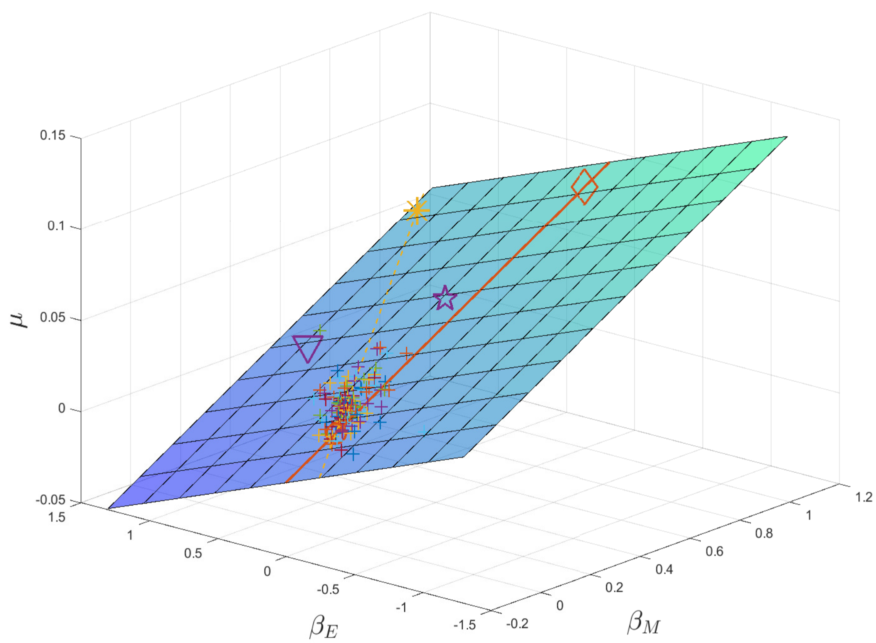
$$(\mu^{\mathbb{O}} - R_f) = -\beta^{\mathbb{O}}_{E} var\left[R_E\right]\theta_M + \beta^{\mathbb{O}}_{M} var\left[R_M\right]\theta_M$$

$$= -\beta_E^{\mathbb{O}} \pi_E^{'} \left[\mu^{\mathbb{D}} + \mathbf{\Sigma}^{\mathbb{O}} \mathbf{\Omega}^{\mathbb{D}} (\mu^{\mathbb{O}} - \mathbb{1} R_f) + \mathbf{\Sigma}^{\mathbb{O}} \mathbf{\Omega}^{\mathbb{D}} \mu^{\mathbb{D}}\right] + \beta_M^{\mathbb{O}} \left((\mu_M^{\mathbb{O}} - R_f) + \gamma_M^{\mathbb{O}}\right)$$

The two-factor model has the advantage of conveniently summarizing all the information into two factors and being easier to represent graphically. However, it also can not distinguish between each type of errors, in particular between mean and variance/covariance expectation errors. From the previous equations, it is clear that precision errors have different effects than mean errors. The latter premium do not depend on the true variance-covariance matrix and can intuitively be thought of as a portfolio of deviations. In contrast, the precision error premium depends on the true covariance matrix in addition to the precision matrix deviations. In addition, the third joint mean-precision term, which combines both mean and precision effects, can not exist in a model with no expectation errors on variances and covariances. It shows that much of the cross-sectional variation is ignored using models with only errors on the first moment. The precision pricing effects are likely to be more important in magnitude and more persistent than mean pricing effects. It is the case in simulations, but it remains to be shown empirically.

Illustration The sample example from Figure 8 is represented as a two-dimensional SMP in Figure 9. It shows that the error-adjusted SML from 2.4.1 is a cut of the SMP at $\beta_E = 0$.

Figure 9: A sample example of expectation errors - the SMP



The market, represented by the yellow asterisk marker, is the return with the most negative alpha. The aggregation corresponding to the market portfolio seems to magnify the expectation errors made on each of the individual assets, rather than offset them. The error-adjusted SML corresponds to the required mean on assets whose covariance with the error portfolio is zero. Therefore, it consists in a return that is not directly available using traded assets. The latter tend have non-zero alphas, especially in the case of belief deviations on the variances and correlations. The most desirable returns are the ones with negative exposures to the error portfolio since it implies positive alphas with respect to the error-adjusted SML.

4.1 The expectation error risk

The expectation error risk is not a true risk factor since it is not in the representative investor's utility function. It does not proxy for fundamental risk as it is the case for a risk factor. Rather, it is an involuntary consequence of the difference between the representative investor's optimization and the true unobservable optimization. That is, the consequence of the expectation errors made on the only one true risk factor, the market risk. The three additional factors are based on a rewriting of the idiosyncratic alphas. The expectation error risk is measured through the beta with respect to the error portfolio return, as it is the case for the market risk. If there are only mean expectation errors, only assets whose means are mis-estimated have non-zero alphas, and are hence exposed to the expectation error risk. When there are errors on volatilities or correlations, all assets have non-zero alphas, even if only a part of the assets' volatilities are mis-estimated. The error portfolios have cross-sectional power in explaining returns ex-post, in the same spirit as risk factors. However, they do not reward in the excess return of the factor portfolio. Rather, the premia are function of the belief biases and the true moments, multiplied by the corresponding factor weights. However, as it is the case for risk factor excess returns, the expectation error premia must be positive. Since they have negative signs, they put a downward pressure on the asset expected returns. Positive error exposures imply negative alphas and vice-versa. The asset exposures inform about the expectation errors made by the market as a representative investor on that asset. Some assets can be more exposed to mean expectation errors than precision expectation errors, in which case $\beta^{\mathbb{O}}_{ME}$ should be more important than $\beta^{\mathbb{O}}_{PE}$. The factor formulation provides a clear method to quantify the effects of the expectation errors in the cross-section. Note that, contrary to risk factors, the market and error portfolios do not have to be independent. In general, they have non-zero correlations with each other. It is especially the case for the mean-precision error portfolio, which is by construction significantly correlated with both the mean and precision error portfolios. It means that the error betas can not be estimated using Ordinary Least Squares (OLS).

4.2 The CAPM and the one-factor linear regression

Estimating a linear regression of the asset returns on the market return corresponds to testing the perfect information assumption of the CAPM. When the latter is relaxed, there are idiosyncratic asset alphas and a systematic adjustment to the market excess return, making the single factor linear regression model too simple. The alphas have a structure which translates to three additional factors in the cross-section. Although it is explained by expectation errors, the cross-sectional variation appears as unexplained when tested using the one-factor linear regression. The excess variation is incorrectly labelled as anomalous or abnormal when it is really the relationship defining the expected variation which is inappropriate. The perfect information assumption is one of the most restrictive assumption in the CAPM, likely violated in practice. The four-dimensional relationship could hence be the one that holds in the data, at least when tested on the mean-variance space. It is the set-up in which many models are tested and this paper makes clear that some variation in the asset returns can simply not be captured by the sole market return. Ignoring three factors arguably has an effect on the pricing performance. Much of the cross-sectional variation is missed in using the wrong model i.e. the sole market excess return. Unless the empiricist believes the perfect information assumption holds in practice, there is no point in using the standard one-factor SML relationship to test the CAPM. The pricing terms depend on both the belief deviations and the true asset moments, making the additional variation difficult to estimate using real data. Empirically, it is hence understandable that the one-factor linear regression model performs poorly and that other models seem to out-perform, factor models in particular. Some known characteristics may be associated with persistent expectation errors, resulting in the characteristic-based factor being correlated with the expectation error portfolio returns. The presence of this effect in the cross-section leads the empiricist to incorrectly conclude that the CAPM is rejected and that the identified variation is an anomaly. The link between expectation errors and factor models/anomalies is studied in Section 4.3

4.3 The link with asset pricing puzzles and factor models

Cross-sectional variation in expected returns is considered as an anomaly only if it is not explained by the benchmark model. In empirical tests, the benchmark model has usually been the one-factor CAPM. When the perfect information assumption is violated, which is arguably the case in practice, the one-factor pricing relationship fails even if the CAPM holds as an equilibrium model. Hence, cross-sectional variation that is considered as anomalous using the usual CAPM might not be an anomaly when adjusting for expectation errors. The widely used three-factor pricing model from Fama-French (1993) is built on empirical evidences of value and size premia in the cross-section. The factors matter in the cross-section but their consistent presence in the data is difficult to explain using equilibrium arguments. The results derived in this paper show that characteristic-based anomalies might be related to persistent expectation errors on the same types of stocks.

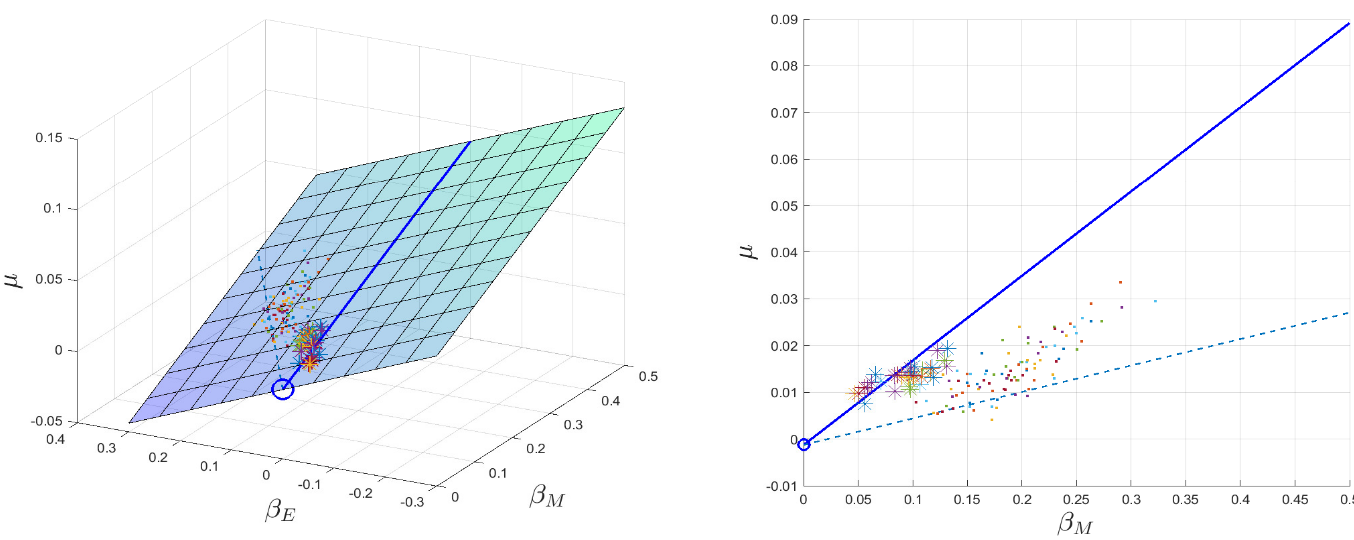
4.3.1 Expectation errors and characteristics-based premia

The factor structure from Ross (1976) comes from arbitrage arguments and does not specify the common sources of risk. The model presented in this paper shows that anomalies with respect to the market excess return factor can be consistent with the CAPM model when relaxing the perfect information assumption. The error pricing terms have cross-sectional power in explaining asset returns, even though they do not relate to any external source of risk. If there is (time-series) persistence in the way the market forms its expectations and that it makes regularly the same types of errors on the same types of stocks, a simili factor structure can be deduced from the data. In other words, if some belief biases are consistent, there can be a consistent structure in the error pricing terms ex-post. The model above depends on arbitrary mean and volatility errors, without any assumptions on the origin of the errors. If the same expectation errors persistently relate to the same stock characteristics over time, long-short portfolios based on these characteristics are positive alpha strategies. For example, to reflect the value premium, the market should be over-invested in growth stocks and under-invested in value stocks, corresponding to respectively positive and negative weights in the error portfolio. It implies negative alphas for the growth stocks and positive alphas for the value stocks. Therefore, a portfolio short-selling growth stocks to invest in value stocks should out-perform the market. A positive systematic impact should correspond to more negative alpha stocks than positive ones, consistent with a stronger impact of growth stocks due to higher prices. Similarly, to reflect the size premium, the error portfolio should be positively invested in big size stocks and negatively invested in small size stocks, reflecting over and under-investment respectively. Therefore, short selling big stocks to invest in small stocks is a positive alpha strategy. The systematic impact is positive because the negative alphas of big stocks exceed the positive alphas of small stocks. A long position in the market and a short position in the error portfolio allows to retrieve the more mean-variance efficient fundamental portfolio. The Fama-French long-short portfolios could be strategies that take profit of some extreme aspects of the expectation errors. Many factors can then be jointly explained, using equilibrium arguments. If the intuition is correct, the error portfolio should be related to stock characteristics like book-to-market or size for example. That is, it should be negatively related to book-to-market and positively related to the market equity characteristic. As explained by Andrei, Cujean and Fournier (2019), many characteristic-based effects can be summarized through one long-short portfolio. That is, factors based on characteristics and other valuation ratios can be proxies for this long-short portfolio. They also conclude that it needs not proxy for fundamental risk. Whereas they can not explain where the portfolio comes from, the analysis conducted in this paper shows that one potential explanation is the expectation errors. It decomposes the additional factor into three separate effects, which correspond to each type of equilibrium expectation errors.

4.3.2 Illustration

In the model, asset alphas with respect to the true SML under $\mathbb{O}$ are related to expectation errors. Mean expectation errors coupled with errors on volatilities and correlations can generate an infinity of possible scenarios for the additional error terms. In Figure 10, the means and the volatilities of a subset of assets are respectively under-estimated and over-estimated. There is a cut-off in the belief biases depending on an (arbitrary) characteristic value. In this case, they are low risk assets, corresponding to the assets in the lowest quartile in terms of volatilities. At the same time, the other asset means are over-estimated. The market excess return slope is represented by the dashed blue line, whereas the SML under $\mathbb{P}$ is represented by the dashed yellow line.

Figure 10: Errors on a subset of assets - SMP & SML



The assets whose volatilities and means are respectively under-estimated and over-estimated are represented by the star markers. The assets whose volatilities are correctly estimated and whose means are over-estimated are represented by dot markers. Star assets are located around low beta values because the biases push their valuations downward, in addition to being low risk assets. They have higher alphas with respect to the SML under $\mathbb{O}$ than the other dot assets, and positive alphas are concentrated around low beta values. All star assets have positive alphas with respect to the excess market return, implying positive out-performance with respect to the market return. However, only a subset of the star assets actually have positive alphas with respect to the true pricing relationship. All dot assets have negative alphas with respect to the SML under $\mathbb{O}$, although some of them concentrated around high beta values have positive alphas with respect to the market excess return. Figure 10 is consistent with low risk anomalies, like the low-beta anomaly. There is a relationship between volatility and expectation errors, which translates directly to the SML. To generate this Figure, for simplicity, only errors on means and volatilities were considered. In addition, the high-dimensional correlation error matrix can bring complexity to the belief combinations. The SMP can can correspond to many market scenarios, with different anomalies based on different characteristics.

5 Empirical test

The theoretical part of the model explicitly links beliefs to equilibrium returns, which have a four dimensional factor structure. The objective of the empirical analysis is to estimate the pricing relationships and assess whether expectation errors can help explaining well-known asset pricing anomalies like the value, size and idiosyncratic volatility premia. The empirical test uses the model structure and allows to study the relevance of any arbitrary beliefs on the mean-variance space, for any test assets. The beliefs are exogenous, and may or may not be an accurate representation of the true representative beliefs, which define the market weights. That is, there can be differences between the endogenous market weights and the observed market weights, corresponding to the true unobservable representative measure. However, the error exposures corresponding to the exogenous beliefs can still be computed and the four-factor pricing relationship can still be estimated. The empirical procedure is similar to Fama-MacBeth regressions, with the exception that it does not use individual linear regressions to estimate the asset exposures. Instead, in the first step, the latter are estimated conditionally using the observed portfolio weights and the return variance-covariance matrix. In the second step, the estimated exposures are used in cross-sectional regressions to determine whether they indeed carry premia out-of-sample. If the exogenous beliefs matter for pricing, the estimated premia corresponding to its error exposures - three in the complete model, and one in the reduced model - should be positive and significantly different from zero. The pricing errors stemming from the out-of-sample cross-sectional regressions should equivalently be lower - lower mean and lower volatility - than the one-factor CAPM. The method requires a belief formation system with exogenous beliefs, characterized by a $N \times 1$ perceived mean vector and a $N \times N$ perceived variance-covariance matrix, where N is the number of considered assets. We use the representative beliefs implied by the Institutional Brokers' Estimate System (I/B/E/S) price targets but other belief system can also be considered, like the sample and equally-weighted beliefs. The procedure requires the estimation of the $N \times N$ objective variance-covariance matrix, which indirectly imposes a constraint on the number of assets that can be considered in the test. At the same time, there should be enough assets such that the endogenous market portfolio is realistic. In order to satisfy both restrictions as much as possible, the test assets consist of characteristic-sorted portfolio returns. The asset weights in the portfolios are defined by the market weights, such that the sum of the portfolios equals the market portfolio. Explaining the characteristic premium corresponds then to explaining the cross-section of test asset returns.

5.1 The objective and representative measures

The results from the theoretical model are valid regardless of the objective and representative measures, which are taken as parameters. In the model, an asset is a return process with finite means and variances/covariances. It can be a stock, a bond, a portfolio or any other types of financial assets. Although the model is evaluated on the mean-variance space, the objective distribution does not have to be normal for the results to hold: the normality assumption is in the beliefs. That is, even if the true returns are not normally distributed, they must be perceived as such by the investors. Whereas the objective distribution can be estimated using observed samples of asset returns, it is not the case for the representative distribution, which is unobservable. The representative beliefs relate to the observed market weights, which have a dimension equal to the number of assets considered. Hence, in theory, the beliefs could be extracted from the market weights. However, due to their high dimension, there are more unknowns than equations and hence an infinity of solutions. However, for a given sample of returns, not all beliefs are equally likely. First, because they do not all correspond to the observed market weights. Some beliefs must be closer to reality than others. The framework is valid for any representative beliefs but an unrealistic belief formation system results in endogenous market weights too far from what is observed. Second, because different belief biases imply different error portfolio weights and hence different effects in the expected returns ex-post. For a given sample of observed returns, more accurate error portfolios should better explain the cross-section of returns. Rather than searching for the representative beliefs that exactly correspond to the market weights, the logic of the empirical test is the following: assuming an arbitrary belief formation system (with errors with respect to the observed market weights) and test its significance for pricing out-of-sample using the model structure. Even if the market weights corresponding to the exogenous beliefs do not equal the market weights in-sample, they still result in estimated error betas that can be used for out-of-sample cross-sectional regressions. The most interesting feature of the model is that it allows to test for the expectation errors on the risk dimension i.e. volatilities and correlations through the second and third error factors. We use the summary price targets from the I/B/E/S database to estimate the representative distribution. That is, we study if the beliefs implied by the analysts' forecasts are relevant for pricing ex-post, more specifically with respect to characteristic-based asset pricing anomalies.

5.2 Data

Characteristics Since they are used to create sorted portfolios, the characteristics considered should correspond to well-known asset pricing anomalies. We consider the book-to-market and market capitalization characteristics together to jointly explain the size and value premia that correspond to the Fama-French three-factor model. The characteristic data come from Compustat. We also consider separately the idiosyncratic volatilities generated by the CAPM. They are computed from the price and return data, using a 5-year estimation period. There are hence two sets of characteristics (size-value and idiosyncratic volatilities), which each define different test assets according to the procedure described in the next section.

Test assets The underlying assets are the stocks covered by the I/B/E/S analysts. The monthly price data come from CRSP and consists in 9640 stocks over a period of slightly more than 20 years (1999-2020). Each month in the sample, the market is composed of all stocks whose prices are available at time t and $t+1$. Stock i's market weights are computed as:

$$\pi_{M,i,t} = \frac{P_{i,t}}{\sum P_{i,t}}$$

The market return at $t+1$ is then computed as:

$$R_{M,t+1} = \pi_{M,t}^{'} \mathbf{R}_{t+1}$$

There are on average 4658 stocks in the market at each period in the sample. At each t, the stocks are regrouped into an arbitrary number of portfolios according to each of the considered characteristics. In the case of the size-value characteristics, the stocks are double-sorted as for the Fama-French 25 portfolios. The market weights are used for each of the portfolios, such that the market-weighted sum of the characteristic-sorted portfolio returns equals the market return at each time period in the sample. The characteristic-based portfolio returns are used then as the test assets in the model. Explaining the associated anomalies corresponds to explaining the cross-section of portfolio returns. The portfolio returns have the advantages of being balanced and of low-dimension, while still representing a high number of individual assets.

Beliefs The forecasts come from the I/B/E/S Summary database and consist in the means and standard deviations of the price targets. The forecasts are forward-looking one-year targets that are updated monthly. The forecast data go from March 1999 to August 2020. At the stock level, the forecasts are converted into monthly expected returns and volatilities using the observed stock price at the date of the forecast announcement. That is, for stock i:

$$\mu_{i,t}^{\mathbb{P}} = \left(\frac{\text{PriceTarget}_{i,t}}{P_{i,t}}\right)^{\frac{1}{12}} - 1$$
$$\sigma_{i,t}^{\mathbb{P}} = \left(\frac{\text{StdPriceTarget}_{i,t}}{P_{i,t}}\right)\sqrt{\frac{1}{12}}$$

The expected moments are converted to a monthly frequency in order to match the observed return data frequency. The conversion is naive but it still results in valid beliefs corresponding to the I/B/E/S price targets. The stock-level mean return forecasts are converted to portfolio-level mean forecasts $\mu_{j,t}^{\mathbb{P}}$ using:

$$\mu_{p,t}^{\mathbb{P}} = \pi_{p,t}' \mu_t^{\mathbb{P}}$$

Where $\pi_{p,t}$ are the weights corresponding to portfolio p for the given characteristic and $\mu_t^{\mathbb{P}}$ the corresponding vector of stock-level mean returns at time t. The IBES beliefs about the portfolio volatilities are then given by:

$$\sigma_{p,t}^{\mathbb{P}} = \left(\pi_{p,t}' \Sigma_{p,t} \pi_{p,t}\right)^{\frac{1}{2}}$$

Where $\Sigma_{p,t}$ is the variance-covariance matrix corresponding to all the stocks into the portfolio:

$$\Sigma_{p,t} = \begin{bmatrix} \left(\sigma^{\mathbb{P}}_{1,t}\right)^2 & \sigma^{\mathbb{P}}_{1,t}\sigma^{\mathbb{P}}_{2,t}\bar{\rho} & \dots \\ \dots & \dots & \dots \\ \sigma^{\mathbb{P}}_{N_p,t}\sigma^{\mathbb{P}}_{1,t}\bar{\rho} & \dots & \left(\sigma^{\mathbb{P}}_{N_p,t}\right)^2 \end{bmatrix}$$

N_p is the total number of assets into portfolio p at time t. Since there are no forecasts about the correlations in the I/B/E/S database, a single coefficient $\bar{\rho}$ is used to aggregate the individual beliefs about the volatilities. The output of the approach consists in beliefs about the means and volatilities of the characteristic-ordered portfolio returns for each month in the sample. It does not provide any information about the perceived correlations between the different portfolios, which are needed to compute the representative variance-covariance matrix. For simplicity, we assume that the I/B/E/S beliefs about the correlations is a diagonal matrix, implying that the I/B/E/S investor believes the portfolio returns to be independent of each other. It is a rather strong assumption but it also seems to correspond well to the observed market weights. Indeed, in the mean-variance set-up, only positive optimal weights in the market portfolio tend to occur when there are zero off-diagonal correlations. Moreover, the assumption does not mean that correlation errors are not taken into account in the test since the objective variance-covariance matrix is non-diagonal. That is, we do not assume that correlations do not matter, but rather that any non-zero correlations in the objective returns is automatically unexpected by the I/B/E/S investor and hence considered as an error.

5.3 The procedure

The idea is to test for exogenous representative beliefs and assess whether they help explaining prices out-of-sample. However, since the beliefs are exogenous, they will not correspond to the observed market weights as it is the case in the theoretical model. Hence, the model equations must be adjusted, leading to modified versions of the four and two-factor pricing relationships that account for the deviations in the market weights.

5.3.1 The adjusted equations

To test for any set of arbitrary beliefs and retain realistic market betas, market weights deviations π_ε are added to the weights decomposition from Section 2.4.1:

$$\begin{aligned} \pi_M &= \frac{\mathbf{\Omega}^{\mathbb{P}}}{\theta_M}(\mu^{\mathbb{P}} - \mathbb{1}R_f) + \pi_\varepsilon \\ &= \frac{1}{\theta_M}\left[\mathbf{\Omega}^{\mathbb{O}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Omega}^{\mathbb{D}}(\mu^{\mathbb{O}} - \mathbb{1}R_f) + \mathbf{\Omega}^{\mathbb{O}}\mu^{\mathbb{D}} + \mathbf{\Omega}^{\mathbb{D}}\mu^{\mathbb{D}}\right] + \pi_\varepsilon \end{aligned}$$

The market weights deviations correspond to the difference between the true observed market weights and the ones stemming from the exogenous beliefs. That is, it corresponds to the exogenous part of the market weights (current prices) that can not be explained using the beliefs. The risk-free rate is the Lagrange multiplier associated with the constraint that the endogenous market weights sum to one. Indeed, regardless of the assumed beliefs, the endogenous market weights must always sum to one. The introduction of the market weights errors results in an adjusted five-factor relationship:

$$\mu^{\mathbb{O}} - R_f = -\beta_{\varepsilon}^{\mathbb{O}} var\left[R_{\varepsilon}\right]\theta_M - \beta_{PE}^{\mathbb{O}} var\left[R_{PE}\right]\theta_M - \beta_{ME}^{\mathbb{O}} var\left[R_{ME}\right]\theta_M - \beta_{MPE}^{\mathbb{O}} var\left[R_{MPE}\right]\theta_M + \beta_M^{\mathbb{O}} var\left[R_M\right]\theta_M$$

As it is the case for the error portfolios, the portfolio π_ε has an impact on the relationships and it results in a fifth factor. However, contrary to the other factor premia, the ε premia can not be defined as a function of the beliefs. It does not exist in the theoretical model. Including the unexplained exposures in the cross-sectional regressions should improve the performance of the model even if the estimated premia may not be significant. Since it is based on observed deviations from the market weights, including the unexplained factor does not require any additional step and may improve the estimation of the error premia. In addition to the cross-sectional regressions, portfolios based on the estimated alphas are computed. That is, the exogenous beliefs result in some representative beliefs associated to some error portfolios and hence perceived alphas. The available assets are sorted according to their alphas, and strategies are created by investing (equally) in some given quantiles of the alpha cross-sectional distribution. It allows to sort the assets from the worst to the best performing ones according to the exogenous beliefs in-sample. If the exogenous beliefs are representative, the strategies invested in high alpha stocks should perform better out-of-sample than strategies invested in low alpha stocks. Or, at least, there should be clear cross-sectional variation in the alpha-sorted portfolio returns. In the opposite, if the beliefs do not correspond well to the reality, the alphas are non-informative and there is not a significant cross-sectional variation in the alpha-sorted portfolios.

5.3.2 The out-of-sample approach

The model is estimated using rolling windows. The subscript t means that the model is re-estimated each month. A first sample of arbitrary size T_{ins} from $[t - T_{ins} + 1; t]$ is used to estimate the true return moments under $\mathbb{O}$ at month t $\mu_t^{\mathbb{O}}$ and $\mathbf{\Sigma}_t^{\mathbb{O}}$. The representative beliefs are computed using the IBES beliefs at t, which is the most recent forecast at that month. The risk-aversion coefficient $\theta_{M,t}$ is then estimated as the one that minimizes the difference between the observed market weights at t and the endogenous weights (i.e. $\pi_{\varepsilon,t}$):

$$\min_{\theta_{M,t}} \quad \left(\pi_{M,t}^{obs} - \frac{\mathbf{\Omega}_t^{\mathbb{P}}}{\theta_{M,t}}(\mu_t^{\mathbb{P}} - \mathbb{1}R_{f,t})\right)^2$$

The risk-free rate is found such that the endogenous market weights sum to one. Using the true estimated moments, the Mean-Error, Precision-Error and Mean-Precision-Error portfolio weights can be computed. The error and market betas are computed from the portfolio weights and the variance-covariance estimate:

$$\beta_{X,t}^{\mathbb{O}} = \frac{\Sigma_t^{\mathbb{O}}\pi_{X,t}}{\pi'_{X,t}\Sigma_t^{\mathbb{O}}\pi_{X,t}}$$

For a given portfolio $\pi_{X,t}$. Estimation of the betas is the first step of the out-of-sample pricing test. In the second step, as it is the case in Fama-MacBeth two-pass procedure, we run a cross-sectional regression of the betas on the returns at $t + 1$ to estimate the factor premia. Several versions of the pricing relationships are tested:

CAPM

$$\mu^{\mathbb{O}}_{e,t+1} = \lambda_{M,t+1}\beta^{\mathbb{O}}_{M,t} + \varepsilon_{t+1}$$

2-Factor model

$$\mu^{\mathbb{O}}_{e,t+1} = \lambda_{M,t+1}\beta^{\mathbb{O}}_{M,t} - \lambda_{E,t+1}\beta^{\mathbb{O}}_{E,t} + \varepsilon_{t+1}$$

4-Factor model

$$\mu^{\mathbb{O}}_{e,t+1} = \lambda_{M,t+1}\beta^{\mathbb{O}}_{M,t} - \lambda_{ME,t+1}\beta^{\mathbb{O}}_{ME,t} - \lambda_{PE,t+1}\beta^{\mathbb{O}}_{PE,t} - \lambda_{MPE,t+1}\beta^{\mathbb{O}}_{MPE,t} + \varepsilon_{t+1}$$

Where $\mu^{\mathbb{O}}_{e,t+1}$ are the excess returns from t to $t+1$. The regressions give estimates for the various premia from t to $t+1$, and the process is reiterated each month until the end of the sample. The output is a distribution for the premia and the pricing errors. The betas stemming from the market weights error $\beta^{\mathbb{O}}_{\varepsilon}$ are added to the independent variables, leading to three and five-Factor models. From the theory, we know that the premia corresponding to each of the factors, including the market, has to be positive. The market premium λ_M is always positive even if the market excess return is negative because it also includes the slope adjustment $\gamma^{\mathbb{O}}_{M}$. A non-negativity constraint is therefore added to the least-square estimation of the regression coefficients. Since the error and market betas have opposite signs, the estimation without the constraints result in noisy unrealistic coefficients. In simulations, adding the constraint has been shown to greatly improve the precision and the stability of the estimates.

5.4 Results

Each set of characteristics result in different test assets and hence different results. The benchmark model is the usual one-factor CAPM relationship whose exposures are estimated using individual linear regressions as in Fama-MacBeth. The Fama-French three-factor model is also considered as it is widely used as a benchmark asset pricing model in practice. The latter is expected to perform better on value and size sorted portfolios than on idiosyncratic volatility sorted portfolios. In contrast, the performance of the error factor models corresponding to the I/B/E/S beliefs should be more flexible and not depend on the sorting of the asset returns. The results from the cross-sectional regressions can be studied along with the pricing error moments corresponding to each of the chosen model alternatives.

5.4.1 Value and size premia

The value-size portfolios are based on a double sorting done each month t. There are 5 quantiles per characteristic, implying 25 value-size portfolios. The sample period used to estimate the objective return moments is equal to four years i.e. 48 monthly observations. Each month, the model is re-estimated until the end of the sample, implying 208 out-of-sample regressions. In Table 1, the results from the cross-sectional regressions are shown. The mean, var and meanvar models consist in two-factor models including only the market and the corresponding three error portfolio exposures. The mean of the premia estimates over the out-of-sample period are shown, along with the standard deviations in parenthesis. The pricing errors P_{ε} are defined as the distance between the true observed returns and the predicted return stemming from the respective models. The four-factor model performs better than the Fama-French model as the pricing errors have lower

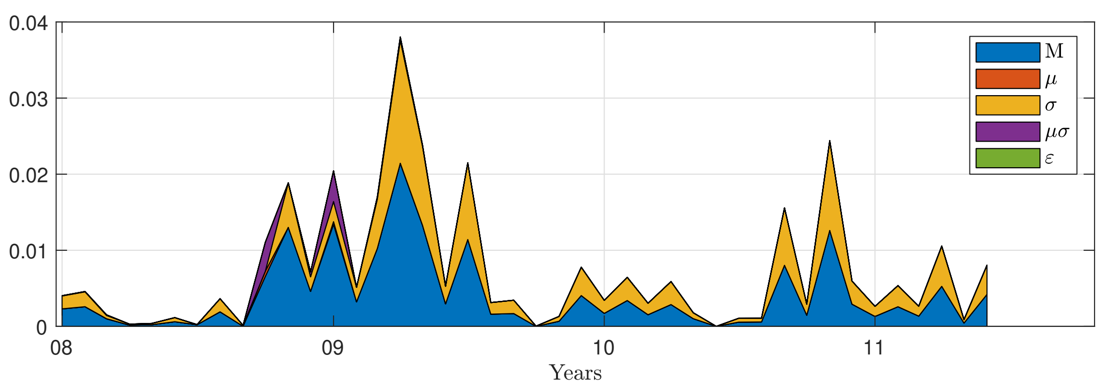
	CAPM	2-param	4-param	mean	var	meanvar	FF	FF+4p
λ_M	0.0064	5.1592	6.0802	0.0271	2.0539	0.1193	0.0059	6.3438
	(0.0615)	(9.7961)	(9.4438)	(0.0155)	(4.1588)	(0.3480)	(0.0793)	(9.4681)
λ_E	-	5.1867	-	-	-	-	-	-
	-	(9.9893)	-	-	-	-	-	-
λ_ε	-	0.0164	0.0178	0.0006	0.0068	0.0020	-	0.0172
	-	(0.0449)	(0.0433)	(0.0012)	(0.0291)	(0.0060)	-	(0.0420)
λ_μ	-	-	0.0103	0.0058	-	-	-	0.0138
	-	-	(0.0543)	(0.0091)	-	-	-	(0.0664)
λ_σ	-	-	5.6715	-	1.9922	-	-	5.9110
	-	-	(9.0176)	-	(4.2657)	-	-	(9.0585)
$\lambda_{\mu\sigma}$	-	-	0.0957	-	-	0.0246	-	0.1277
	-	-	(0.3739)	-	-	(0.1013)	-	(0.4754)
λ_{SMB}	-	-	-	-	-	-	0.0036	0.0250
	-	-	-	-	-	-	(0.0436)	(0.0329)
λ_{HML}	-	-	-	-	-	-	0.0038	0.0236
	-	-	-	-	-	-	(0.0509)	(0.0414)
$E[P_\varepsilon]$	0.0166	0.0138	0.0114	0.0168	0.0136	0.0182	0.0129	0.0106
$\sigma[P_\varepsilon]$	0.0132	0.0111	0.0092	0.0128	0.0109	0.0132	0.0106	0.0084

Table 1: Value & size cross-sectional regressions - I/B/E/S beliefs

mean and volatility, by about 0.15% each. Naturally, there are two additional factors in the four-factor model, compared to the Fama-French model. The two-factor model, which has actually three factors since beliefs models always includes the market weights error ε, has pricing errors with slightly higher mean and volatility compared to the Fama-French model. The latter is designed based on the size and value premia, hence it is not surprising that it performs well on value-size portfolio returns. Concerning the expectation error premia, most of the explaining power seems to come from the variance factor. The magnitude of the coefficient is higher in all the models, and the var model (with only the market and variance factors) performs better in terms of pricing errors than the two-factor model which includes all types of expectation errors into one factor. In contrast, including only the mean-variance factor results in a worse pricing performance than the regular CAPM. The variance premia value is close to the market premia value in all models in which both factors are included. It suggests that the variance error portfolio, which makes most of the overall error portfolio, relates to the market portfolio. That is, according to the I/B/E/S beliefs with the zero correlation assumption, most of the market portfolio seems to be related to variance expectation errors. As it can be seen in the last column on the right, adding the SMB and HML factors from the Fama-French model to the four-factor model does not significantly improve the pricing error statistics. However, adding the four belief factors (five including the market weights error) allows to significantly lower the mean and volatility of P_ε. It suggests that the belief factors are not restricted to but include size and value cross-sectional effects. Note that the Fama-French factors, as it is the case for the regular CAPM factor, is defined based on a given cross-section of asset returns different than the ones considered in the test. In the opposite, the expectation error factors as well as the market factor stemming from the belief models are endogenously created using the given test assets. It is a significant difference that likely favours the performance of the belief models. None of the estimated premia are significant by the t-stat criterion, including the ones from the CAPM and Fama-French models. Note that the latter are unconstrained, whereas the former are constrained to be only positive. However, the means and volatilities are time-series statistics computed using the full sample of 208 monthly estimated premia i.e. about 17 years. Each premium is estimated using cross-sectional regressions with the next monthly observed test asset returns as dependent variables. That is, as it is the case in the Fama-Macbeth method, the dependent variables are not mean returns but observed returns. The latter are more volatile than the former, and it leads to volatile estimated premia. The sample encompasses many market conditions and the t-stat might under-estimate the coefficient significance. Morever, since they relate to expectation errors, it may be that the true premia vary much over time, although always remaining positive.

In Figure 11, the estimated (scaled) premia are represented for a given subsample including the 2008 financial crisis. That is, the coefficients estimated from the regressions in Table 1 are divided at each t by the market estimated risk-aversion coefficient. In the period preceding the financial crisis the estimated premia are close to zero, but they increase in 2008 to attain their maximum in 2009. They then decrease in magnitude but still tend to stay positive until the end of the sample, with a local peak in the end of 2010. As the t-stats suggested, they are indeed unstable and move much from one month to the next but with local tendencies. Aside from the market premium and the mean-variance premium in the beginning of the crisis, the variance expectation error premium is the only one visible in Figure 11. That is, as opposed to the variance errors, the mean expectation errors from the I/B/E/S beliefs do not seem to explain much of the cross-sectional variation in expected returns.

Figure 11: Value-size estimated prices of risk over time - 4-Factor model



The values of the market and variance premia seem to be highly correlated and related to each other. Remember that the market and error premia have opposite signs in the pricing relationship. Hence, exposures to the market factor (rewarded positively in equilibrium) are partly compensated by the exposure to the variance factor (rewarded negatively in equilibrium). However, although they can be similar, the asset market beta is generally different than the error beta, implying that the net effect is different than zero. The expectation error betas stemming from the belief deviations from $\mathbb{P}$ to $\mathbb{O}$ depend on the exogenous beliefs, in this case the I/B/E/S beliefs. Although it has no impacts on the market betas, different beliefs result in different error betas and thus different estimated premia. The most relevant error portfolio for pricing is the one related to the variance expectation errors. Since the correlations between test assets are exogenously assumed to be zero under the representative measure whereas they are always different than zero under the objective measure, it is not surprising that the variance expectation error portfolio is the most important. However, as mentioned earlier, this assumption results in low deviations between the endogenous market weights and the observed market weights. That is, the market weights errors π_ε are generally small in magnitude, implying that the zero correlation assumption fits well to the observed price data in the mean-variance set-up.

	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\beta}_M$	$\widehat{\alpha}_M$	$\widehat{\sigma}_{\varepsilon,M}$	$\widehat{\beta}_E$	$\widehat{\alpha}_E$	$\widehat{\sigma}_{\varepsilon,E}$	$\bar{\text{mktcap}}$	$\bar{\text{btm}}$
$\pi_{\alpha,1}$	0.917	7.994	1.261	0.151	2.546	1.282	0.208	2.453	870.884	0.883
	-	-	(0.000)	(39.845)	-	(0.000)	(22.688)	-	-	-
$\pi_{\alpha,2}$	0.737	6.816	1.107	0.064	1.477	1.130	0.112	1.229	2002.859	0.635
	-	-	(0.000)	(53.568)	-	(0.000)	(19.337)	-	-	-
$\pi_{\alpha,3}$	0.564	6.624	1.080	-0.093	1.315	1.098	-0.044	1.185	1839.932	0.503
	-	-	(0.000)	(31.223)	-	(0.000)	(59.456)	-	-	-
$\pi_{\alpha,4}$	0.726	6.358	1.046	0.090	0.945	1.060	0.140	0.929	5008.327	0.444
	-	-	(0.000)	(17.399)	-	(0.000)	(3.251)	-	-	-
$\pi_{\alpha,5}$	0.535	5.684	0.934	-0.032	0.909	0.938	0.016	1.135	21594.760	0.421
	-	-	(0.611)	(61.059)	-	(0.000)	(84.200)	-	-	-
π_M	0.608	6.009	1.000	0.000	0.000	1.009	0.050	0.528	12548.204	0.428
	-	-	(0.000)	(100.000)	-	(0.000)	(17.889)	-	-	-
π_E	0.554	5.935	0.984	-0.045	0.521	1.000	-0.000	0.000	9278.323	0.471
	-	-	(0.000)	(22.187)	-	(0.000)	(0.000)	-	-	-
π_*	0.004	0.337	0.027	-0.012	0.294	0.025	-0.009	0.302	1268.432	0.004
	-	-	(0.000)	(55.295)	-	(0.000)	(65.640)	-	-	-

Table 2: Value & size portfolio statistics - I/B/E/S beliefs

In order to better understand the error portfolio corresponding to the I/B/E/S beliefs and its relationship with the market portfolio, the out-of-sample statistics corresponding to different relevant portfolios are shown in Table 2. They are based on the 208 out-of-sample monthly returns. The three bottom portfolios are the observed market portfolio π_M, the exogenous error portfolio π_E and the efficient part of the market portfolio π_*. The five portfolios in the first lines of the Table are the portfolios constructed based on the estimated alphas, corresponding to the I/B/E/S beliefs. They are sorted from the lowest alphas to the highest alphas. That is, $\pi_{\alpha,1}$ is the equally-weighted portfolio invested in the 5 test assets with the lowest estimated alphas, whereas $\pi_{\alpha,1}$ is invested in the opposite in the 5 test assets with the highest estimated alphas. Concerning the statistics, $\widehat{\mu}$ and $\widehat{\sigma}$ are respectively the means and volatilities of the portfolio returns, in percentage points. $\widehat{\beta}_M$, $\widehat{\alpha}_M$ and $\widehat{\sigma}_{\varepsilon,M}$ are the estimated beta, alpha and volatility of the residuals in a contemporary time-series regression of the portfolio returns on the market return. The figures in parenthesis are the p-values, in percentage points. Similarly, $\widehat{\beta}_E$, $\widehat{\alpha}_E$ and $\widehat{\sigma}_{\varepsilon,E}$ correspond to the estimated coefficients resulting from a time-series regression with the error return as independent variable rather than the market return. Hence, all the regression statistics consist in single scalar coefficients. Finally, $\bar{\text{mktcap}}$ and $\bar{\text{btm}}$ are the average market capitalization (in millions) and book-to-market values for each of the considered portfolios. The error and market portfolios are indeed similar, highlighted by their corresponding statistics and the similarity between the two betas for all the portfolios. It is the case even if the market and error portfolios are different, as the latter can for example have short positions. It means that most of the observed market portfolio is due to I/B/E/S expectation errors, in particular to variance expectation errors (including variances and correlations). The efficient part of the market portfolio π_* has small positions in magnitude as both its mean and volatility are close to zero. It means that few of the market variation relates to true moments, whereas much of the variation is related to expectation errors. Assets with low estimated alphas and hence not desirable according to the I/B/E/S beliefs, are high beta assets with high means and idiosyncratic volatilities. In the opposite, assets with high estimated alphas are low-beta assets, with low market betas inferior to one and low absolute/idiosyncratic volatilities. $\pi_{\alpha,5}$ is also the portfolio with the lowest mean out of all the alpha-sorted portfolios. It implies that there are systematic biases in the I/B/E/S beliefs, favouring low-beta assets with low risk. Assets with low co-variation with the error portfolio are also assets with low co-variation with the market portfolio. Regardless of the assumed beliefs, high alpha assets should have low beta with the error portfolio, since the latter has a negative impact in the cross-section. In the I/B/E/S beliefs case, low beta with the error portfolio also corresponds to low beta with the market portfolio. The relationship between alphas and market/error betas is stable across alpha-sorted portfolios. In the regressions, all beta coefficients are significant but only the alpha corresponding to portfolio $\pi_{\alpha,4}$'s return with respect to the error portfolio is significant at the 5% level. Although not significant, the portfolio $\pi_{\alpha,1}$ has a positive alpha with respect to the market portfolio, contrary to what is expected by the I/B/E/S investor. In the opposite, the portfolio $\pi_{\alpha,5}$, expected to have the highest alpha according to the I/B/E/S investor, has a negative alpha with respect to the market return. Therefore, the alpha predictions stemming from the I/B/E/S beliefs do not seem to correspond to what happens ex-post. The average characteristics in the last two columns on the right show that assets with low alphas according to the I/B/E/S beliefs are systematically small stocks with high book-to-market ratios whereas assets with high alpha stocks are systematically big stocks with low book-to-market ratios. To bet on the I/B/E/S beliefs, one should take a long position in portfolio $\pi_{\alpha,5}$ with high market capitalization/low book-to-market and simultaneously take a short positions in $\pi_{\alpha,1}$ with low market capitalization/high book-to-market. This strategy generates a negative alpha with respect to the market return. In the opposite, if ones wishes to bet against the I/B/E/S beliefs, one should take short positions in big stocks with low book-to-market ratios and take long positions in small stocks with high book-to-market ratios. The latter strategy generates positive alpha and is the idea behind the SMB and HML factors in the Fama-French three-factor model. The size and value anomalies are perpetuated by the I/B/E/S beliefs in the sense that they are over-optimistic on big stocks with low book-to-market ratios and low market betas, and over-pessimistic on small stocks with high book-to-market ratios and high market betas/idiosyncratic volatilities. That belief does not correspond to the reality as it is actually the contrarian strategy that generates positive alpha ex-post.

5.4.2 Idiosyncratic volatility premium

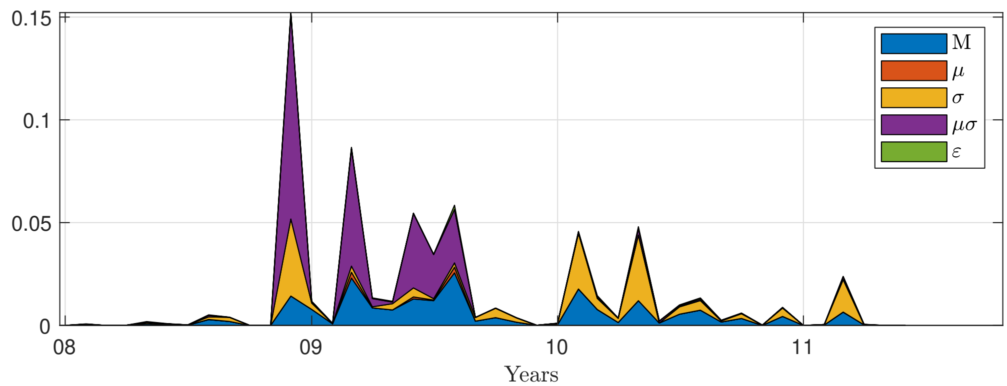
	CAPM	2-param	4-param	mean	var	meanvar	FF	FF+4p
λ_M	0.0031	2.2579	2.9970	0.1008	0.8139	0.1774	0.0073	3.1936
	(0.0665)	(4.9731)	(5.0816)	(0.0577)	(3.1379)	(0.5956)	(0.1626)	(5.2498)
λ_E	-	2.7480	-	-	-	-	-	-
	-	(5.2877)	-	-	-	-	-	-
λ_ε	-	0.2099	0.3469	0.0251	0.0779	0.0194	-	0.3812
	-	(0.3315)	(0.5855)	(0.0214)	(0.2797)	(0.0438)	-	(0.6381)
λ_μ	-	-	0.0221	0.0131	-	-	-	0.0242
	-	-	(0.0631)	(0.0203)	-	-	-	(0.0648)
λ_σ	-	-	4.7494	-	0.9770	-	-	4.8651
	-	-	(9.8728)	-	(3.3818)	-	-	(9.7428)
$\lambda_{\mu\sigma}$	-	-	1.0930	-	-	0.2087	-	1.1410
	-	-	(3.3729)	-	-	(1.3116)	-	(3.4026)
λ_{SMB}	-	-	-	-	-	-	-0.0067	0.0174
	-	-	-	-	-	-	(0.1087)	(0.0326)
λ_{HML}	-	-	-	-	-	-	-0.0094	0.0149
	-	-	-	-	-	-	(0.0967)	(0.0295)
$E[P_\varepsilon]$	0.0181	0.0149	0.0135	0.0228	0.0163	0.0222	0.0187	0.0131
$\sigma[P_\varepsilon]$	0.0174	0.0140	0.0117	0.0226	0.0152	0.0168	0.0156	0.0112

Table 3: Idiosyncratic volatility cross-sectional regressions - I/B/E/S beliefs

The idiosyncratic volatilities for each of the stocks in-sample are computed using time-series regression with the market return as independent variables, using 24 months for the estimation. It results in a characteristic that can be used to generate characteristic-ordered portfolio returns. Each t, the volatility characteristic is decomposed into 25 bins used to generate 25 volatility-sorted portfolio returns. As it is the case for the value-size premia, the in-sample period is equal to 48 months, implying 184 out-of-sample periods. In Table 3, the results from the cross-sectional regressions are shown, using the I/B/E/S beliefs with the zero correlation assumption as exogenous beliefs. As expected, the pricing performance of the Fama-French model on idiosyncratic volatility sorted portfolios is not as accurate as it is on value-size sorted portfolios. The Fama-French model performs slightly worse than the CAPM on average pricing errors but significantly better in terms of volatility. The best performing model is the four-factor model, with an improvement of about 0.50% in average pricing errors and close to 0.6% in volatility compared to the CAPM. Although it is less clear than with the value-size portfolios, the most significant error portfolio is again the variance expectation error portfolio. Its coefficients in all models tend to be higher in magnitude than the ones of the market factor. Contrary to the value-size portfolios, the pricing performance of the two-factor belief model is better than the one using only the variance error portfolio. It is also better than the Fama-French model, with an improvement by about 0.40% in expected pricing error and 0.16% in its volatility. Moreover, the mean-variance premium seems to be higher in Table 3 than in Table 1. The models with only the mean and mean-variance expectation errors portfolio result in pricing performances significantly worse than the usual CAPM. As shown on the last column on the right, adding the Fama-French factors to the four-factor belief model does not bring significant improvements in terms of pricing errors. The regression results suggest that the four-factor belief model (corresponding to the I/B/E/S beliefs) is a more general approach than the one of the Fama-French model since it can price accurately both value-size and idiosyncratic volatility portfolios.

In Figure 12, the estimated premia are shown from January 2008 to June 2011. They have low values in the beginning of the period, before the financial crisis. Contrary to the case of the value-size portfolios, the mean-variance expectation error is the most important premium during the financial crisis. Moreover, the sum of the premia is higher, about 0.15, compared to a maximum of about 0.04 in Figure 11. However, it does not appear much the rest of the period, in which the variance expectation error premium is the most important aside from the market premium.

Figure 12: Idiosyncratic volatility estimated prices of risk over time - 4-Factor model



The relevant portfolio statistics are shown in Table 4. As it is the case for the value-size sorted portfolio returns, the alpha-sorted market betas are similar to their error counterpart. It is not surprising since the exogenous beliefs are the same, only applied to different returns. Assets with low expected alphas by the I/B/E/S beliefs have high betas and idiosyncratic volatilities, whereas assets with high expected alphas have low betas and low idiosyncratic volatilities. The alpha-sorted portfolio moments show that the Sharpe ratio tends to indeed increase with the expected alpha stemming from the I/B/E/S beliefs, and portfolio $\pi_{\alpha,1}$ has a monthly Sharpe ratio equal to 0.037 whereas portfolio $\pi_{\alpha,5}$ has a monthly Sharpe ratio of 0.127. The error beta of the latter portfolio is lower than its market beta, the most important difference amongst all portfolios. That is, the net exposure to the factors ($\beta_M^{\mathbb{O}}$ - $\beta_E^{\mathbb{O}}$) is positive leading to a positive effect on $\pi_{\alpha,5}$'s mean. The I/B/E/S beliefs seem to better predict the alphas of the idiosyncratic volatility sorted portfolio returns. Although insignificant, portfolios one and two have negative alphas whereas portfolios four and five have positive alphas. The volatilities of the residuals of the regressions show that the error portfolio does significantly better at explaining the alpha returns than the market portfolio, except for portfolio $\pi_{\alpha,5}$. The only significant regression alpha coefficient at the 5% level in the regressions is the (negative) of $\pi_{\alpha,2}$ on the error portfolio. The efficient part of the market portfolio π_* has higher volatility and mean than in the value-size case in Table 2. It also has a significant negative beta with respect to the error portfolio and a beta insignificantly different from zero with respect to the market portfolio. The error portfolio has again statistics similar to the market portfolio, but it

	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\beta}_M$	$\widehat{\alpha}_M$	$\widehat{\sigma}_{\varepsilon,M}$	$\widehat{\beta}_E$	$\widehat{\alpha}_E$	$\widehat{\sigma}_{\varepsilon,E}$	$\bar{\text{vol}}$
$\pi_{\alpha,1}$	0.347	9.334	1.337	-0.390	5.664	1.406	-0.352	3.523	31.283
	-	-	(0.000)	(35.508)	-	(0.000)	(18.013)	-	-
$\pi_{\alpha,2}$	0.220	7.270	1.121	-0.398	3.765	1.142	-0.348	1.875	12.167
	-	-	(0.000)	(15.660)	-	(0.000)	(1.333)	-	-
$\pi_{\alpha,3}$	0.526	6.975	1.110	-0.087	3.269	1.098	-0.020	1.756	8.886
	-	-	(0.000)	(72.142)	-	(0.000)	(87.738)	-	-
$\pi_{\alpha,4}$	0.593	6.187	1.012	0.035	2.599	0.968	0.112	1.683	6.530
	-	-	(0.000)	(85.456)	-	(0.000)	(37.035)	-	-
$\pi_{\alpha,5}$	0.639	5.012	0.842	0.175	1.819	0.754	0.264	1.900	4.535
	-	-	(0.197)	(19.695)	-	(0.000)	(6.239)	-	-
π_M	0.551	5.550	1.000	-0.000	0.000	0.788	0.160	2.702	6.181
	-	-	(0.000)	(100.000)	-	(0.000)	(42.672)	-	-
π_E	0.497	6.149	0.968	-0.037	2.993	1.000	0.000	0.000	6.004
	-	-	(0.000)	(86.915)	-	(0.000)	(100.000)	-	-
π_*	0.072	1.119	0.004	0.070	1.118	-0.046	0.095	1.083	3.451
	-	-	(0.766)	(40.390)	-	(0.001)	(23.999)	-	-

Table 4: Idiosyncratic volatility portfolio statistics - I/B/E/S beliefs

is slightly less mean-variance efficient since both its mean and volatility are respectively lower and higher. The I/B/E/S beliefs seem to favour assets with low idiosyncratic volatilities compared to asset with high idiosyncratic volatilities. It is a sound decision because the former assets perform better than the latter ex-post. A strategy that bets on the I/B/E/S beliefs i.e. that takes a long position $\pi_{\alpha,1}$ and a short position in $\pi_{\alpha,5}$ allows to generate a positive mean return for a zero net investment. The strategy is then long low idiosyncratic volatility assets and short high idiosyncratic volatility assets, leading to an overall negative exposure to the idiosyncratic volatility characteristic.

6 Conclusion

In this paper, we fully relax the perfect information assumption in the context of the CAPM. That is, there are heterogeneous and biased beliefs on all moments of the return distribution, resulting in equilibrium expectation errors on the asset means, variances and correlations. Market weights are mean-variance inefficient and decomposed as the sum of four portfolios. The first one relates to the true asset moments, whereas the three other ones relate to each type of expectation errors: Mean-Errors (ME), Precision-Errors (PE) and combined Mean-Precision-Errors (MPE). It leads to an error-adjusted SML relationship, with idiosyncratic asset alphas and a systematic slope adjustment with respect to the market excess return. All the pricing terms retain the same structure as the portfolio decomposition, and the errors with respect to risk have different and more complex consequences than errors with respect to mean. The idiosyncratic alphas can be expressed as exposures to the three expectation error portfolios, implying that the pricing relationship is four-dimensional: the Security Market Space (SMS). The expectation error factors do not relate to outside sources of risk, but rather are consequences of mistakes on the only true source of risk, the market return. It suggests that the perfect information assumption of the CAPM has been tested in the literature, rather than the market model itself. Cross-sectional effects considered as anomalous in the standard CAPM can be explained using the SMS. Fama-French portfolios and other factor models are based on empirical performance but they can not be explained using equilibrium arguments. Cross-sectional effects related to characteristics in the data are consistent with persistent belief deviations on the underlying assets.

In order to test the results developed in the theoretical part, the SMP must be estimated using real return data. While the objective distribution parameters can be estimated using observed returns, the representative distribution parameters is only indirectly observed through the market weights. There are more unknowns than equations and hence an infinity of solutions. Rather than searching for the optimal representative measure, we develop a procedure that allows to test for the pricing relevance of arbitrary exogenous beliefs on any sample of asset returns. It requires as inputs the representative variance-covariance matrix and mean vector, but the latter do not have to correspond to the observed market weights. The procedure is similar to Fama-Macbeth two-pass regressions but without linear regressions in the first stage. Different assumed beliefs result in different error betas, with different out-of-sample pricing performances. The approach works for any arbitrary beliefs, irrespective of how realistic they are. However, the out-of-sample pricing performance should be lower for unrealistic than for realistic beliefs. For the illustration, we use the analysts' forecasts from the I/B/E/S database as exogenous representative beliefs. Since the correlation beliefs are not given by the I/B/E/S database, they are for simplicity assumed to be zero across portfolios. The zero correlation assumption seems to match well to the only positive observed market weights in the mean-variance preferences set-up. To reduce the test asset dimension and still be able to test for well-known asset pricing premia, we regroup the assets into characteristic-sorted portfolio returns. We focus on the size-value premia, as well as the idiosyncratic volatility premium. The results from the cross-sectional regressions show that the four-factor model associated to the I/B/E/S beliefs performs better than the CAPM and the Fama-French model in terms of pricing errors. In the value-size case, the latter model, which is specifically designed to price the value and size premia, has a similar performance to the four-factor belief model. However, in the idiosyncratic volatility case, there is a significant pricing error difference between both models. There is a preference for low-beta and hence low-risk assets in the I/B/E/S analysts' forecasts. Moreover, in both datasets, amongst the three expectation error portfolios, the variance error premium seems to be the most important. The I/B/E/S beliefs seem to do better at pricing the idiosyncratic volatility sorted portfolios as the expected alphas better correspond to the realized performances ex-post. It suggests that the biases stemming from the analysts' forecasts might be less exposed to the idiosyncratic volatility anomaly than to the value and size anomalies. The error portfolio seems to be similar to the error portfolio, implying that, according to the I/B/E/S beliefs and mean-variance optimality, most of the market covariance is due to errors.

The empirical results show the model structure to be relevant as the four-factor model allows to better price assets than the standard CAPM and the Fama-French model. The empirical test is done on the value-size and idiosyncratic volatility premia, but many other premia can be considered. Moreover, while the results are informative with respect to the beliefs corresponding to the I/B/E/S forecasts, they also highly depend on the assumed beliefs. Different beliefs give different results, and some are more realistic and hence accurate than others. The pricing performance of the four-factor model can likely be improved by searching for more realistic beliefs, especially on the correlation dimension. To that purpose, additional work must be done on the estimation of the representative measure, which is a high-dimensional object. Regarding the theoretical results, they are valid for a static model under the mean-variance utility assumption. It could be interesting to use the same weights decomposition method to derive the error-adjusted versions of more complicated models, for example dynamic models. Moreover, the three additional error term structure relates to the mean-variance utility assumption. If more than two moments were involved in the utility function of the representative investor, more additional error terms would appear in the equations. Hence, it would also be interesting to use the same method to relax the perfect information assumption with other utility functions. These extensions are left for further research.

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