Parameter Learning in Production Economies ☆

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Abstract

We examine how parameter learning amplifies the impact of macroeconomic shocks on equity prices and quantities in a standard production economy where a representative agent has Epstein-Zin preferences. An investor observes technology shocks that follow a regime-switching process, but does not know the underlying model parameters governing the short-term and long-run perspectives of economic growth. We show that rational parameter learning endogenously generates long-run productivity and consumption risks that help explain a wide array of dynamic pricing phenomena. The asset pricing implications of subjective long-run risks crucially depend on the introduction of a procyclical dividend process consistent with the data.

Keywords: Parameter Learning, Equity Premium, Business Cycles, Markov Switching
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1. Introduction

Parameter learning has recently been proposed as an amplification mechanism for the pricing of macroeconomic shocks used to explain standard asset pricing moments. In the endowment economy, parameter uncertainty helps explain the observed equity premium, the high volatility of equity returns, the market price-dividend ratio and the equity Sharpe ratio (Collin-Dufresne, Johannes and Lochstoer, 2016; Johannes, Lochstoer and Mou, 2016). In contrast to the consumption-based approach, a production dynamic stochastic general equilibrium (DSGE) model endogenously generates consumption and dividends and, as a result, it becomes more challenging to explain asset pricing puzzles in a production-based setting while simultaneously matching the moments of macroeconomic fundamentals. In this paper, we study how the macroeconomic risks arising from parameter uncertainty improve the performance of a standard DSGE model in jointly reproducing salient features of the macroeconomic quantities and equity returns.

Kaltenbrunner and Lochstoer (2010) and Croce (2014) argue that the presence of a small but persistent long-run risk component in the productivity growth process can endogenously generate long-run risks in consumption growth that help boosting up moments of financial variables. However, these long-run risk components are difficult to identify in the data.[1] In contrast, we demonstrate that rational pricing of parameter uncertainty is a source of these subjective long-run risks in productivity growth. This emphasizes the importance of accounting for parameter uncertainty in the productivity growth process. It is not clear, however, whether macroeconomic risks associated with rational learning about productivity growth amplify the moments of financial variables. If so, what is the magnitude of the effect? In this paper, we document a considerable amplification mechanism of rational parameter learning on asset prices.

We introduce parameter uncertainty in the technology growth process of an otherwise standard production-based asset pricing model. We depart from the extant macro-finance literature by presuming that the representative investor does not know

[1] Croce (2014) empirically demonstrates the existence of such a predictable component; however, the results are not robust to estimation method and sample choice. Moreover, low values for goodness-of-fit statistics lead to a conclusion that there is considerable uncertainty about the model specification for productivity growth.

the parameters of the technology process and learns about true parameter values from the data. In each period, he updates his beliefs in a Bayesian fashion upon observing newly arrived data. Rational learning about unknown parameters together with recursive preferences gives rise to subjective long-lasting macroeconomic risks. Coupled with endogenous long-run consumption risks due to consumption smoothing (Kaltenbrunner and Lochstoer, 2010) these risks are priced under the investor's preference for early resolution of uncertainty. The model generates higher equity Sharpe ratios, risk premia and volatility, as well as lower interest rates and price-dividend ratios relative to the standard framework. Additionally, the model with rational belief updating reproduces the excess return predictability patterns observed in the data. We show that under certain calibrations of capital adjustment cost parameters, parameter learning significantly magnifies propagation of productivity shocks and hence helps to match the moments and comovements of macroeconomic variables.

In our analysis, we restrict our attention to uncertainty about parameters governing the magnitude and persistence of productivity growth over the various phases of the business cycle. In particular, we examine the implications of learning about the transition probabilities and mean growth rates in a two-state Markov-switching process for productivity growth, where volatility of productivity growth is homoskedastic and known.[2] We consider two approaches to dealing with parameter uncertainty in the equilibrium models: anticipated utility (AU) and priced parameter uncertainty (PPU). The AU approach is common for most existing models, and assumes that economic agents learn about unknown parameters over time, but treat their current beliefs as true and fixed parameter values in the decision-making. For the PPU case, the representative investor calculates his utility and prices in the current period, assuming that posterior beliefs can be changed in the future. We quantify the impact of each type of parameter uncertainty pricing by comparing the results of AU and PPU with the full information (FI) model.

We illustrate the economic importance of parameter uncertainty in a standard

[2]There is a large strand of the literature emphasizing the importance of time-varying macroeconomic uncertainty (see, for example, Justiniano and Primiceri (2008); Bloom (2009); Fernandez-Villaverde, Guerron-Quintana, Rubio-Ramirez and Uribe (2011); Born and Pfeifer (2014); Christiano, Motto and Rostagno (2014); Gilchrist, Sim and Zakrajsek (2014); Liu and Miao (2014) and more recent studies by Leduc and Liu (2016); Basu and Bundick (2017); Bloom, Floetotto, Jaimovich, Saporta-Eksten and Terry (2018)). We leave the investigation of learning about volatility risks for future research.

real business-cycle model with Epstein-Zin preferences and asymmetric quadratic capital adjustment costs. The increased uncertainty due to unknown parameters in the productivity growth process creates a stronger precautionary saving motive, which leads to a lower risk-free rate. Fully rational learning about unknown parameters generates endogenous long-run risks in the economy, which in turn increase the mean and volatility of levered returns to the firm's payouts (Jermann, 1998). In contrast, fluctuations in parameter beliefs are not priced in the AU case. Thus, the PPU approach leads to around a two-fold increase in the risk premium (in addition to higher return volatility) on a levered firm's dividends, relative to the FI and AU cases. The combination of time-varying posterior beliefs and rational parameter learning is crucial for generating long-term predictability of excess returns by Tobin's Q, investment-capital, price-dividend and consumption-wealth ratios, as found in the empirical literature. The time-variation in beliefs leads to fluctuations in the equity risk premium and hence generates more predictability in the models with parameter uncertainty relative to the known parameter frameworks. Fully rational learning further magnifies the impact of belief revisions on the conditional equity premium and therefore there is more significant return predictability with PPU compared to AU. Specifically, the model with PPU closely replicates the increasing patterns (in absolute terms) of the regression coefficients and R^2's. In contrast, both the FI and AU models generate less predictability power and cannot match the magnitude of slope coefficients. In terms of the macroeconomic variables, the model with parameter learning has a sizable effect on the unconditional second moments of investment but smaller effects on volatilities of consumption and output. It also improves comovements between macroeconomic variables.

The main mechanism of this paper is closely related to the work of Collin-Dufresne, Johannes and Lochstoer (2016) who study a similar learning problem in the endowment economy. Our analysis differs from theirs in the following ways. First, we extend their methodology to a production economy setting and explore joint implications of parameter uncertainty for macroeconomic quantities and asset prices. Second, relative to the endowment model, one needs to generate procyclical dividends in the production economy to obtain a significant amplification of equity moments by parameter learning. We document this finding in the model with costly reversibility, which generates endogenous procyclical firm's payoffs. Further, we show the robustness of this feature to more common convex adjustment costs by pricing a claim to exogenous calibrated dividends. Third, rather than exploring the impact of learning in a rare events model (Rietz, 1988; Barro, 2006), we instead estimate the production parameters by the expectation maximization algorithm from the postwar U.S. data. Even though the estimated process for productivity growth does not reflect rare states that are naturally difficult to be learned about due to their rareness, fully rational parameter learning still matches well financial moments in our setting with more frequent states. The main reason for this is that long-run consumption risks generated by consumption smoothing (see Kaltenbrunner and Lochstoer, 2010) magnify the impact of endogenous long-run productivity risks originating from belief revisions on asset prices; therefore, less is needed in terms of the speed of parameter learning.

Our paper speaks to macro-finance research in the production-based economies. Cagetti, Hansen, Sargent and Williams (2002) is one of the first examples of the real business-cycle model with parameter learning. In their paper, they consider a signal extraction problem about the unobservable mean growth rate of technology shocks. However, they do not study the implications of incomplete information for quantities and asset prices, a key focus of our analysis. Chen (2017) achieves amplifications of financial moments by incorporating external habit in the standard real business-cycle model. The agent in the model is assumed to know true values of parameters of the productivity process. In a recent paper, Jahan-Parvar and Liu (2014) examine a production economy with learning about a latent state in a productivity growth process following a two-state hidden Markov chain. Their paper is an adaption of the endowment economy with ambiguity preferences (Ju and Miao, 2012) to a production setting. The key differentiator of our study from Jahan-Parvar and Liu (2014), as well as the extant literature on learning in a real business-cycle model, is a multidimensional learning problem and rational pricing of parameter beliefs. Hirshleifer et al. (2015) demonstrate that introducing extrapolative bias into the production-based model also helps to reconcile stylized facts about business-cycle fluctuations and financial markets. We show that these salient stylized features of the data can be generated within rational framework without resorting to behavioral biases.

Our paper is also related to the long-run risks models introduced by Bansal and Yaron (2004) in the consumption-based setting. Kaltenbrunner and Lochstoer (2010) and Croce (2014) investigate the implications of long-run productivity risks in the production-based economy. In relation to these studies, we do not explicitly incorporate long-run risks in productivity growth by directly adopting the specification of Bansal and Yaron (2004). In our paper, Bayesian learning about unknown parameters in the productivity growth process gives rise to the subjective long-run macroeconomic risks. Therefore, our approach is complementary to the existing long-run risks literature and in fact provides the empirical investigation of possible origins of long-run risks.

The paper proceeds as follows. Section 2 presents the formal model. Section 3 investigates the quantitative implications of parameter learning for quantities and asset prices. Section 4 performs sensitivity analysis. Section 5 concludes.

2. Model

In this section, we present a production-based asset pricing model similar to Jermann (1998), Campanale, Castro and Clementi (2010), Kaltenbrunner and Lochstoer (2010), and Croce (2014). The model is a standard real business-cycle framework (Kydland and Prescott, 1982; Long and Plosser, 1983) populated by a representative firm with Cobb-Douglas production technology and a representative household with Epstein-Zin preferences. The firm produces a single consumption-investment good using labor and capital as inputs subject to productivity shocks and capital adjustment costs. The household participates in the production process by working for the firm and providing investment for capital. Additionally, the household trades firm shares and risk-free bonds to maximize lifetime utility of a consumption stream subject to a sequential budget constraint. Ultimately, the firm maximizes its value by choosing labor and investment demand. Our objective is to investigate the impact of rational learning about unknown parameters in the productivity process on the moments of macroeconomic quantities and equity returns as well as predictability patterns observed in the data.

2.1. Household

A representative household has recursive preferences of Epstein and Zin (1989):

$$U_t = \left\{ (1-\beta) V_t^{1-1/\psi} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}} \tag{1}$$

where $E_t[\cdot]$ is the expectation operator, $\beta \in (0,1)$ is the discount factor, $\psi > 0$ is the elasticity of inter-temporal substitution (EIS), and $\gamma > 0$ is the risk aversion parameter. The utility index, V_t, depends on consumption, C_t, and hours worked, N_t, and takes a standard Cobb-Douglas form:

$$V_t = C_t (1 - N_t)^{\nu},$$

where $\nu > 0$ is the leisure preference. It is straightforward to show that the stochastic discount factor of the economy is defined as:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \left(\frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-1/\psi)\nu} \left(\frac{U_{t+1}}{\left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right)^{1/\psi - \gamma}. \tag{2}$$

A calibration with $1/\psi \neq \gamma$ implies the utility function is not time-additive. This feature of recursive preferences allows for a separation of agent's relative risk aversion from the elasticity of inter-temporal substitution. Further, we set $\gamma > \frac{1}{\psi}$ and hence the household prefers earlier resolution of uncertainty. When the household's continuation utility, U_{t+1}, is below the certainty equivalent, the last multiplier in the pricing kernel increases, raising a premium for future low utility states.

2.2. Firm

The representative firm produces the consumption good using a constant returns to scale Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} (A_t N_t)^{1-\alpha}, \tag{3}$$

where Y_t is the output, K_t is the capital stock, N_t is labor hours, and A_t is an exogenous, labor-enhancing technology level (which we also refer to as productivity). The firm's capital accumulation equation incorporates capital adjustment costs and is given by:

$$K_{t+1} = (1-\delta) K_t + \varphi(I_t / K_t) K_t,$$

where $\delta \in (0,1)$ is the capital depreciation rate, $I_t = Y_t - C_t$ is gross investment, and $\varphi(\cdot)$ is the capital adjustment cost function.

Following Abel and Eberly (1994, 1996) and Zhang (2005), we include investment frictions in the form of costly reversibility: firms face higher adjustment costs for contracting than expanding their capital stock. We model costly reversibility by adopting an asymmetric capital adjustment cost function, which takes a quadratic form:

$$\varphi(x_t) = x_t - \frac{\theta_t}{2} \cdot (x_t - x_0)^2,$$

where

$$\theta_t = \theta^+ \cdot \mathbb{I}(x_t \geq x_0) + \theta^- \cdot \mathbb{I}(x_t < x_0)$$

and $\mathbb{I}(\cdot)$ denotes the indicator operator that equals 1 if the condition is satisfied and 0 otherwise. We choose the constant x_0 such that there are no adjustment costs in the non-stochastic steady state, which implies $x_0 = \exp(\bar{\mu}) - 1 + \delta$. The remaining two parameters θ^+ and θ^- satisfy the condition $0 < \theta^+ < \theta^-$ to capture the idea of costly reversibility: the representative firm faces higher capital adjustment costs for the investment decisions leading to the capital stock being below a non-stochastic steady state value.

2.3. Technology

We consider a parsimonious two-state Markov switching model for the productivity growth rate $\Delta a_t = \ln\left(\frac{A_t}{A_{t-1}}\right)$:

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$$

where s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{bmatrix},$$

with $\pi_{ii} \in (0,1)$. We label $s_t = 1$ the "good" regime with high productivity growth and $s_t = 2$ the "bad" regime with low productivity growth. In this paper, we investigate the impact of learning about unknown parameters governing the technology process. Our aim is to quantitatively evaluate the degree of the model improvement due to parameter learning in explaining salient features of the macroeconomic and financial data.

2.4. Asset Prices

In the competitive equilibrium of the economy, the representative household works for the firm and trades its shares to maximize the lifetime utility over a consumption stream. The representative firm chooses labor and capital inputs to maximizes the firm's value, the present value of its future cash flows. The firm's maximization problem implies the following equilibrium conditions for gross return $R_{j,t+1}$:

$$E_t\left[M_{t+1}R_{j,t+1}\right] = 1. \tag{4}$$

In particular, the equation above is satisfied by the investment return, $R_{I,t+1}$:

$$R_{I,t+1} = \frac{1}{Q_t}\left[Q_{t+1}\left(1-\delta+\varphi\left(\frac{I_{t+1}}{K_{t+1}}\right)\right) + \frac{\alpha Y_{t+1} - I_{t+1}}{K_{t+1}}\right] \tag{5}$$

where Q_t is Tobin's marginal Q defined as $Q_t = \frac{1}{\varphi'\left(\frac{I_t}{K_t}\right)}$. The return on investment can be interpreted as the return of an equity claim to the unlevered firm's payouts (Restoy and Rockinger, 1994). As the firm behaves competitively, the labor input is chosen at a level equal to its marginal product:

$$w_t = \partial Y_t/\partial N_t = (1-\alpha)A_t^{1-\alpha}K_t^{\alpha}N_t^{-\alpha} = (1-\alpha)Y_t/N_t. \tag{6}$$

The unlevered firm value, FV_t, is given by $FV_t = Q_t K_{t+1}$, and the firm's unlevered dividends, D_t^u, are defined by:

$$D_t^u = Y_t - w_t N_t - I_t \overset{(6)}{=} \alpha Y_t - I_t. \tag{7}$$

Since the observed aggregate stock market dividends are not directly comparable to the endogenous payouts defined above, we consider pricing levered equity claims.[3] We introduce financial leverage in the spirit of Jermann (1998) by presuming that in each period the firm issues long-term bonds for a fixed fraction of capital and pays the outstanding debt from previous periods. Note that Modigliani and Miller conditions hold in this setting and hence the financial leverage does not change the equilibrium allocations. It only influences the dynamics of a firm's payouts and the way we report the returns on a claim to the firm's dividends. In particular, the financial leverage increases volatility of dividends and makes equity returns more risky.

[3] As noted by other studies, unlevered cash flows and investment returns are not directly observed in reality. Additionally, the equity prices observed on the market are for leveraged corporations, in contrast to unlevered dividend payments of production companies in the model.

Following Jermann (1998), we assume that the firm issues n period discount bonds and pays back its outstanding debt of n period maturity in each period. The fraction ω of the firm's capital K_t at time t is invested in long-term bonds. Denoting the price of the n period discount bonds at time t by $B_{t,n}$, the levered dividends are:

$$D_t = \alpha Y_t - I_t + \omega \left(K_t - K_{t-n}/B_{t-n,n} \right), \tag{8}$$

where the first part, $\alpha Y_t - I_t$, represents the operating cash flow of an unlevered claim, whereas the second part, $\omega \left(K_t - K_{t-n}/B_{t-n,n} \right)$, is the difference between proceeds from newly issued bonds in period t at the price $B_{t,n}$ and repayments of the bonds purchased in period $t-n$ at the price $B_{t-n,n}$.

The prices of the n-period bonds are defined recursively by:

$$B_{t,n} = E_t \left[M_{t+1} B_{t+1,n-1} \right], \tag{9}$$

with the boundary condition $B_{t,0} = 1$ for any t. We denote the price of the levered equity claim by P_t and the levered equity return by $R_{t+1} = (P_{t+1} + D_{t+1})/P_t$. As is well known, one can readily compute the equity price as $P_t = FV_t - DV_t$, where FV_t is a firm's value and DV_t is a net balance of the long-term debt issued over the period from $t-n+1$ to t. The quantities FV_t and DV_t satisfy the conditions:

$$FV_t = Q_t K_{t+1} \quad \text{and} \quad DV_t = \sum_{j=1}^{n} \frac{B_{t,j} \omega K_{t-n+j}}{B_{t-n+j,n}}.$$

3. Calibration

3.1. Parameter Values

Panel A in Table 1 reports the parameter values of an investor's preferences, production and capital adjustment cost functions. The coefficient of relative risk aversion (γ) is equal to 10, an upper bound of the interval considered plausible by Mehra and Prescott (1985). The subjective discount factor (β) is set to 0.995. This value allows the benchmark calibration to match the low risk-free rate in the data. There is no consensus in the literature about the value of the elasticity of inter-temporal substitution. We follow the disaster risk literature (Gourio, 2012) and long-run risks models (Bansal and Yaron, 2004; Ai, Croce and Li, 2013; Bansal, Kiku, Shaliastovich and Yaron, 2014) by setting EIS (ψ) to 2. Consistent with the existing real business-cycle literature, the constant capital share in a Cobb-Douglas production

Table 1
Benchmark Calibration

Parameter	Description	Value
Panel A: Preferences, Production and Capital Adjustment Cost Functions		
β	Discount factor	0.995
γ	Risk aversion	10
ψ	EIS	2
α	Capital share	0.36
δ	Depreciation rate	0.025
θ^{+}	Adjustment cost coefficient	5
θ^{-}/θ^{+}	Asymmetry of adjustment costs	20
Panel B: Markov-switching Model of Productivity Growth		
π_{11}	Transition probability from expansion to expansion	0.961
π_{22}	Transition probability from recession to recession	0.625
μ_1	Productivity growth in expansion	0.52
μ_2	Productivity growth in recession	-1.86
σ	Productivity volatility	1.47

This table reports the parameter values in the benchmark calibration. Panel A presents preferences parameters, values in the production and adjustment costs functions. Panel B shows the maximum likelihood estimates of parameters in a two-state Markov-switching model for productivity growth. We obtain these estimates by applying the expectation maximization algorithm (Hamilton, 1990) to quarterly total factor productivity growth rates from 1947:Q2 to 2016:Q4.

function (α) is 0.36. We set the quarterly depreciation rate (δ) to 0.025, which implies an annual rate of 10% (Favilukis and Lin, 2016).

We further calibrate the adjustment cost coefficient (θ^{+}) and the degree of asymmetry (θ^{-}/θ^{+}). We jointly set values of these parameters by using the estimates from prior studies and by matching key moments in the data. Specifically, we target the volatility of investment and the correlation between consumption and levered dividends. The empirical estimates of θ^{+} vary from 2 to 8 in quarterly frequency. We choose a middle point of this range and set $\theta^{+} = 5$ as in Zhang (2005). There is limited empirical evidence on the degree of asymmetry. In the benchmark calibration, we set $\theta^{-}/\theta^{+} = 20$. These parameter choices allow the model to match the large volatility of investment and, most importantly, to generate procyclical levered dividends of the firm.

Following the methodology of Stock and Watson (1999), we use the macroeconomic data to construct the cumulative Solow residuals. We further scale these residuals by the labor share $(1-\alpha)$ in order to interpret them as labor-augmenting technology. We estimate a two-state Markov switching process of quarterly productivity growth rates by applying the expectation maximization algorithm developed by Hamilton (1990). Panel B in Table 1 reports the maximum likelihood estimates for the transition probabilities (π_{ii}), productivity growth rates (μ_i) as well as the constant volatility (σ). Productivity is estimated to grow at the quarterly rate of about 0.52 percent in expansions and about -1.86 percent in recessions. The productivity volatility comes out around 1.47 percent. The transition probability to the expansion (recession) conditional on being in the expansion (recession) is estimated around 0.961 (0.625). These numbers imply the average duration of the high-growth expansion state of about 25.64 quarters and the average duration of the low-growth recession of about 2.67 quarters. Our maximum likelihood estimates are consistent with the values reported by Hamilton (1989) and Cagetti, Hansen, Sargent and Williams (2002).

In terms of the financial leverage, we assume that the firm issues long-term bonds with a maturity of $n = 60$ quarters. For each model, we choose the leverage parameter (ω) to match the average debt-to-equity ratio of around 1:1 similar to Gourio (2012) and Jahan-Parvar and Liu (2014). The calibrated values of ω across different models are in the interval $[0.75\%, 1.00\%]$. The model-implied leverage ratio is consistent with the empirical estimates of Rauh and Sufi (2010) and Jermann and Quadrini (2012), who document the average total debt to capital ratio of around 50%.

3.2. Parameter Uncertainty and Initial Beliefs

There are five parameters in the productivity growth process of the economy. We employ conjugate priors for each unknown parameter in order to obtain conjugate posteriors via Bayesian updating. If all parameters are assumed to be unknown for the household, we obtain a 10-dimensional vector of state variables including the current regime of the Markov chain, capital stock, time and hyperparameters of prior distributions. In addition to the curse of dimensionality, the numerical solution methodology in the production-based setting requires solving the agent's maximization problem for all combinations of state variables in each period. This makes the model solution especially slow. To mitigate this complexity in the model solution, we investigate the impact of uncertainty about the transition probabilities and mean growth rates, whereas a volatility parameter is assumed to be known.[4] Furthermore,

[4]We motivate our choice of unknown parameters by the results in the consumption-based asset pricing model. Collin-Dufresne, Johannes and Lochstoer (2016) conclude that uncertainty about variance has a negligible effect on asset prices. Meanwhile, learning about transition probabilities has long-lasting asset pricing implications. Mean growth rates are harder to learn than volatilities, though

our analysis assumes homoskedastic volatility of productivity growth, though a large strand of the macroeconomic literature documents the importance of time-varying uncertainty on macroeconomic variables and asset returns. We leave the important investigation of the implications of learning about volatility risk and regime switches in volatility of productivity growth for future research.

Having decided which parameters the household does not know, we compare two approaches to dealing with unknown parameters: anticipated utility and priced parameter uncertainty. Anticipated utility assumes investors learn about unknown parameters over time but in each period they treat their current beliefs as "true" values. In this case, agents ignore the possibility that beliefs will change later. In contrast, priced parameter uncertainty implies economic agents learn about unknown parameters from the data and acknowledge that their current beliefs will be updated in the future. Consequently, agents take into account future belief revisions while calculating utility and asset prices. The forward-looking nature of rationally accounting for parameter uncertainty gives rise to subjective long-run risks.[5] These parameter learning-generated, permanent risks are priced in the economy with recursive utility, where agents have a preference for early resolution of uncertainty.

To quantitatively evaluate the impact of these additional macroeconomic risks, we consider different specifications of the production economy. Specifically, we solve three frameworks: a model with full information about parameters in the productivity growth process and an identical calibration with unknown parameters when the investor applies either priced parameter uncertainty or anticipated utility pricing. This comparison illustrates the contribution of rational parameter learning relative to the cases when the investor has full structural knowledge of the economy or has incomplete information about the true parameters while applying anticipated utility pricing. Furthermore, we study the role of a household's prior knowledge by injecting different training samples into the model. For instance, we assume 100, 150, and 200 years of prior learning before reporting model-generated results. By experimenting with the prior samples, we can evaluate the persistence of the impact of priced parameter uncertainty on the macroeconomic and financial variables.

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the implications are less pronounced compared to learning about unknown transition probabilities.

[5]This is due to the fact that posterior beliefs of parameters are martingales and hence Bayesian learning produces permanent shocks to agents' expectations.

In sum, we compare the performance of the model assuming different information settings. In the case of incomplete investor knowledge, we distinguish between rational parameter learning and more commonly assumed anticipated utility. We use standard, conjugate priors distributions for the unknown parameters: beta and normal distributions for the transition probabilities and mean growth rates, respectively. We center initial beliefs at the true values of unknown parameters estimated from the post-war sample and further fine-tune hyperparameters of prior distributions to embody various samples of prior learning. Calibrating initial beliefs based on the historical data is a more realistic procedure and would likely improve the model's performance due to pessimism induced by the Great Depression and both World Wars. We refrain from doing this to illustrate that our results do not require pessimistic beliefs and are robust to "look-ahead" priors. Finally, we numerically solve the production economy using the methodology outlined in the Appendix.

3.3. Results

Unconditional Moments. Panel A in Table 2 presents the quarterly moments of macroeconomic variables from different models and the data. The data column shows that output is more volatile than consumption and hours worked but less volatile than investment. Also, there is the positive but not perfect correlation between the series. Comparing the empirical moments with the model-generated statistics, all the models with priced parameter uncertainty, anticipated utility, and known parameters explain the empirical moments reasonably well. In general, rational parameter learning increases investment growth volatility, lowers consumption growth volatility, and brings the correlations between the quantities closer to the data. However, parameter learning has quantitatively marginal effects on the macro dynamics.

Panel B in Table 2 shows that priced parameter uncertainty improves significantly the performance of the model in terms of matching financial moments. The last two columns show that the production economy with known parameters, or with unknown parameters but AU pricing, generates a too high average risk-free rate and price-dividend ratio as well as a too low mean and volatility of excess equity returns compared to the data.[6] Columns 3 to 6 shows that rationally taking into account

[6] For the AU case, we report the results only with a prior period of 100 years as the results remain similar across different prior specifications.

Table 2
Sample Moments

	Data	PPU				AU	FI
		100 yrs	150 yrs	200 yrs	∞ yrs	100 yrs	
Panel A: Macroeconomic Quantities							
$E(\Delta c)$	0.47	0.30	0.30	0.30	0.29	0.28	0.29
$\sigma(\Delta c)$	0.72	0.97	0.98	0.97	0.98	0.97	0.99
$ar1(\Delta c)$	0.21	0.14	0.14	0.14	0.14	0.14	0.13
$\sigma(\Delta i)$	2.39	2.53	2.51	2.49	2.34	2.28	2.36
$\sigma(\Delta y)$	1.28	1.21	1.21	1.21	1.19	1.18	1.18
$\sigma(\Delta n)$	0.67	0.55	0.53	0.52	0.48	0.45	0.49
$\rho(\Delta i, \Delta y)$	0.62	0.86	0.86	0.86	0.85	0.86	0.84
$\rho(\Delta c, \Delta y)$	0.45	0.73	0.75	0.77	0.80	0.82	0.79
$\rho(\Delta c, \Delta i)$	0.35	0.32	0.33	0.35	0.39	0.44	0.35
$\rho(\Delta n, \Delta i)$	0.62	0.88	0.87	0.87	0.85	0.86	0.87
$\rho(\Delta n, \Delta y)$	0.68	0.54	0.53	0.53	0.48	0.51	0.47
Panel B: Financial Variables							
Debt/Equity	1.00	1.01	1.01	1.01	1.00	1.00	1.01
$E(R_f) - 1$	0.23	0.20	0.27	0.31	0.42	0.45	0.46
$\sigma(R_f)$	0.40	0.43	0.36	0.32	0.22	0.20	0.20
$E(R - R_f)$	1.59	1.84	1.54	1.40	0.92	0.80	0.83
$\sigma(R - R_f)$	7.75	5.63	5.16	5.07	4.99	4.30	4.59
$SR(R - R_f)$	0.21	0.34	0.31	0.29	0.19	0.19	0.16
$E(pd)$	4.38	4.13	4.29	4.37	4.72	4.78	4.83
$\sigma(pd)$	0.34	0.22	0.23	0.24	0.27	0.27	0.30
$ar1(pd)$	0.96	0.88	0.86	0.85	0.80	0.82	0.77
$E(\Delta d)$	0.49	0.23	0.25	0.27	0.30	0.30	0.31
$\sigma(\Delta d)$	5.25	7.66	9.36	10.01	13.56	14.08	16.14
$ar1(\Delta d)$	0.01	−0.10	−0.10	−0.10	−0.12	−0.12	−0.15
$\rho(\Delta c, \Delta d)$	0.09	0.14	0.15	0.16	0.18	0.13	0.20

This table reports the average sample moments from 1,000 simulations of 279 quarters of the data from the production economy considered in this paper. The historical data moments are reported in the "Data" column and correspond to the U.S. data from 1947:Q2 to 2016:Q4. The "PPU" column refers to the production economy with priced parameter uncertainty, whereas the "AU" column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The "FI" column presents the results of the model in which the parameters are known. $E(x)$, $\sigma(x)$, and $SR(x)$ denote the average sample mean, standard deviation, and Sharpe ratio of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in quarterly terms.

parameter uncertainty in the productivity growth process leads to a lower risk-free rate and price-dividend ratio. The risk premium, the Sharpe ratio of excess equity returns, and the volatility of the risk-free rate are more than two times higher with rational parameter learning, whereas the equity volatility also increases substantially. Overall, the production economy with priced parameter uncertainty and 100 years of prior learning is able to match the first and second moments of interest rates, to generate a large equity premium and around three quarters of its volatility, while capturing large equity Sharpe ratios and low price-dividend ratios.

Furthermore, the financial moments remain amplified in the model with parameter learning and compare well with the data even after 200 years of prior learning. This impact is long-lasting despite the conservative amount of parameter uncertainty in the calibration of the model and its simulations. Indeed, using 200-year prior in 1947 effectively implies that households had access to the productivity data from the beginning of the Industrial Revolution back in 1750s. In reality, however, one would expect a much higher degree of parameter uncertainty due to a shorter sample of the productivity data. Also, there is a considerable amount of uncertainty faced by investors when calibrating prior beliefs, which are set at the true parameter values in our simulations. Finally, the household in our model faces uncertainty about parameters governing business-cycle fluctuations, which are relatively frequent. Augmenting the productivity growth process with a rare state is likely to amplify the impact and persistence of parameter learning.

The bottom of Panel B in Table 2 shows standard moments of levered dividends from the data and different models. According to our calibration strategy, all specifications reasonably capture the positive correlation between consumption and dividends, a hard-to-match moment in the real business-cycle model. Also, the specification with rational parameter learning better captures the volatility of dividends. Intuitively, the impact of investment frictions on levered dividends works as follows. In bad times, it is more difficult for a representative firm to reduce investment, due to higher costs that would lead to a smaller drop in investment compared to the symmetric capital adjustment cost. Thus, net profits after deducting investment appear less countercyclical. With the financial leverage, a firm's dividends are the sum of a firm's profits and the net balance of the long-term debt. The latter is proportional to capital and therefore declines in the recession. The overall sum of the profits and net issuance of the long-term debt results in procyclical dividends.

It is well known that a frictionless production economy cannot capture a large equity premium and equity volatility mainly due to countercyclical dividends of the firm (Kaltenbrunner and Lochstoer, 2010). In this paper, we show that a combination of a financial leverage and asymmetric adjustment cost is able to match the observed procyclical dividends in the data.[7] In the sensitivity analysis, we further evaluate the importance of this feature to warrant a sizable amplification mechanism for productivity shocks by priced parameter uncertainty.

Return Predictability. A large strand of the empirical literature documents that excess returns at an aggregate level can be predicted by variables like the investment-capital ratio (Cochrane, 1991; Bansal and Yaron, 2004), Tobin's Q (Pontiff and Schall, 1998; Lewellen, 2004), the dividend-price ratio (Campbell and Shiller, 1988; Fama and French, 1989), and the consumption-wealth ratio (Lettau and Ludvigson, 2001). The conclusion of the extensive empirical literature is that high dividend yields, high book-to-market, and consumption-wealth ratios predict high future excess returns, whereas high investment rates forecast low future excess returns. Furthermore, the predictive regressions suggest that the slope coefficients (in absolute terms) and R^2's are relatively large and tend to increase over the forecast horizon. These regularities pose a significant challenge for the standard real business-cycle model. In this section, we compare the long-term predictability patterns generated by the production economy with parameter uncertainty (both the PPU and AU cases, which embody 100 years of prior learning) and known parameters to the predictability observed in the post-war data.

Tobin's Q, the investment-capital and consumption-wealth ratios are endogenously specified in our production economy. Furthermore, we follow Epstein and Zin (1989) and calculate the wealth-consumption ratio as:

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{U_t}{C_t}\right)^{1-1/\psi},$$

where the equilibrium allocations of the agent's utility and consumption are endogenously determined. Using these model-generated quantities, we run the aforementioned predictive regressions and report results in Table 3. We find that all models can generate monotonic patterns in the slope coefficients and R^2's over the forecast horizon, but the magnitudes differ across different frameworks.

[7] A number of studies (Uhlig, 2007; Belo, Lin and Bazdresch, 2014; Favilukis and Lin, 2016) introduce wage rigidity in the standard production model in order to generate more volatile and procyclical dividends. This extension of the model can further improve our results and possibly magnify the effect of parameter learning. However, we leave the investigation of the interplay between sticky wages and parameter uncertainty for future research.

Table 3
Return Predictability

h	Data Slope	Data R^2	PPU Slope	PPU R^2	AU Slope	AU R^2	FI Slope	FI R^2
Panel A: Investment-capital ratio (ik)								
1Y	−0.38	0.07	−0.35	0.07	−0.25	0.04	−0.35	0.04
2Y	−0.62	0.12	−0.59	0.12	−0.47	0.08	−0.57	0.07
3Y	−1.27	0.25	−0.78	0.16	−0.65	0.11	−0.75	0.10
4Y	−1.82	0.33	−0.96	0.20	−0.82	0.14	−0.92	0.12
5Y	−2.31	0.37	−1.12	0.24	−0.98	0.16	−1.07	0.14
Panel B: Tobin's Q								
1Y	−0.45	0.08	−0.45	0.10	−0.31	0.06	−0.43	0.09
2Y	−1.57	0.13	−0.72	0.16	−0.54	0.10	−0.65	0.13
3Y	−1.95	0.16	−0.94	0.22	−0.73	0.14	−0.84	0.16
4Y	−2.47	0.19	−1.14	0.27	−0.91	0.17	−1.00	0.19
5Y	−2.75	0.25	−1.32	0.31	−1.06	0.20	−1.14	0.22
Panel C: Dividend-price ratio (dp)								
1Y	0.13	0.09	0.13	0.09	0.04	0.02	0.07	0.04
2Y	0.23	0.17	0.20	0.14	0.06	0.03	0.09	0.04
3Y	0.28	0.20	0.26	0.17	0.07	0.04	0.10	0.05
4Y	0.33	0.23	0.30	0.20	0.09	0.04	0.10	0.05
5Y	0.39	0.27	0.35	0.23	0.10	0.05	0.10	0.06
Panel D: Consumption-wealth ratio (cw)								
1Y	1.51	0.08	3.16	0.09	2.96	0.07	6.29	0.04
2Y	4.41	0.16	4.92	0.14	5.38	0.13	10.46	0.07
3Y	5.25	0.24	6.31	0.18	7.48	0.19	14.02	0.10
4Y	6.47	0.28	7.55	0.22	9.35	0.24	17.24	0.12
5Y	7.63	0.31	8.65	0.25	11.01	0.28	20.06	0.14

This table reports the results of univariate regressions of cumulative excess log equity returns on several valuation and macroeconomic variables over various forecasting horizons (h years; 1 to 5). We use investment-capital ratio, Tobin's Q, dividend-price and consumption-wealth ratios as the right-hand side variable (x_t) in the linear projection:

$$r^{ex}_{t+1\to t+h} = \text{Intercept} + \beta(h) \times x_t + \varepsilon_{t+h},$$

where $r^{ex}_{t+1\to t+h}$ are h-year future excess log equity returns. The empirical statistics are for the U.S. data from 1947:Q2 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. For each model, we simulate 1,000 economies at a quarterly frequency with a sample size equal to the empirical counterpart. We obtain the slope coefficients and R^2's for each simulation and report average sample statistics over all 1,000 artificial series.

Panels A and B show that, in the regressions with investment rates and Tobin's Q, priced parameter uncertainty generates larger R^2's in both cases, while there is no noticeable difference between the slope coefficients across three models. Panel C shows that, in the regression with dividend yields, the model with known parameters generates too small slope coefficients and R^2's at around 0.1 and 5%, respectively. Surprisingly, Bayesian learning in the AU model leads to slightly worse results. In contrast, the model with priced parameter uncertainty displays significant return predictability with the magnitudes of coefficient estimates and R^2's being comparable to the empirical results. Panel D shows that, in the regression with consumption-wealth ratios, both PPU and AU frameworks dominate the model with full information in terms of R^2's, while rational parameter uncertainty better captures the coefficient estimates by producing the lowest slopes among the three models.

In sum, rational pricing of the risks generated by Bayesian learning not only helps match unconditional moments of the macroeconomic and financial variables, but it also helps explain excess return predictability observed in the data. Intuitively, dynamic updating of beliefs about unknown parameters generates the time-variation in the equity risk premium, leading to stronger return predictability in the parameter learning models. Furthermore, rational pricing of parameter uncertainty amplifies the impact of belief revisions on the equilibrium quantities and risk premium, and hence it increases (in absolute terms) the slope coefficients and R^2's compared to anticipating utility pricing. Finally, priced parameter uncertainty has a stronger effect on financial variables than macroeconomic quantities, and therefore there is generally more significant improvement in the regression results using dividend yields and consumption-wealth ratios as predictors.

4. Inspecting the Mechanism

4.1. Capital Adjustment Costs

The benchmark calibration shows that asymmetric adjustment costs coupled with the financial leverage can generate procyclical dynamics of leveraged dividends of the firm. We examine the sensitivity of our results to the choice of the two parameters (θ^+, θ^-) in the capital adjustment cost function. We present the results of the models with different degrees of asymmetry as well as symmetric quadratic adjustment costs. Specifically, we choose three pairs of (θ^+, θ^-) : $(15, 15), (6, 90), (8, 80)$. Also, we present the results of the model with the more common convex capital adjustment costs (Jermann, 1998):

$$\varphi(x) = a_1 + \frac{a_2}{1 - 1/\xi} x^{1-1/\xi}, \tag{10}$$

in which ξ is the elasticity of the investment rate to Tobin's Q and it is equal to 2.5.[8] To put the models considered on a comparable footing, the adjustment cost parameters are chosen to deliver similar investment volatility across different calibrations.

Table 4 presents the results of a sensitivity analysis. For convenience, we report summary statistics of the PPU and AU models based on a 100-year prior period. Several observations are noteworthy. First, smaller degree of asymmetry of adjustment costs reduces a positive correlation between consumption and levered dividends, as expected. Consequently, the average equity premium and excess return volatility are smaller compared to the benchmark calibration, though priced parameter uncertainty still provides a comparable amplification of moments. Second, shutting down costly reversibility by considering symmetric adjustment costs leads to strongly countercyclical firm's dividends, and therefore the impact of rational pricing of posterior beliefs becomes negligible. Finally, the model with convex adjustment costs provide similar results by generating countercyclical firm's cash flows, too small risk premium and its volatility, too high risk-free rate and price-dividend ratio.

4.2. Priced Parameter Uncertainty with Jermann Adjustment Costs

The previous subsection demonstrates that convex adjustment costs of Jermann (1998) result in countercyclical levered firm's dividends. Although dividends remain quite volatile in the economy, the wrong comovement between dividends and consumption significantly undermines the model-implied moments of financial variables. This subsection shows that the amplification mechanism provided by priced parameter uncertainty is robust to Jermann adjustment costs once a firm's dividends become procyclical. For illustrative purposes, we consider a simplified version of the benchmark model with the fixed amount of labor hours ($N_t = 1$) and convex adjustment costs ($\xi = 4$). Also, instead of adding additional ingredients, we directly

[8] We follow Boldrin, Christiano and Fisher (2001) and choose the constants a_1 and a_2 such that there are no adjustment costs in the non-stochastic steady state:

$$a_1 = \frac{1}{\xi - 1}\left(1 - \delta - \exp(\bar{\mu})\right), \quad a_2 = \left(\exp(\bar{\mu}) - 1 + \delta\right),$$

where $\bar{\mu}$ is the unconditional mean of μ_{s_t}.

Table 4
Sensitivity Analysis: Sample Moments

	Data	$\theta^+ = 15$, $\theta^- = 15$		$\theta^+ = 6$, $\theta^- = 90$		$\theta^+ = 8$, $\theta^- = 80$		Convex $\xi = 2.5$	
		PPU	AU	PPU	AU	PPU	AU	PPU	AU
Panel A: Macroeconomic Quantities									
$E(\Delta c)$	0.47	0.30	0.31	0.30	0.29	0.30	0.30	0.30	0.31
$\sigma(\Delta c)$	0.72	0.93	0.89	0.96	0.96	0.95	0.95	0.87	0.88
$ar1(\Delta c)$	0.21	0.08	0.14	0.14	0.14	0.14	0.14	0.09	0.14
$\sigma(\Delta i)$	2.39	2.04	1.80	2.30	2.24	2.05	2.07	2.16	1.99
$\sigma(\Delta y)$	1.28	1.18	1.17	1.20	1.18	1.17	1.17	1.22	1.19
$\sigma(\Delta n)$	0.67	0.38	0.22	0.46	0.44	0.38	0.37	0.39	0.28
$\rho(\Delta i, \Delta y)$	0.62	0.92	0.99	0.88	0.87	0.90	0.89	0.94	0.97
$\rho(\Delta c, \Delta y)$	0.45	0.88	0.99	0.82	0.84	0.88	0.88	0.90	0.96
$\rho(\Delta c, \Delta i)$	0.35	0.62	0.95	0.46	0.48	0.59	0.58	0.70	0.87
$\rho(\Delta n, \Delta i)$	0.62	0.90	0.97	0.87	0.87	0.86	0.86	0.93	0.95
$\rho(\Delta n, \Delta y)$	0.68	0.65	0.92	0.55	0.53	0.55	0.55	0.76	0.85
Panel B: Financial Variables									
Debt/Equity	1.00	1.03	1.01	0.99	1.00	0.98	1.00	0.99	0.95
$E(R_f) - 1$	0.23	0.31	0.46	0.20	0.45	0.20	0.46	0.31	0.46
$\sigma(R_f)$	0.40	0.33	0.20	0.43	0.20	0.43	0.20	0.33	0.20
$E(R - R_f)$	1.59	0.58	0.38	1.71	0.78	1.60	0.74	0.64	0.38
$\sigma(R - R_f)$	7.75	2.24	1.99	5.14	4.08	4.80	3.96	2.41	2.11
$SR(R - R_f)$	0.21	0.26	0.20	0.34	0.20	0.34	0.19	0.27	0.19
$E(pd)$	4.38	5.14	5.32	4.20	4.80	4.25	4.83	5.07	5.34
$\sigma(pd)$	0.34	0.35	0.32	0.21	0.27	0.20	0.25	0.36	0.34
$ar1(pd)$	0.96	0.92	0.95	0.89	0.84	0.90	0.87	0.93	0.94
$E(\Delta d)$	0.49	0.17	0.27	0.23	0.30	0.23	0.31	0.19	0.25
$\sigma(\Delta d)$	5.25	12.44	8.04	7.09	13.20	5.95	9.98	11.99	9.48
$ar1(\Delta d)$	0.01	−0.11	0.02	−0.08	−0.11	−0.08	−0.08	−0.06	−0.00
$\rho(\Delta c, \Delta d)$	0.09	−0.17	−0.64	0.07	0.08	−0.02	0.00	−0.31	−0.57

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for correlations and autocorrelations expressed in quarterly terms.

calibrate and price an exogenous dividend process to match salient moments of stock market dividends.

Following Bansal and Yaron (2004), we price a levered consumption claim with a leverage factor λ. We formally define quarterly log dividend growth as follows:

$$\Delta d_t^M = g_d + \lambda \Delta c_t + \sigma_d \varepsilon_t^d, \tag{11}$$

Table 5
Sensitivity Analysis : Return Predictability

	Data		$\theta^+ = 15$, $\theta^- = 15$				Convex $\xi = 2.5$			
			PPU		AU		PPU		AU	
h	Slope	R^2	Slope	R^2	Slope	R^2	Slope	R^2	Slope	R^2
Panel A: Investment-capital ratio (ik)										
1Y	−0.38	0.07	−0.09	0.04	−0.08	0.03	−0.10	0.05	−0.09	0.04
2Y	−0.62	0.12	−0.15	0.06	−0.14	0.05	−0.18	0.09	−0.16	0.06
3Y	−1.27	0.25	−0.21	0.09	−0.19	0.07	−0.25	0.11	−0.21	0.08
4Y	−1.82	0.33	−0.27	0.11	−0.24	0.09	−0.31	0.15	−0.26	0.10
5Y	−2.31	0.37	−0.33	0.14	−0.30	0.10	−0.37	0.17	−0.31	0.12
Panel B: Tobin's Q										
1Y	−0.45	0.08	−0.21	0.04	−0.18	0.03	−0.26	0.05	−0.22	0.03
2Y	−1.57	0.13	−0.37	0.06	−0.33	0.05	−0.46	0.08	−0.39	0.06
3Y	−1.95	0.16	−0.52	0.09	−0.45	0.06	−0.62	0.11	−0.52	0.08
4Y	−2.47	0.19	−0.67	0.11	−0.58	0.08	−0.79	0.14	−0.65	0.10
5Y	−2.75	0.25	−0.82	0.14	−0.71	0.10	−0.95	0.17	−0.78	0.12
Panel C: Dividend-price ratio (dp)										
1Y	0.13	0.09	0.01	0.02	0.01	0.01	0.02	0.04	0.01	0.02
2Y	0.23	0.17	0.02	0.04	0.02	0.02	0.03	0.06	0.02	0.03
3Y	0.28	0.20	0.03	0.05	0.02	0.03	0.04	0.07	0.03	0.04
4Y	0.33	0.23	0.04	0.06	0.02	0.04	0.05	0.09	0.03	0.05
5Y	0.39	0.27	0.05	0.08	0.03	0.05	0.06	0.11	0.03	0.06
Panel D: Consumption-wealth ratio (cw)										
1Y	1.51	0.08	0.98	0.04	1.11	0.03	1.17	0.05	1.29	0.04
2Y	4.41	0.16	1.67	0.06	2.03	0.06	2.02	0.08	2.29	0.08
3Y	5.25	0.24	2.18	0.08	2.78	0.09	2.65	0.10	3.07	0.10
4Y	6.47	0.28	2.74	0.10	3.53	0.11	3.25	0.12	3.82	0.13
5Y	7.63	0.31	3.33	0.12	4.27	0.13	3.85	0.15	4.53	0.15

This table reports univariate regressions of cumulative excess log equity returns on several valuation and macroeconomic variables over various forecasting horizons (h years; 1 to 5). We use investment-capital ratio, Tobin's Q, dividend-price and consumption-wealth ratios as the right-hand side variable (x_t) in the linear projection:

$$r^{ex}_{t+1\to t+h} = \text{Intercept} + \beta(h) \times x_t + \varepsilon_{t+h},$$

where $r^{ex}_{t+1\to t+h}$ are h-year future excess log equity returns. The empirical statistics are for the U.S. data from 1947:Q2 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. For each model, we simulate 1,000 economies at a quarterly frequency with a sample size equal to the empirical counterpart. We obtain the slope coefficients and R^2's for each simulation and report average sample statistics over all 1,000 artificial series.

where $\varepsilon^d_t \overset{\text{iid}}{\sim} N(0,1)$, g_d and σ_d are the dividend growth rate and volatility, respectively. We calibrate the parameters g_d, σ_d, and λ to make model implied statistics of dividend growth consistent with the historical data. We set the mean adjustment $g_d = -0.50$ and the idiosyncratic dividend volatility $\sigma_d = 6.5$ to match the observed quarterly mean growth (0.49 percent) and volatility (5.25 percent) of dividends for the considered period. The leverage parameter (λ) is equal to 3.5, a midpoint of the range from 2.5 to 4.5 used in other studies.

Let R^M_{t+1} denote the return on a claim delivering stochastic dividends given by (11). Then:

$$R^M_{t+1} = \frac{P^M_{t+1} + D^M_{t+1}}{P^M_t} = \frac{P^M_{t+1}/D^M_{t+1} + 1}{P^M_t/D^M_t} \cdot \frac{D^M_{t+1}}{D^M_t}.$$

Substituting this expression into the equilibrium condition (4), the price-dividend ratio of a claim on the aggregate stock market dividends satisfies the equation:

$$\frac{P^M_t}{D^M_t} = E_t\left[M_{t+1}\left(1 + \frac{P^M_{t+1}}{D^M_{t+1}}\right)\frac{D^M_{t+1}}{D^M_t}\right] \tag{12}$$

Unconditional Moments. Now we take a closer look at the equity claim paying stochastic dividends as a leverage on consumption similarly to Bansal and Yaron (2004). The numerical methods used to solve for the equilibrium price-dividend ratio are presented in the Appendix. Table 6 shows the model-implied statistics of dividend growth, excess equity returns, the Sharpe ratio and the price-dividend ratio.

The calibrated dividends closely replicate the empirical first and second moments as well as a positive correlation between dividends and consumption observed in the data. Our conservative choice of a leverage parameter produces a slightly higher correlation between dividend and consumption growth rates, but it is crucial that the correlation remains positive in all models. Turning to equity moments, parameter uncertainty with AU pricing produces similar results to the production model with known parameters. Relative to the FI and AU cases, a priced parameter uncertainty approach significantly improves the fit of the model with the data. The model with parameter uncertainty and a prior sample of learning of 100 years match the sample equity premium, its volatility, the equity Sharpe ratio and the level of the price-dividend ratio well. Furthermore, the volatility of the price-dividend ratio comes out two to three times its value with fixed parameters, though it still remains lower than in the data. In the data, the log price-dividend ratio is highly persistent and the model with parameter learning reconciles this feature. Furthermore, looking

Table 6
Calibrated Stock Market Dividend Claim

	Data	PPU				AU	FI
		100 yrs	150 yrs	200 yrs	∞ yrs	100 yrs	
$E(R_f) - 1$	0.23	0.20	0.26	0.30	0.42	0.43	0.44
$\sigma(R_f)$	0.40	0.37	0.34	0.29	0.20	0.20	0.18
$E(R^M - R_f)$	1.59	1.68	1.49	1.26	0.83	0.75	0.72
$\sigma(R^M - R_f)$	7.75	7.98	7.73	7.55	7.45	7.40	7.25
$SR(R^M - R_f)$	0.21	0.22	0.20	0.18	0.10	0.08	0.09
$E(pd^M)$	4.38	4.33	4.44	4.56	5.05	5.11	5.14
$\sigma(pd^M)$	0.34	0.07	0.05	0.04	0.03	0.05	0.02
$ar1(pd^M)$	0.96	0.90	0.87	0.84	0.79	0.90	0.76
$E(\Delta d^M)$	0.49	0.49	0.47	0.45	0.42	0.40	0.40
$\sigma(\Delta d^M)$	5.25	5.68	5.70	5.71	5.71	5.71	5.72
$ar1(\Delta d^M)$	0.01	0.01	0.01	0.01	0.01	0.01	0.01
$\rho(\Delta c, \Delta d^M)$	0.09	0.32	0.32	0.32	0.32	0.32	0.32

This table reports the average moments from 1,000 simulations of 260 quarters of the data from the production economy considered in this paper, where the transition probabilities and mean growth rates are assumed to be unknown. As in Bansal and Yaron (2004), equity is a claim to an exogenous dividend stream. The historical data moments are reported in the data column and correspond to the U.S. data from 1947:Q2 to 2016:Q4. The PPU column refers to the production economy with rational pricing of parameter uncertainty, whereas the AU column refers to the production economy with AU pricing. In both cases, parameter uncertainty includes unknown transition probabilities and mean growth rates. The FI column presents the results of the full information case where the parameters are known. $E(x)$ and $\sigma(x)$ denote the average sample mean and standard deviations of x, respectively. $ar1(x)$ and $\rho(x, y)$ denote the average sample autocorrelation of x and correlation between x and y, respectively. All statistics are expressed in annualized terms, except for correlations and autocorrelations expressed in quarterly terms.

at the results based on different training samples, one can see that Bayesian learning and rational pricing of an investor's subjective beliefs generates permanent shocks in the production economy.

Conditional Dynamics. To better understand of the source of the model improvement, we focus on the conditional dynamics of the risk premium and key prices in response to a regime switch in mean productivity growth. For a better illustration, the results are presented in the economy with unknown transition probabilities. In particular, we consider three typical recessions lasting for 1 quarter, 3 quarters and 2 years. The economy is assumed to grow at the mean growth μ_1 and μ_2 in each state. Before the economy enters the recession, the representative investor holds unbiased beliefs about the uncertain parameters (the transition probabilities π_{11} and π_{22}) assuming a 100-year prior period. We feed these simulated paths of beliefs and productivity growth series into the model and calculate the equilibrium quantities as described in Appendix C.

The top panels in Figure 1 show the mean beliefs about the transition probabilities. Upon the onset of the recession, the mean belief about staying in the good regime falls sharply and stays at the same level during the recession. Once the economy returns back to the high growth state, the investor gradually updates his beliefs about π_{11} upward. In contrast, learning about π_{22} happens only in the recession. The random durations of 1 quarter, 3 quarters and 2 years correspond to the realization of a short economic decline, an average recession and a long downturn, respectively. When the agent experiences an average duration of the recession, his belief about π_{22} increases but then returns back to the initial value. The mean belief about π_{22} remains permanently lower (higher) relative to the initial belief in the case of the recession that is shorter (longer) than the average downturn.

Figure 1 further plots the responses of several key variables to a bad state realization, that lasts for 1 quarter, 3 quarters and 2 years. The sharp decline in beliefs about the probability of staying in the good state leads to a reduction in the interest rate, a decline in the price-dividend ratio as well as an increase in the risk premium and equity volatility. As long as the economy stays in the low productivity growth regime, the agent learns about the persistence of the bad state by revising his beliefs upward. During this period, the interest rates are low, the price-dividend ratio keeps declining, while the equity Sharpe ratio, the conditional equity premium and volatility remain elevated. Although both AU and PPU pricing predict similar paths of financial variables in response to a negative long-run risk shock to the expected productivity growth, the magnitude of their responses is substantially different.

For the anticipated utility case, one can observe very moderate responses in the returns, prices and conditional moments upon the onset of the bad state. Before the regime switch, both the conditional equity premium and the conditional Sharpe ratio are too low relative to the data and then they approximately double in response to the negative shock. Meanwhile, the conditional equity volatility increases only marginally in this case. In contrast, rationally priced parameter uncertainty predicts around 6-fold and 3-fold increases in the conditional risk premium and the equity Sharpe ratio, respectively. The equity volatility turns out to be highly countercyclical

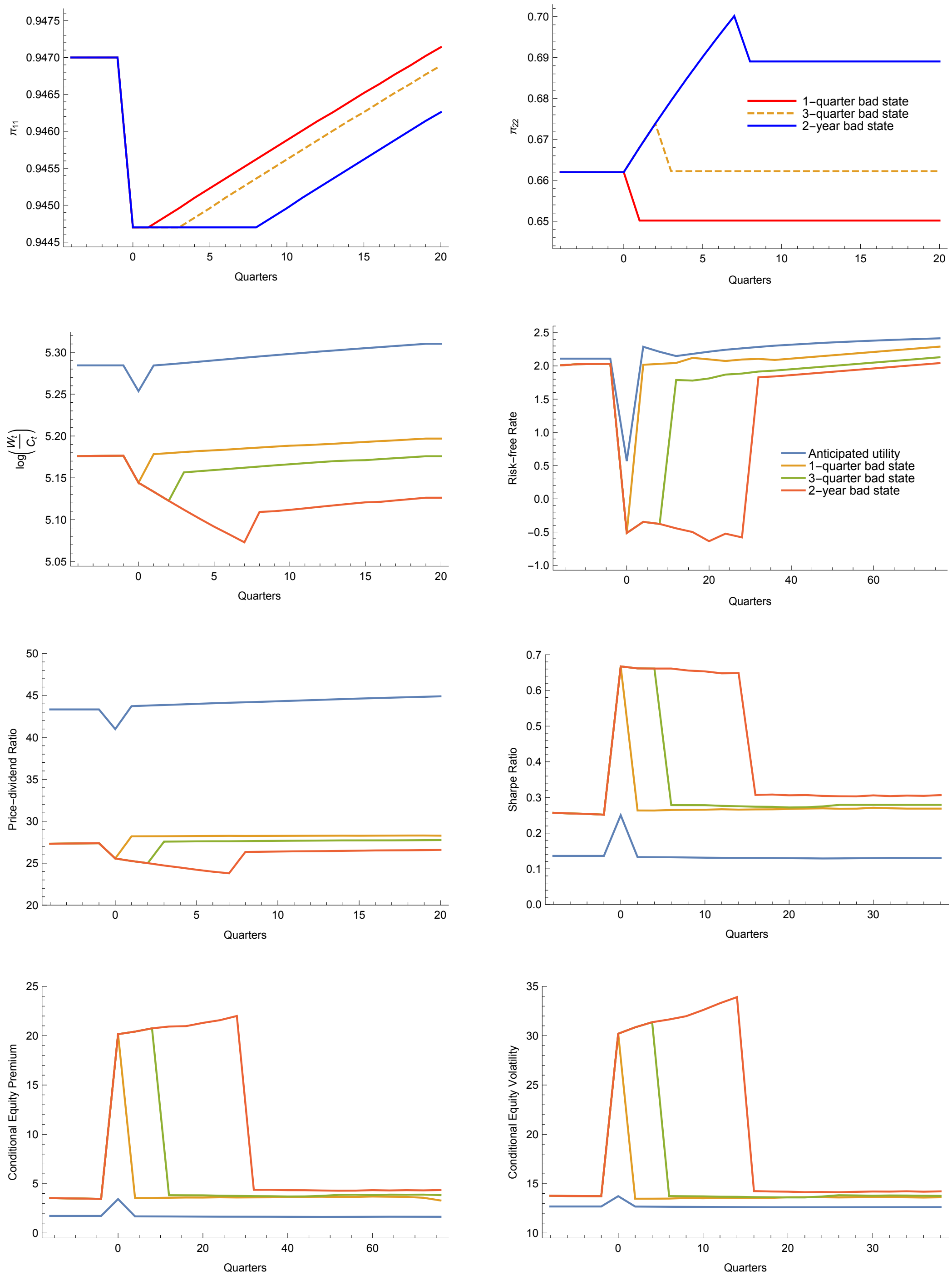

Figure 1: Conditional Prices and Moments. This figure shows the conditional risk-free rate, the price-dividend and equity Sharpe ratios, as well as the conditional equity premium and its volatility. The simulated variables are impulse response functions to the realization of a bad state of 1 quarter, 3 quarters and 2 years in the production economy, considered in this paper for the case of a 100-year prior. The economy is assumed to stay in the high productivity growth steady-state for a long period, and the representative agent holds unbiased initial mean beliefs. We report the conditional dynamics of the variables for the AU and PPU cases. For the sake of a convenient exposition, the former one includes only the responses to a 1-quarter bad state realization. The Appendix describes the numerical approach used.

as it increases by a factor of about 2.5. The interest rate drops more in bad times with parameter uncertainty, while the realized equity returns are more volatile. The price-dividend ratio experiences about the same percentage decline upon the realization of the low productivity growth state in both cases. However, the level of the price-dividend ratio is substantially higher with AU, while parameter learning generates reasonable levels of the price-dividend ratio.

In sum, rational pricing of parameter uncertainty is able to reproduce salient features of the risk-free rate, the price-dividend ratio and equity returns in the economy with convex adjustment costs as long as the dividends are procyclical.

5. Conclusion

We show that introducing parameter uncertainty into an otherwise standard real business cycle framework improves the model's ability to match salient moments of macroeconomic and asset return data. Combined with investment frictions in the form of costly reversibility, parameter uncertainty gives rise to additional macroeconomic risks that help to capture the low mean and volatility of the risk-free rate, the large equity premium and around three quarters of excess return volatility, the large equity Sharpe ratio and the level of the price-dividend ratio, while matching investment volatility and other moments of macroeconomic quantities. Furthermore, time-varying posterior beliefs about unknown parameters reproduce the long-horizon predictability of excess returns by macroeconomic and valuation variables as observed in the data. The asset pricing implications of subjective long-run risks crucially depend on the introduction of a procyclical dividend process consistent with the data.

Future research may consider extending our mechanism to a richer model with sticky prices and financial frictions. In particular, modeling wage rigidity in the spirit of Favilukis and Lin (2016) can help endogenously generate procyclical dividend growth in the model. The interaction between sticky prices and learning effects may have additional interesting implications for the labor market. Motivated by a large strand of the literature on time-varying macroeconomic uncertainty, it is interesting and straightforward to extend our methodology to learning about volatility risks. This might have additional asset pricing implications, especially for volatility sensitive assets, as well as interesting effects for the real economy.

Appendix

A. Numerical Algorithm: Anticipated Utility

In the AU case, the representative household learns about the unknown parameters by updating his beliefs upon the realization of new data, but ignores parameter uncertainty when making decisions. Thus, although the beliefs vary over time, the household centers the "true" parameters at the current posterior means and keeps these subjective estimates constant while solving for the continuation utility and equilibrium asset prices in each period.

In this paper, we focus on two learning about parameters economies with unknown transition probabilities, and unknown transition probabilities and mean growth rates.[9] The numerical solution for both models under AU pricing simplifies to solving for the equilibrium pricing ratios when all parameters are actually known by the household. We find the solution of these simplest economies on a dense grid for unknown parameters (that is, unknown transition probabilities in the former model; unknown transition probabilities and mean growth rates in the latter model). Then the household uses these equilibrium pricing functions for the decision making and asset pricing based on the current beliefs.

A.1. All Known Parameters

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma\varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} N(0,1),$$

where s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

Here, we give details on how the continuation utility is computed for the economy with all parameters known. We define the following stationary variables:

$$\{\tilde{C}_t, \tilde{I}_t, \tilde{Y}_t, \tilde{K}_t, \tilde{U}_t \quad = \left\{ \frac{C_t}{A_t}, \frac{I_t}{A_t}, \frac{Y_t}{A_t}, \frac{K_t}{A_t}, \frac{U_t}{A_t} \right\}.$$

[9] The methodology for the AU case (as well as the priced parameter uncertainty case in Appendix B) can be further extended for learning about the volatility of productivity growth. However, we leave this investigation for the future research.

The household's problem is:

$$\tilde{U}_t = \max_{\tilde{C}_t, \tilde{I}_t, N_t} \left\{ (1-\beta)\tilde{V}_t^{1-\frac{1}{\psi}} + \beta \quad E_t \left[\tilde{U}_{t+1}^{1-\gamma} \cdot \left(\frac{A_{t+1}}{A_t} \right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}} \tag{A1}$$

subject to the constraints:

$$\tilde{V}_t = \tilde{C}_t (1-N_t)^{\nu} \tag{A2}$$

$$\tilde{C}_t + \tilde{I}_t = \tilde{K}_t^{\alpha} N_t^{1-\alpha} \tag{A3}$$

$$e^{\Delta a_{t+1}} \tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \varphi \left(\frac{\tilde{I}_t}{\tilde{K}_t} \right) \tilde{K}_t \tag{A4}$$

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0,1). \tag{A5}$$

Since all parameters are assumed known, s_t and $\tilde{K}_t$ are the only state variables in the economy. Ultimately, the recursive equation (A1) can be rewritten as:

$$\tilde{U}_t(s_t, \tilde{K}_t) \tag{A6}$$

$$= \max_{\tilde{C}_t, \tilde{I}_t, N_t} \left\{ (1-\beta)\tilde{V}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1} \left(s_{t+1}, \tilde{K}_{t+1} \right)^{1-\gamma} \cdot e^{(1-\gamma)\Delta a_{t+1}} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}$$

To solve the recursion (A6), we use the the value function iteration algorithm. In particular, the numerical algorithm proceeds as follows:

1. We find the de-trended steady state capital $\tilde{K}_{ss}$, assuming the productivity growth equals the steady state level predicted by a Markov-switching model. The state space for capital normalized by technology is set at $[0.5\tilde{K}_{ss}, 3.5\tilde{K}_{ss}]$. We further use $n_k = 50$ points on a grid for capital in the numerical computation. A denser grid does not lead to significantly different results.

2. For any level of capital $\tilde{K}_t$ at time t, we construct a grid for $\tilde{I}_t$ with uniformly distributed points between 0 and $\tilde{K}_t^{\alpha}$. Specifically, we use $n_i = 100$ points.

3. For each pair $(\tilde{K}_t, \tilde{I}_t)$ on the grids for $\tilde{K}_t$, and $\tilde{I}_t$, we find $N_t = N_t(\tilde{K}_t, \tilde{I}_t)$ that solves

$$\max_{N_t} (\tilde{K}^{\alpha} N^{1-\alpha} - \tilde{I}_t)(1-N_t)^{\nu}$$

4. For the expectation, we use the Gauss-Hermite quadrature with $n_{gh} = 8$ points. Using the quadrature weights and nodes, we can calculate the expression on the right hand side.

5. We solve the optimization problem in the Bellman equation (A6) subject to the constraints (A2)-(A5) and update a new value function $\tilde{U}_t = \tilde{U}_t(s_t, \tilde{K}_t)$ given an old one $\tilde{U}_{t+1} = \tilde{U}_{t+1}(s_{t+1}, \tilde{K}_{t+1})$. Note that the optimal labor hours N_t corresponding to a given capital $\tilde{K}_t$ and investment $\tilde{I}_t$ are taken from Step 3.

6. We iterate Steps 2-5 by updating the continuation utility on each iteration until a suitable convergence is achieved. Specifically, the stopping rule is that the distance between the new and old value functions satisfies $|\tilde{U}_{t+1} - \tilde{U}_t| / |\tilde{U}_t| < 10^{-12}$.

B. Numerical Algorithm: Priced Parameter Uncertainty

The numerical solution for the case of priced parameter uncertainty consists of two steps.[10] First, we solve for the equilibrium pricing ratios when true parameters are actually known by the household (by assumption, these are learned at $T = \infty$). We find the solution of this simplest limiting economy on a dense grid of state variables. Second, we use the known parameters boundary economies as terminal values in the backward recursion to obtain the equilibrium function at time t. For the first step, Appendix A outlines details of the numerical algorithm for all known parameters. In this section, we present details of the solution methodology employed at the second step for two models with unknown transition probabilities, and unknown transition probabilities and mean growth rates.

B.1. Unknown Transition Probabilities

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, 1),$$

[10] Johnson (2007) uses this solution methodology in a case with parameter learning and power utility. Collin-Dufresne, Johannes and Lochstoer (2016) apply this approach to the case of Epstein-Zin utility in the endowment economy. We further extend the numerical solution to the production-based setting with Epstein-Zin preferences of the representative household.

where s_t is a two state Markov chain with a transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

with $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

In the case of unknown transition probabilities, the representative household knows true values of the parameters within each state (μ_1, μ_2, σ) and observes states (s_t) but does not know the transition probabilities (π_{11}, π_{22}). At time $t = 0$, the household holds priors about uncertain probabilities in the transition matrix and updates beliefs each period upon realization of new series and regimes.

We assume a Beta distributed prior for unknown transition probabilities and, hence, posterior beliefs are also Beta distributed. The Beta distribution has the probability density function of the form:

$$p(\pi|a,b) = \frac{\pi^{a-1}(1-\pi)^{b-1}}{B(a,b)},$$

where $B(a,b)$ is the Beta function (a normalization constant), a and b are two positive shape parameters. We are particularly interested in the expected value of the Beta distribution defined by:

$$E[\pi|a,b] = \frac{a}{a+b}.$$

We use two pairs of hyperparameters parameters (a_1, b_1) and (a_2, b_2) for unknown transition probabilities π_{11} and π_{22}, respectively. At time t, the household uses Bayes' rule and the fact that states are observable to update hyperparameters for each state i as follows:

$$a_{i,t} = a_{i,0} + \#(\text{state } i \text{ has been followed by state } i), \tag{B7}$$

$$b_{i,t} = b_{i,0} + \#(\text{state } i \text{ has been followed by state } j), \tag{B8}$$

given the initial prior beliefs $a_{i,0}$ and $b_{i,0}$.

Once we find the limiting boundary economies on the first step, we perform a backward recursion using the following state variables:

$$\tau_{1,t} = a_{1,t} - a_{1,0} + b_{1,t} - b_{1,0} \tag{B9}$$

$$\lambda_{1,t} = E_t[\pi_{11}] = \frac{a_{1,t}}{a_{1,t} + b_{1,t}} \tag{B10}$$

$$\tau_{2,t} = a_{2,t} - a_{2,0} + b_{2,t} - b_{2,0} \tag{B11}$$

$$\lambda_{2,t} = E_t[\pi_{22}] = \frac{a_{2,t}}{a_{2,t} + b_{2,t}} \tag{B12}$$

To define the equilibrium recursion, note that $X_t = \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ are sufficient statistics for the agent's priors. We can update X_{t+1} using the equations (B7)-(B12), the next period regime, and sufficient statistics:

$$X_{t+1} = f(s_{t+1}, s_t, X_t).$$

For notational purposes, it might be useful to denote $X_t^s \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ and $X_t^{\Delta a} \equiv \{\tilde{K}_t\}$, where the superscripts s and Δa indicate that variables in the vectors X_t^s and $X_t^{\Delta a}$ are a function of the observed state realization s_t and a function of the realized productivity growth, respectively. Denote $X_t = \left[X_t^s, X_t^{\Delta a}\right]$ Using these notations, we can rewrite

$$\tilde{U}_{t+1}(s_{t+1}, X_{t+1}) = \tilde{U}_{t+1}(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a})$$

to better indicate the dependence of state variables on specific shocks. Ultimately, the recursive equation (A1) can be rewritten as:

$$\tilde{U}_t(s_t, X_t) \tag{B13}$$

$$= \max_{\tilde{C}_t, \tilde{I}_t, N_t} \left\{ \left((1-\beta)\tilde{V}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a} \right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \;\; s_t, X_t \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right)^{\frac{1}{1-\psi}} \right\},$$

where the expectation on the right hand side is equivalent to:

$$
\begin{aligned}
& E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \,\Big|\, s_t, X_t\right] \\
= \; & E_t\left[E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \,\Big|\, s_{t+1}, s_t, X_t\right] \,\Big|\, s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2} \mathbb{P}(s_{t+1}|s_t, X_t^s) \quad \dots \\
\times \; & E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \,\Big|\, s_{t+1}, s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t}|s_t, X_t^s) \quad \dots \\
\times \; & E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \,\Big|\, s_{t+1}, s_t, X_t\right]
\end{aligned} \tag{B14}
$$

where the first and second equalities follow from the independency of the regime changes and the Gaussian shocks to productivity growth (s_{t+1} and ε_{t+1}). Let the conditional density of π_{s_{t+1},s_t} be $g(\pi_{s_{t+1},s_t}|s_t, X_t^s)$, then the third equality follows from:

$$
\mathbb{P}(s_{t+1}|s_t, X_t^s) = \int_0^1 \pi_{s_{t+1},s_t} g(\pi_{s_{t+1},s_t}|s_t, X_t) d\pi_{s_{t+1},s_t} = E_t(\pi_{s_{t+1},s_t}|s_t, X_t)
$$

Furthermore, using the definition of our state variables, this last conditional expectation equals $\lambda_{s_t,t}$ or $1 - \lambda_{s_t,t}$.

Note that before choosing the optimal consumption, investment and labor hours in (B13), we need to solve numerically first the inner expectation, which is equivalently represented by (B14). Hopefully, we have an analytical expression for the conditional expectation of transition probabilities in (B14), which is either $\lambda_{s_t,t}$ or $1 - \lambda_{s_t,t}$. For the second conditional expectation in (B14), we do not have a closed form since the continuation utility depends on the realized productivity growth through $\tilde{K}_{t+1}$. Therefore, we use quadrature-type numerical methods to evaluate this expectation as follows:

$$
\begin{aligned}
& E_t\left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \,\Big|\, s_{t+1}, s_t, X_t\right] \\
\approx \; & \sum_{j=1}^{J} \omega_\varepsilon(j) \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a(j), X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a(j)} \,\Big|\, s_{t+1}, s_t, X_t\right]
\end{aligned} \tag{B15}
$$

where $\omega_\varepsilon(j)$ is the quadrature weight corresponding to the quadrature node $n_\varepsilon(j)$ used for the integration of a standard normal shock ε_{t+1} in productivity growth. The observed realized productivity growth, $\Delta a(j)$, and a state variable, $X_{t+1}^{\Delta a}(j) = \tilde{K}_{t+1}(j)$, are updated as follows:

$$\Delta a(j) = \mu_{s_{t+1}} + \sigma \cdot n_{\varepsilon}(j) \tag{B16}$$

$$e^{\Delta a(j)} \tilde{K}_{t+1}(j) = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t, \tag{B17}$$

where

$$\tilde{I}_t = \tilde{K}_t^{\alpha} N_t^{1-\alpha} - \tilde{C}_t. \tag{B18}$$

Finally, the numerical backward recursion can be performed by using (B13)-(B18). The boundary conditions are defined by the limiting economies $\tau_{1,\infty}$ and $\tau_{2,\infty}$, where the transition probabilities π_{11} and π_{22} are known.

B.1.1. Solving for a Dividend Claim

We also solve for the price-dividend ratio of the equity claim written on aggregate dividends, which are defined as a leverage to aggregate consumption. Let exogenous aggregate dividends be given by:

$$\Delta d_{t+1} = g_d + \lambda \Delta c_{t+1} + \sigma_d \varepsilon_{d,t+1},$$

where $g_d = \left(1-\lambda\right)\left(E(\mathbb{P}(s_\infty = 1|\pi_{11},\pi_{22}))\mu_1 + E(\mathbb{P}(s_\infty = 2|\pi_{11},\pi_{22}))\mu_2\right)$ and $\mathbb{P}(s_\infty = i|\pi_{11},\pi_{22})$ is the ergodic probability of being in state i conditional on the transition probabilities π_{11} and π_{22}. Note that the long run mean of dividends growth, g_d, is changing under the household's filtration, though the true long run growth is constant. The subjective beliefs about the true parameter values induce fluctuations in g_d, which can be expressed as $g_d = g_d(s_{t+1}, s_t, X_t)$.

The equilibrium condition for the price-dividend ratio is standard in the Epstein-Zin economy and is given by:

$$PD_t = E_t\left[\beta\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{-\frac{1}{\psi}}\left(\frac{A_{t+1}}{A_t}\right)^{-\frac{1}{\psi}}\left(\frac{\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)\right)}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{D_{t+1}}{D_t}\right)\left(PD_{t+1}+1\right)\right] \tag{B19}$$

Similarly to the solution for the value function, we rewrite all variables in the recursion (B19) as a function of the state variables and further use quadrature-type numerical methods to evaluate expectations on the right hand side of (B19). Additionally, we update the long run dividends growth, $g_d(s_{t+1}, s_t, X_t)$, which is in fact random. Consequently, the equilibrium recursion used to solve the model is then:

$$
\begin{aligned}
& PD_t(s_t, X_t,) \\
= \;& E_t\left[\left(\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \dots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1},s_t,X_t^s,\Delta a_{t+1},X_t^{\Delta a}\right)+1\right) \end{array}\right) \middle| s_t, X_t\right] \\
= \;& E_t\left[E_t\left[\left(\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \dots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1},s_t,X_t^s,\Delta a_{t+1},X_t^{\Delta a}\right)+1\right) \end{array}\right) \middle| s_{t+1}, s_t, X_t\right] \middle| s_t, X_t\right] \\
= \;& \sum_{s_{t+1}=1}^{2} \mathbb{P}(s_{t+1}|s_t, X_t^s) \quad \dots \\
\times \;& E_t\left[\left(\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \dots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1},s_t,X_t^s,\Delta a_{t+1},X_t^{\Delta a}\right)+1\right) \end{array}\right) \middle| s_{t+1}, s_t, X_t\right] \\
= \;& \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t}|s_t, X_t^s) \quad \dots \\
\times \;& E_t\left[\left(\begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \dots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1},s_t,X_t^s,\Delta a_{t+1},X_t^{\Delta a}\right)+1\right) \end{array}\right) \middle| s_{t+1}, s_t, X_t\right]
\end{aligned}
$$

Again, the conditional expectation of transition probabilities under the household's filtration permits an analytical formula, while the inner expectation in the expression above can be evaluated using the quadrature-type integration methods.

B.1.2. Limiting Economies - Boundary Values for General Case

The key assumption of the numerical solution is that the household eventually learns the true values of all uncertain parameters in the productivity growth. Thus, the simplest limiting economy is the one where all parameters are known, including both transition probabilities π_{11} and π_{22}. In this case, s_t and K_t are the only state variables in the economy. We employ the numerical solution methodology outlined for AU pricing for this limiting economy. Specifically, we find the continuation utility (and the price-dividend ratio of the equity claim) for a grid on π_{11} and π_{22}.

B.2. Unknown Transition Probabilities and Unknown Mean Growth Rates

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma \varepsilon_t,$$

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with the transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

As before, we assume that the representative household does not know the transition probabilities (π_{11}, π_{22}). Additionally, the mean growth rates within each state (μ_1, μ_2) are assumed to be unknown, while the realization of states (s_t) and productivity volatility (σ_t) remain observable. Due to the limitations of the numerical solution algorithm under the prices parameter uncertainty case, we are unable to extend the economy to unobservable regimes, while it is still possible to assume that the household does not know a volatility parameter. Nevertheless, the extension to the case with all parameters unknown, including volatility except for states, is quite straightforward, and we leave the investigation of learning about volatility parameters for future research.

Regarding priors, we assume a conjugate prior for transition probabilities and mean growth rates within each state i : the Beta distributed prior and the truncated normal distributed prior, respectively. The updating equations for two pairs of hyperparameters (a_1, b_1) and (a_2, b_2) remain as before. Additionally, we denote hyperparameters of the truncated normal distributed prior for mean growth in state i by $\mu_{i,t}$ and $\sigma_{i,t}$, which are updated by the Bayes' rule as follows:

$$\mu_{i,t+1} = \mu_{i,t} + \mathbf{1}_{s_{t+1}=i} \frac{\sigma_{i,t}^2}{\sigma_i^2 + \sigma_{i,t}^2} (\Delta a_{t+1} - \mu_{i,t}) \tag{B20}$$

$$\sigma_{i,t+1}^{-2} = \mathbf{1}_{s_{t+1}=i} \cdot \sigma_i^{-2} + \sigma_{i,t}^{-2}, \tag{B21}$$

where $\mathbf{1}$ is an indicator function that equals 1 if the condition in subscript is true and 0 otherwise.

Note that since the variance hyperparameters $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are a function of the time, the following 6-dimensional vector $X_t \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}, \mu_{1,t}, \mu_{2,t}\}$ is sufficient statistics for the priors. Thus, we can define X_{t+1} using the equations (B7)-(B12), (B20)-(B21), the next period regime, and sufficient statistics at time t :

$$X_{t+1} = f(s_{t+1}, s_t, X_t).$$

Following the notations of a previous section, we define $X_t^s \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ and $X_t^{\Delta a} \equiv \{\tilde{K}_t, \mu_{1,t}, \mu_{2,t}\}$, where the superscripts s and Δa indicate that variables in the vectors X_t^s and $X_t^{\Delta a}$ are a function only of the observed state realization s_t and a function of (also) the realized productivity growth, respectively. Thus, $X_t = \left[X_t^s, X_t^{\Delta a}\right]$. Using these notations, we can rewrite

$$\tilde{U}_{t+1}(s_{t+1}, X_{t+1}) = \tilde{U}_{t+1}(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a})$$

to better indicate the dependence of state variables on specific shocks. Ultimately, the recursive equation (A1) is of the same form:

$$\tilde{U}_t(s_t, X_t) \tag{B22}$$

$$= \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a} \right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \mid s_t, X_t \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}},$$

where the expectation on the right hand side is equivalent to:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a} \right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \mid s_t, X_t \right] \\ = & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t} | s_t, X_t^s) \quad \ldots \\ \times & E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a} \right) e^{(1-\gamma)\Delta a_{t+1}} \mid s_{t+1}, s_t, X_t \right] \end{aligned} \tag{B23}$$

In this case, we compute the conditional expectation in (B23) by integrating over conditional distribution of mean growth rates as well as Gaussian distribution of the error term in productivity growth. In particular:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a} \right) e^{(1-\gamma)\Delta a_{t+1}} \mid s_{t+1}, s_t, X_t \right] \\ \approx & \sum_{j=1}^{J} \omega_\varepsilon(j) \left[\sum_{k=1}^{K} \omega_{\mu_{s_{t+1}}}(k) \cdot \tilde{U}_{t+1}^{1-\gamma} \left(s_{t+1}, s_t, X_t^s, \Delta a(j,k), X_t^{\Delta a} \right) e^{(1-\gamma)\Delta a(j,k)} \mid s_{t+1}, s_t, X_t \right] \end{aligned} \tag{B24}$$

where $\omega_\varepsilon(j)$ is the quadrature weight corresponding to the quadrature node $n_\varepsilon(j)$ used for the integration of a standard normal shock ε_{t+1} in productivity growth, and

$\omega_{\mu_{s_{t+1}}}(k)$ is the quadrature weight corresponding to the quadrature node $n_{\mu_{s_{t+1}}}(k)$ used for the integration of a truncated standard normal variable $\mu_{s_{t+1}}$. The observed realized productivity growth, $\Delta a(j,k)$, and a state variable, $X^{\Delta a}_{t+1}(j,k) = \tilde{K}_{t+1}(j,k)$, are updated as follows:

$$\Delta a(j,k) = n_{\mu_{s_{t+1}}}(k) + \sigma \cdot n_{\varepsilon}(j) \tag{B25}$$

$$e^{\Delta a(j,k)}\tilde{K}_{t+1}(j,k) = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t, \tag{B26}$$

where

$$\tilde{I}_t = \tilde{K}_t^{\alpha}\bar{N}^{1-\alpha} - \tilde{C}_t. \tag{B27}$$

Finally, the numerical backward recursion can be performed by using (B22)-(B27). The boundary conditions are defined by the limiting economies $\tau_{1,\infty}$ and $\tau_{2,\infty}$, where the transition probabilities π_{11} and π_{22}, and mean growth rates μ_1 and μ_2, are known.

B.2.1. Solving for a Dividend Claim

We also solve for the price-dividend ratio of the equity claim written on aggregate dividends, which are defined as a leverage to aggregate consumption. Let exogenous aggregate dividends be given by:

$$\Delta d_{t+1} = g_d + \lambda \Delta c_{t+1} + \sigma_d \varepsilon_{d,t+1},$$

where $g_d = \left(1-\lambda\right)\left(E(\mathbb{P}(s_\infty = 1|\pi_{11},\pi_{22}))\mu_1 + E(\mathbb{P}(s_\infty = 2|\pi_{11},\pi_{22}))\mu_2\right)$ and $\mathbb{P}(s_\infty = i|\pi_{11},\pi_{22})$ is the ergodic probability of being in state i conditional on the transition probabilities π_{11} and π_{22}.

Note that the long run mean of dividends growth, g_d, is changing under the household's filtration, though the true long run growth is constant. The subjective beliefs about the true parameter values induce fluctuations in g_d, which can be expressed as $g_d = g_d(s_{t+1}, s_t, X_t)$. The equilibrium condition for the price-dividend ratio and the equilibrium recursion remain the same as in the "unknown transition probabilities" model. The only difference between the two models lie in the way we calculate the conditional expectations. With unknown transition probabilities and mean growth rates in the productivity growth process, we employ quadrature-type integration methods analogous to solving for the continuation utility in this economy.

B.2.2. Limiting Economies - Boundary Values for General Case

The key assumption of the numerical solution is that the household eventually learns the true values of all uncertain parameters in the productivity growth. Thus, the simplest limiting economy is the one where all parameters are known, including both transition probabilities π_{11} and π_{22}, mean growth rates μ_1 and μ_2. In this case, s_t and K_t are the only state variables in the economy. We employ the numerical solution methodology outlined for AU pricing for this limiting economy. Specifically, we find the continuation utility (and the price-dividend ratio of the equity claim) for a grid on $\pi_{11}, \pi_{22}, \mu_1$ and μ_2.

B.3. Existence of Equilibrium

Similarly to Collin-Dufresne, Johannes and Lochstoer (2016) and Johannes, Lochstoer and Mou (2016), the existence of the equilibrium in our production-based economy relies on the fact that the value function is concave and finite for all parameters known economies. Therefore, we verify that these conditions are satisfied for all limiting boundary economies.

C. Impulse Responses

In this section, we consider the numerical procedure used to obtain impulse responses of key macroeconomic and financial variables to a regime switch in the mean growth rate of productivity. In particular, we assume that the economy stays in the high growth state for a long period and then moves to a low growth regime at time 0. We further consider three possible scenarios where the economy remains in the bad regime for one quarter, three quarters, or two years before returning to the good state. The details of the numerical algorithm look as follows.

First, we find the steady state of capital, $\tilde{K}$, in the high growth regime, $s_t = 1$, assuming unbiased parameter beliefs, X_t, which are centered at the true values. Formally, $\tilde{K}$ solves the equation:

$$\tilde{K} = f^k(s_{-1} = 1, X_{-1}, \tilde{K}),$$

where $f^K(\cdot)$ is the policy function for capital assuming the productivity growth is high forever.

Second, suppose that the economy starts in the high growth steady state before time 0 and the investor holds unbiased parameter beliefs. Then unexpectedly the economy shifts to to the bad state at time 0 and stays there for τ periods. Using the policy function, capital is computed recursively as:

$$\begin{aligned}
\tilde{K}_{-1} &= \tilde{K}, \\
\tilde{K}_{0} &= f^{k}(s_0 = 2, X_0, \tilde{K}_{-1}), \dots \\
\tilde{K}_{\tau} &= f^{k}(s_\tau = 2, X_\tau, \tilde{K}_{\tau-1}), \\
\tilde{K}_{\tau+1} &= f^{k}(s_{\tau+1} = 1, X_{\tau+1}, \tilde{K}_{\tau}), \dots \\
\tilde{K}_{t} &= f^{k}(s_t = 1, X_t, \tilde{K}_{t-1}), \quad \forall t,
\end{aligned}$$

where investor's parameter beliefs are updated in each period.

Third, we use policy functions for investment and consumption to obtain equilibrium values of $\tilde{I}_t$ and $\tilde{C}_t$. Finally, we calculate the remaining macroeconomic and financial variables using the updated state variables.

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