# Quantile Spectral Beta: A Tale of Tail Risks, Investment Horizons, and Asset Prices*

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**Abstract**

We examine how extreme market risks are priced in the cross-section of asset returns at various horizons. Based on the decomposition of covariance between indicator functions capturing fluctuations of different parts of return distributions over various frequencies, we define a *quantile spectral* beta representation that characterizes asset's risk generally. Nesting the traditional frameworks, the new representation explains *tail*-specific as well as horizon-, or frequency-specific *spectral* risks. Further, we work with two notions of frequency-specific extreme market risks. First, we define tail market risk that captures dependence between extremely low market and asset returns. Second, extreme market volatility risk is characterized by dependence between extremely high increments of market volatility and extremely low asset return. Empirical findings based on the datasets with long enough history, 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market support our intuition. We reach the same conclusion using stock-level data as well as daily data. These results suggest that both frequency-specific tail market risk and extreme volatility risk are priced and our final model provides significant improvement over specifications considered by previous literature.

**Keywords**: Asset pricing, downside risk, tail risk, frequency, spectral risk, investment horizons
**JEL**: C21; C58; G12

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# 1 Introduction

Classical result of asset pricing literature states that price of an asset should be equal to its expected discounted payoff. In the Capital Asset Pricing Model (CAPM) introduced by Sharpe (1964), Lintner (1965), Black (1972), we assume that stochastic discount factor can be approximated by return on market portfolio and thus expected excess returns can be fully described by their market betas based on covariance between asset return and market return. Yet, decades of the consequent research show that we are unable to sufficiently explain the cross-section of asset returns with this notion. Instead, literature calls for more accurate characterization of risks associated with assets that will better reflect preferences of investors. We aim to show that in order to understand formation of expected returns, one has to look deeper into the features of asset returns that are crucial in terms of preferences of a representative investor. We argue that the two important features are risk related to tail events, and frequency-specific (spectral) risk capturing behavior at different investment horizons. To characterize the risks, we derive novel *quantile spectral* representation of beta. Our work nests classical representation that simply averages beta with equal weights over different quantile levels, as well as frequencies.

Economists has long recognized that decisions under risk are more sensitive to changes in probability of possible extreme events compared to probability of a typical event. The expected utility might not reflect this behavior since it weights probability of outcomes linearly. Consequently, CAPM beta as an aggregate measure of risk may fail to explain the cross-section of asset returns. Several alternative notions emerged in the literature. Mao (1970) presents survey evidence showing that decision makers tend to think of risk in terms of the possibility of outcomes below some target. For an expected utility maximizing investor, Bawa and Lindenberg (1977) has provided a theoretical rationale for using lower partial moment as a measure of portfolio risk. Based on the rank-dependent expected utility due to Yaari (1987), Polkovnichenko and Zhao (2013) introduce utility with probability weights and derive corresponding pricing kernel. More recently, Ang et al. (2006); Lettau et al. (2014) argue that downside risk – risk of negative returns – is priced across asset classes and is not captured by CAPM betas. Further, Farago and Tédongap (2018) extend the results using general equilibrium model based on generalized disappointment aversion and show that downside risks in terms of market return and market volatility are priced in the cross-section of asset returns.[1]

The results described above leads us to question the role of the expected utility maximizers in asset pricing. A recent strand of literature solves the problem by considering quantile of the utility instead of expectation. This literature complements the previously described empirical findings focusing on downside risk as it highlights the notion of economic agents particularly averse to outcomes below some threshold compared to outcomes above this threshold. The concept of a quantile maximizer and its features was pioneered by Manski (1988), and later axiomatized by Rostek (2010). Most recently, de Castro and Galvao (2019) develop a model of quantile optimizer in a dynamic setting. A different approach to emphasizing investor's aversion towards least favorable outcomes defines theory based on Choquet expectations. This

[1] In addition, it is interesting to note that equity and variance risk premium are also associated with compensation for jump tail risk (Bollerslev and Todorov, 2011). More general exploration of asymmetry of stock returns is provided by Ghysels et al. (2016), who propose a quantile-based measure of conditional asymmetry and show that stock returns from emerging markets are positively skewed. Conrad et al. (2013) use option price data and find a relation between stock returns and their skewness. Another notable approach uses high frequency data to define realized semivariance as a measure of downside risk (Barndorff-Nielsen et al., 2008). From a risk-measure standpoint, dealing with negative events, especially rare events, is highly discussed theme in both practice and academics. The most prominent example is Value-at-Risk (Adrian and Brunnermeier, 2016; Engle and Manganelli, 2004).

approach is based on distortion function that alters probability distribution of future outcomes by accentuating probabilities associated with least desirable outcomes. This approach was utilized in finance, for example, by Bassett Jr et al. (2004).

Whereas aggregating linearly weighted outcomes may not reflect the sensitivity of investors to tail risk, aggregating linearly weighted outcomes over various frequencies, or economic cycles may not reflect risk specific to different investment horizons. One can suspect that an investor cares differently about short-term and long-term risk according to their preferred investment horizon. Distinguishing between long-term and short-term dependence between economic variables was proven to be an insightful approach since the introduction of co-integration (Engle and Granger, 1987). Frequency decomposition of risk thus uncovers another important feature of risk which cannot be captured solely by market beta which captures risk averaged over all frequencies. This recent approach to asset pricing enables to shed light on asset returns and investor's behaviour from a different point of view highlighting heterogeneous preferences. Empirical justification is brought by Boons and Tamoni (2015) and Bandi and Tamoni (2017) who show that exposure in long-term returns to innovations in macroeconomic growth and volatility of matching half-life is significantly priced in variety of asset classes. The results are based on decomposition of time series into components of different persistence proposed by Ortu et al. (2013). Piccotti (2016) further sets portfolio optimization problem into frequency domain using matching of utility frequency structure and portfolio frequency structure, and Chaudhuri and Lo (2016) present approach to constructing mean-variance-frequency optimal portfolio. This optimization yields mean-variance optimal portfolio for a given frequency band, and thus optimizes portfolio for a given investment horizon.

From a theoretical point of view, preferences derived by Epstein and Zin (1989) enables to study frequency aspects of investor's preferences, and quickly became a standard in the asset pricing literature. With the important results of Bansal and Yaron (2004), long-run risk started to enter asset pricing discussions. Dew-Becker and Giglio (2016) investigate frequency-specific prices of risk for various models and conclude that cycles longer than business cycle are significantly priced in the market. Other papers utilize frequency domain and Fourier transform to facilitate estimation procedures for parameters hard to estimate using conventional approaches. Berkowitz (2001) generalizes band spectrum regression and enables to estimate dynamic rational expectations models matching data only in particular ways, for example, forcing estimated residuals to be close to white noise. Dew-Becker (2016) proposes spectral density estimator of long-run standard deviation of consumption growth, which is a key component for determining risk premiums under Epstein-Zin preferences, and shows its superior performance compared to the previous approaches. Crouzet et al. (2017) develop model of multi-frequency trade set in frequency domain and show that restricting trading frequencies reduces price informativeness at those frequencies, reduces liquidity and increases return volatility.

The debate clearly indicates that the standard assumptions leading to classical asset pricing models do not correspond with reality. In this paper, we suggest that more general pricing models have to be defined and they should take into consideration both asymmetry of dependence structure among stock market, and different behavior of investors at various investment horizons.

The main contribution of this paper is threefold. First, based on the frequency decomposition of covariance between indicator functions, we define the *quantile spectral beta* of an asset capturing tail-specific as well as frequency-specific risks and corresponding ways of measuring the beta. The newly defined notion of beta can be viewed as disaggregation of a classical beta to a frequency-, and tail- specific beta. With this notion, we examine how extreme market risks are priced in the cross-section of asset returns at various horizons. We define frequency-specific tail market risk that captures dependence between extremely low market and asset returns, as well as extreme market volatility risk that is characterized by dependence between extremely high increments of market volatility and extremely low asset return.

Second, we motivate emergence of these types of risks in a simple theoretical model in which the representative investor cares differently about long- and short-term risk associated with an asset. Moreover, we incorporate notion of aversion to losses into the investor's decision making in such framework. This model then leads to the four-factor representation of the risk premium and is used for building our empirical model.

Third, based on the quantile spectral betas, we estimate models that provides considerable improvements in explaining cross-section of asset returns. Results on a 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market suggest extreme market risk is priced in cross-section of asset returns and it is differently priced for long and short horizon. This extreme market risk is characterized by the risk of extremely low returns or extremely high volatility. Since most of the state of the art models do not perform very well on a daily sampling frequency, we also challenge our models with daily data for the same portfolios, and finally with individual stock data from the CRSP database. We document substantial improvements in both additional datasets.

The rest of the paper has the following structure. Section 2 introduces the concept of quantile spectral betas later employed in the empirical analysis. Section 3 defines a simple theoretical model which incorporates both investment horizons and aversion to losses. Section 4 defines the empirical models used for testing significance of extreme risks. Section 5 conducts the empirical analysis of the extreme risks and provides definition of tested robustness checks. Section 6 concludes. In Appendix A, we provide some details on the derivation of the theoretical model, in Appendix B, we shed light on the estimation procedure of the quantile spectral betas, and the rest of the Appendix report results from the robustness checks.

## 2 Quantile spectral beta: measuring the tail risks across horizons

The empirical search for explanation of why different assets earn different average returns centers around return factor models arising from the Euler equation. With the only assumption of 'no arbitrage', a stochastic discount factor $m_{t+1}$ exists and, for the $i$th excess return $r_{i,t+1}$ satisfies $\mathbb{E}[m_{t+1}r_{i,t+1}] = 0$, hence

$$\mathbb{E}[r_{i,t+1}] = \frac{\mathbb{C}ov(m_{t+1}r_{i,t+1})}{\mathbb{V}ar(m_{t+1})}\left( - \frac{\mathbb{V}ar(m_{t+1})}{\mathbb{E}[m_{t+1}]}\right) = \beta_i \lambda \tag{1}$$

where loading $\beta_i$ can be interpreted as exposure to systematic risk factors, and $\lambda$ as the risk price associated with factors.

Empirical literature centering around this expression assumes silently that the risk factors aggregate information over the distribution of returns as well as investment horizons. Part of the literature tracing back to early work by Roy (1952); Markowitz (1952); Hogan and Warren (1974); Bawa and Lindenberg (1977) argue that the reason we do not empirically find the support for the above thinking is that pricing relationship is fundamentally too simplistic. If investors are averse to volatility only when it leads to losses, not gains, the total variance as a relevant measures of risk should be disaggregated. Later work by Ang et al. (2006); Lettau et al. (2014); Farago and Tédongap (2018) show that investors require additional premium as a compensation for exposures to disappointment-related risk factors called downside risk.

Recently, Lu and Murray (2019) argue that bear risk capturing the left tail outcomes is even more important, and Bollerslev et al. (2019) introduce betas based on semi-covariances. In contrast to the promising results, Levi and Welch (2019) conclude that estimated downside betas do not provide superior predictions compared to standard aggregated beta, partially due to the difficulties of accurately determining downside betas from daily returns.

With a similar argument of too simplistic pricing relation, another part of the literature looks at frequency decomposition and explores the fact that risk factors being claims on the consumption risk should be frequency dependent since consumption has strong cyclical components (Bandi et al., 2018; Dew-Becker and Giglio, 2016).

These studies however fail to fully account for the horizon specific information in tails while one of the main reasons turns to be inability to measure such risks. Here we propose robust methods for measurement of such risks, and we argue that exploring the risk related to tail events as well as frequency-specific risk is crucial.

First, we define quantile risk measure based on covariance between indicator functions, which has natural economic interpretation in terms of probabilities. Second, we introduce frequency decomposition, and combine these two frameworks into quantile spectral risk measure, which is the building block for our empirical model. This measure enables to robustly test for the presence of extreme market risks over various horizons in the asset prices. The aim is not to convince the reader that the functional form of the preferences follows precisely our model, but to show that there is a heterogeneity in the weights that investors put to the risk for different investment horizons and different parts of the distribution of their future wealth. By estimating prices of risk for short- and long-term part, we are able to identify the horizon the investor care most about. Moreover, by estimating prices of risk for various threshold values, we are able to identify the part of the joint distribution towards which is the investor the most risk averse. This is done by controlling for CAPM beta and the influence of these new measures is measured as an incremental information over simplifying assumptions that lead to the CAPM beta asset pricing models.

## 2.1 Tail risk

As argued above, risk premium of an asset or a portfolio can be explained by its covariance with some reference economic or financial variable such as consumption growth or return on market portfolio. This measure may not be sufficient in the cases in which the investor cares about different parts of the distribution of his future wealth differently. Hence, the most widely used measure of dependence between stochastic discount factor $m_{t+1}$ and $r_{i,t+1}$, cross-covariance at lag $k$,

$$\gamma_{i,m}^{k} = \mathbb{C}ov\big(m_{t+k}, r_{i,t}\big) \equiv \mathbb{E}[(m_{t+k} - \bar{m})(r_{i,t} - \bar{r}_i)], \tag{2}$$

is unable to describe asymmetry features of dependence structure between two variables due to its averaging nature unless the variables are jointly normal. In case we are interested to measure dependence separately in different parts of a joint distribution, we need to employ more flexible measures. Since we are interested in pricing extreme negative events, we want to measure dependence and risk in lower quantiles of the joint distribution, and propose a quantity of the following form

$$\gamma_{i,m}^{k}(\tau) \equiv \mathbb{C}ov\big(I\{m_{t+k} \leq q_m(\tau)\}, I\{r_{i,t} \leq q_m(\tau)\}\big), \tag{3}$$

where $m_t$ and $r_{i,t}$ are two time series of strictly stationary random variables, $q_m(\tau)$ is a quantile function of $m_t$ for $\tau \in (0, 1)$, and $I\{A\}$ is indicator function of event $A$. The measure is given by the covariance between two indicator functions and can fully describe joint distribution of the pair of random variables $m$ and $r_i$. If distribution functions of the variables are continuous, the quantity is essentially difference between copula of pair $m$ and $r_i$ and independent copula, i.e., the following quantity $Pr\{m_{t+k} \leq q_m(\tau), r_{i,t} \leq q_m(\tau) \quad - \tau\tau_i$ where $\tau_i = F_{r_i}\{q_m(\tau)\}$ and $F_{r_i}$ is cdf of $r_i$. Thus, covariance between indicators measures additional information from the copula over independent copula about how likely is that the series are jointly less or equal to a given quantile of the variable $m$. It enables to flexibly measure both cross-sectional structure and time-series structure of the pair of random variables.

Note that the quantity introduced in Eq. 3 can be further generalized in the way that one can replace $q_m(\tau)$ by some general threshold values derived from distribution of a reference variable. Being below the threshold value corresponds to an inconvenient event for the investor and thus this measure of dependence adequately captures the corresponding risk. In our model, we set threshold values to be equal. In case the stochastic discount factor is linear in factors and we consider the market return as a risk factor, we further look at the dependence between asset returns and market returns $r_{m,t}$, and the threshold values are based on quantiles of market returns $q_m(\tau) = q_{r_m}(\tau)$. A market beta associated with the tail risk is then defined using quantity given in 3 for $k = 0$ and normalized by variance of the indicator function of the market return

$$\beta_i(\tau) \equiv \frac{\mathbb{C}ov\big(I\{r_{m,t} \leq q_{r_m}(\tau))\}, I\{r_{i,t} \leq q_{r_m}(\tau)\}\big)}{\mathbb{V}ar(I\{r_{m,t} \leq q_{r_m}(\tau)\})}. \tag{4}$$

Note that $\mathbb{V}ar(I\{r_{m,t} \leq q_{r_m}(\tau)\}) = \tau(1-\tau)$.

## 2.2 Frequency-specific (spectral) risk

It is further natural to assume that economic agents care not only about different parts of the wealth distribution, but they care differently about long-, and short-term investment horizon in terms of expected returns and associated risks. Investors may be interested in long-term profitability of their portfolio and do not concern with short-term fluctuations. Frequency-dependent features of an asset return then play an important role for an investor. Bandi and Tamoni (2017) argue that covariance between two returns can be decomposed into covariance between disaggregated components evolving over different time scales, and thus the risk on these components can vary. Hence, market beta can be decomposed into linear combination of betas measuring dependence at various scales, i.e., dependence between fluctuations with various half-lives. Frequency specific risk at given time plays an important role for determination of asset prices, and the price of risk in various frequency bands may differ, this means that the expected return can be decomposed into linear combination of risks in various frequency bands.

The most simple and natural way how to decompose covariance between two assets into dependencies over different horizons is via its Fourier and inverse Fourier transform. Frequency domain counterpart of cross-covariance is obtained as Fourier transform of the cross-covariance functions. Conversely, cross-covariance can be obtained from inverse Fourier transform of its cross-spectrum in the following way

$$S_{i,m}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_{i,m}^k e^{-\mathrm{i}k\omega}$$
$$\gamma_{i,m}^k = \int_{-\pi}^{\pi} S_{i,m}(\omega)e^{\mathrm{i}k\omega}d\omega$$

where $S_{i,m}(\omega)$ is cross-spectral density of random variables $r_{i,t}$ and $m_t$, $\gamma^k$ is cross-covariance function given by equation 2 and $\mathrm{i} = \sqrt{-1}$. It is important to note that cross-covariance can be decomposed into frequencies, more specifically, for $k = 0$, we can decompose covariance between two time series into the covariance components at each frequency $\omega$

$$\mathbb{C}ov(m_t, r_{i,t}) = \int_{-\pi}^{\pi} S_{i,m}(\omega) d\omega.$$

Following the same logic decomposition of variance follows as

$$\mathbb{V}ar(m_t) = \int_{-\pi}^{\pi} S_m(\omega) d\omega.$$

where $S_m(\omega)$ is spectrum of $m_t$.

Since we can decompose cross-covariance between two returns into covariances at each frequency, we can disentangle the dependence at short- and long-term components. Then, beta for an asset $i$ and factor $m$ can be decomposed to $\beta_i(\omega)$ at a given frequency using spectral decomposition as

$$\beta_i \equiv \frac{\mathbb{C}ov(m_t, r_{i,t})}{\mathbb{V}ar(m_t)} = \int_{-\pi}^{\pi} w(\omega) \frac{S_{i,m}(\omega)}{S_m(\omega)} d\omega = \int_{-\pi}^{\pi} w(\omega) \beta_i(\omega) d\omega$$

where $w(\omega) = \frac{S_m(\omega)}{\int_{-\pi}^{\pi} S_m(\omega) d\omega}$ represent weights. The decomposition is important step since it provides decomposition of classical beta into the weighted frequency-specific betas. Using similar approach, Bandi and Tamoni (2017) estimate price of risk for different investment horizons and show that investors posses heterogeneous preferences over various economic cycles instead of looking only on averaged quantities over the whole frequency spectrum.

## 2.3 Quantile spectral beta

Since we argue that both tail risk as well as frequency-specific (spectral) risk are important in explaining formation of asset returns, we aim to combine these risks into a single model. We start by defining measure of risk associated with various combinations of quantile and frequency in order to determine the most important combination priced across assets.

Our measures of risk in the quantile spectral domain are based on the dependence measures recently introduced by Baruník and Kley (2019). To quantify risk premium across frequencies and across the joint distribution, we use the quantile spectral densities to build a quantile spectral beta.

The cornerstone of the new beta representation lies in quantile cross-spectral density kernels which are defined as

$$f_{i,m}(\omega;\tau) \equiv \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{i,m}^{k}(\tau) e^{-\mathrm{i}k\omega} \tag{5}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathbb{C}ov\big(I\{m_{t+k} \leq q_m(\tau)\}, I\{r_{i,t} \leq q_m(\tau)\}\big) e^{-\mathrm{i}k\omega}. \tag{6}$$

with $= \tau \in (0,1)$. A quantile cross-spectral density kernel is obtained as a Fourier transform of covariances of indicator functions defined in Equation 3, and will allow us to define beta that will capture the tail risks as well as spectral risks.

A quantile spectral (QS) betas for a given $\tau$ quantile of market returns are defined as

$$\beta_i(\omega;\tau) \equiv \frac{f_{i,m}(\omega;\tau)}{f_m(\omega;\tau)} \equiv \frac{\sum_{k=-\infty}^{\infty}\gamma_{i,m}^k(\tau)e^{-\mathrm{i}k\omega}}{\sum_{k=-\infty}^{\infty}\gamma_m^k(\tau)e^{-\mathrm{i}k\omega}}. \tag{7}$$

QS betas for given asset quantify the dependence between $i$th asset and market factor $m$ for a given frequency $\omega$ at chosen quantiles $\tau \in (0,1)$ of the joint distribution.

For better interpretability, we construct beta for a given frequency band corresponding to reasonable economic cycles as

$$\beta_i(\Omega;\tau) \equiv \int_{\Omega} \frac{f_{i,m}(\omega;\tau)}{f_m(\omega;\tau)} d\omega \tag{8}$$

where $\Omega \equiv [\omega_1, \omega_2)$, $\omega_1, \omega_2 \in [-\pi, \pi]$, $\omega_1 < \omega_2$ is a frequency band. This definition is important since it allows to define short-run, or long-run bands covering corresponding frequencies, and hence disaggregate beta based on the specific demands of a researcher. The justification for assuming that this definition of beta gives relevant quantity of risk for given assets provides theoretical model from Section 3. We will see that the risk premium of an asset can be characterized as two $\beta$'s and $\lambda$'s for an investment horizons - risk factors present in the economy. Moreover, each of them has two parts for values below and above the threshold value given by some quantile of consumption reflecting the investor's aversion to (large) losses.

### 2.3.1 Quantile spectral beta under Gaussianity

Before we continue and use the new beta representation, it is important to note how newly defined quantity relates to a classical beta under the assumption of Gaussian distribution, as commonly assumed by many asset pricing models. Assuming that returns of an asset and returns of market portfolio are jointly normal random variables independently distributed through the time (correlated Gaussian white noises), QS betas would be in the following form

$$\beta_i^{\text{Gauss}}(\omega;\tau) = \frac{C^{\text{Gauss}}(\tau, \tau_i; \rho) - \tau\tau_i}{\tau(1-\tau)} \tag{9}$$

where $C^{\text{Gauss}}$ is Gaussian copula with correlation coefficient $\rho$. This stems from the fact that quantile cross-spectral density corresponds to a difference of probabilities $Pr\{r_{i,t} \leq q_{r_m}(\tau), r_{m,t} \leq q_{r_m}(\tau) \quad - \tau\tau_i$, where $\tau$ and $\tau_i$ are probability levels under Gaussian distribution.

In our case, the threshold values are given by the market quantile. So, if we want to compute beta for an asset $i$ and market under the Gaussian distribution assumption, first, value of $\tau_i$ has to be estimated. We do that by using the empirical distribution function $\widehat{F}_{r_i}$ of asset $i$'s returns, i.e. $\tau_i = \widehat{F}_{r_i}\{q_m(\tau)\}$.

QS betas for Gaussian variables are important since the quantity is constant over frequencies, and depend only on chosen quantiles and correlation coefficient between asset and market return. Hence Eq. 9 provides the quantile spectral counterpart to classical CAPM beta. We will use this fact to construct our model later. In the spirit of Ang et al. (2006) and Lettau et al. (2014), we define relative QS betas which capture additional information not contained in the classical CAPM beta.

Finally, we note that for serially uncorrelated variables (no matter of their joint or marginal distributions), the Frećhet/Hoeffding bounds gives the limits that QS beta can attain $\frac{\max\{\tau+\tau_i-1,0\}-\tau\tau_i}{\tau(1-\tau)} \leq \beta_i(\omega;\tau) \leq \frac{\min\{\tau,\tau_i\}-\tau\tau_i}{\tau(1-\tau)}$ where $\tau_i$ is derived as described above.

# 3 Quantile spectral risk and its cross-sectional implications

Knowing how to estimate risk in various parts of the joint distribution and over various investment cycles, we would like to provide a notion of how this type of risk may be priced cross-sectionally in an equilibrium setting. Frequency part is important because of the fact that the risk premium in the setting described above is determined by the covariance between asset return and two parts of the wealth process - short- and long-term. The measure of dependence between asset return and wealth in specific part of the distribution is important because agent is highly averse to extremely low outcomes and thus requires a premium for assets that posses high covariance with wealth. We motivate this discussion using simple economic model and show how the aversion to tail market risk, which varies with different investment horizons, may emerge. Instead of developing a general asset pricing model, we aim to discuss the pricing implications in a simple setting that will later lead to a simple empirical framework for testing the implications on data.

We propose a simple endowment-type model of the economy with two sources of aggregate risk: long-term and short-term dividend streams of the endowment cashflows. The consumption (wealth) process is based on the two-tree model of Cochrane et al. (2007) that extends a general equilibrium model of Lucas Jr (1978). We utilize these two endowment trees and interpret them as long-term and short-term risk factors in the economy. The first tree, corresponding to the long-term risk, generates dividend that is more persistent and this cash-flow bears the long-term risk in the economy. On the other hand, the overall variability of this process is much smaller than the variability of the short-term process. The second tree generates much less persistent dividend returns and represents the short-term risk in the economy. The variability of short-term cash-flows is considerably higher than the variability of the long-term part. This means that the dividend share of the short-term part is higher.

We define our model in continuous-time setting for two main reasons. First, as noted in Cochrane (2012), we do not have to worry about the timing; in the discrete time setting, there is an ambiguity whether investment made at time $t$ directly joins the capital stock and generate a return at time $t+1$ (finance timing), or whether the investment made at time $t$ sits for one period and joins the capital stock at time $t+1$. Second, it enables to handle various nonlinearities and obtain intuitive closed-form formulas of the equity premium.

The representative investor has the following utility over the stream of consumption

$$U_t = E_t \int_0^\infty e^{-\delta\tau} u(c_{t+\tau}) d\tau.$$

We start the discussion without specific functional form of the instantaneous utility function of the representative investor. Later, we discuss the case of power utility and then the case of asymmetric utility which generates the aversion to tail risk.

In the economy, there are two trees which generate dividend and they follow geometric Brownian motion with different values of mean and variance parameters

$$\frac{dD_\ell}{D_\ell} = \mu_\ell dt + \sigma_\ell dZ_\ell, \quad \ell = 1, 2$$

where $dZ_\ell$ is a standard Brownian motion and the processes are orthogonal. Because this is an endowment economy, the prices adjust until consumption equals the sum of dividends, $c = D_1 + D_2$. We argue that the natural choice is that these two cash-flows represent long-term and short-term investment, respectively. Thus, the long-term dividend is characterized by more persistent process with small time variance, and short-term dividend corresponds to a less persistent process with higher variance, $\mu_1 > 0$, $\mu_2 = 0$ and $\sigma_1 < \sigma_2$[2].

Relative sizes of these trees determine state variable for the economy. Let's denote

$$s = \frac{D_1}{D_1 + D_2}$$

the relative size of the first dividend - long-term part of the consumption. Applying Itôs lemma, we obtain the dynamics for the consumption process

$$\frac{dc_t}{c_t} = [\mu_1 s + \mu_2 s(1-s)]dt + \sigma_1 s dZ_1 + \sigma_2(1-s)dZ_2. \tag{10}$$

Let's suppose that the investor is offered to purchase an asset for price $p_t$ which pays a dividend stream $d_t$. We assume that the price follows geometric Brownian motion

$$\frac{dp_t}{p_t} = \mu dt + \sigma dZ.$$

We do not directly link cashflow of the aggregate consumption with the cashflow generated by this asset. Rather, we model consumption cash-flows as claims on certain risk factors in the consumption process[3].

Let's define the instantaneous total excess return of the asset as

$$R = \frac{dp_t}{p_t} + \frac{d_t}{p_t}dt.$$

Then, it can be shown[4] that the expected excess return is equal to

$$\mathbb{E}_t(R) - r_t^f = \gamma_t \mathbb{E}_t\Big(\frac{dc_t}{c_t}\frac{dp_t}{p_t}\Big) \tag{11}$$

where $\gamma_t$ is utility curvature parameter (coefficient of relative risk aversion)

$$\gamma_t = -\frac{c_t u''(c_t)}{u'(c_t)}.$$

The functional form of the instantaneous utility function $u(c)$ determines the nature of $\gamma_t$. For power utility, the parameter is constant for all values of consumption, yielding constant relative risk aversion over all states of consumption. On the other hand, in the case of the loss-averse utility, the relative risk parameter differs across the distribution of the consumption.

The risk premium of an asset from Equation 11, can be expressed in terms of covariance rather than expectations as

$$\mathbb{E}_t(R) - r_t^f = \gamma_t \mathbb{C}ov_t\Big(\frac{dc_t}{c_t}, \frac{dp_t}{p_t}\Big). \tag{12}$$

We see that the risk premium is proportional to the covariance between growth rate of the asset's price and consumption growth multiplied by the curvature parameter. By substituting

[2]This is very similar to the long-run risk model of Bansal and Yaron (2004), in which consumption growth contains a small persistent part which has a significant asset pricing implications.

[3]This is not an unusual approach; the same assumption is adopted in pricing dividend claim in various models, e.g., Bansal and Yaron (2004) or Backus et al. (2011).

[4]See, for example, Cochrane (2009).

for the consumption growth from Equation 10 into the Equation 12, we can rewrite the risk premium in the form of

$$\mathbb{E}_t(R) - r_t^f dt = \beta_t^1 \lambda_t^1 + \beta_t^2 \lambda_t^2 \tag{13}$$

where we define the betas and lambdas for long and short horizon risk, respectively, as

$$\begin{aligned} \beta_t^1 \equiv \frac{\rho_1 \sigma dt}{s_t \sigma_1 dt}, \quad \beta_t^2 \equiv \frac{\rho_2 \sigma dt}{(1-s_t)\sigma_2 dt} \\ \lambda_t^1 \equiv \gamma_t s_t^2 \sigma_1^2 dt, \quad \lambda_t^2 \equiv \gamma_t (1-s_t)^2 \sigma_2^2 dt. \end{aligned} \tag{14}$$

For more detailed derivation, see Appendix A. We stress that the coefficients $\beta_t^i$ (quantities of risk) are specific for every asset, and $\lambda_t^i$ (prices of risk) are common for every asset in the economy representing the risk premium of the asset. Parameter $\beta_t^1$ corresponds to the covariance between asset's return and dividend growth of the first tree (long-term part of the consumption), and the parameter $\beta_t^2$ to the covariance between asset's return and dividend growth of the second tree. Moreover, we see that the prices of risk are dependent on the share size and variance of a given tree. If the variance and/or share of a tree is small, then the corresponding risk premium is also small.

From the previous, we see that the risk premium is given by short-term dependence between consumption process and asset's return process, and long-term dependence. As stated in Bandi et al. (2018), frequency is a dimension of risk and thus investors care differently about its various parts.

## 3.1 Power utility

Until now we worked with a general notion of a utility function. Let's consider more specific instantaneous utility function of a representative investor

$$u(c) = \frac{c^{1-\xi}}{1-\xi}, \quad \xi > 0, \xi \neq 1 \tag{15}$$

where $\xi$ is relative risk aversion coefficient; if $\xi = 1$, then the we obtain log utility, which was studied in Cochrane et al. (2007). Power utility is widely used due to its simplicity and number of desirable properties (e.g., homogeneity) although it generates many puzzles in the asset pricing literature (e.g., equity premium puzzle).

Moreover, the constant curvature parameter $\gamma_t = \xi$ makes the aversion to risk constant throughout the whole distribution of consumption. If we substitute $\xi$ into 14 we obtain the formulas for both quantities of risk and prices of risk and these values are independent of the current value of consumption (only on the proportions of the consumption due to the long- and short-term dividends). Hence we arrive to a model that assumes frequency-specific risk, but also the same risk across all points of return distribution.

## 3.2 Asymmetric utility

To introduce loss (tail risk) aversion into the model, we employ asymmetric utility function over instantaneous consumption[5]. We employ utility function based on the prospect theory

[5]The use of behavioral finance in continuous time setting is quite rare. Two examples are Berkelaar et al. (2004) and Jin and Yu Zhou (2008)

(PT) of Tversky and Kahneman (1979) and Tversky and Kahneman (1992) which posses a kink at some reference point[6]. This approach was shown to resolve many puzzles generated by conventional models which do not correspond to empirical facts present in the real-world data. Generally, PT is combined with some other theory to match an observed phenomenon. One of the first successful attempts to introduce it into the asset pricing literature was Benartzi and Thaler (1995), proposing myopic loss aversion as a possible solution to equity premium puzzle. Barberis et al. (2001) employ prospect theory model in conjunction with habit formation in an intertemporal setting and show that their model can help explain high mean, excess volatility, and predictability of excess returns. Others combine PT, for example, with disposition effect (Li and Yang, 2013), narrow framing (Barberis et al., 2006). etc[7]. In a way, we follow this stream by combining PT with investment horizons.

The concept of PT is based on the notion that investor does not care about the absolute value of his final wealth, but he compares it with some benchmark value - reference point, and thus he specifically cares about gain or loss induced by the risk. Based on that, investor is more sensitive to reduction in his well-being than to increases. This feature is mimicked by modeling the investor's utility being convex in the realm of outcomes below the reference point, and concave in the realm above the reference point[8]. Later in the empirical part, we show that the data support the notion that the benchmark corresponds to the left-tail values of market returns.

The utility function of Tversky and Kahneman (1979) relative to some reference point $c_0$ for gains and losses can be written in terms of consumption as

$$u(c) = (c - c_0)^{\alpha} I\{c \geq c_0\} - \delta(-(c - c_0))^{\epsilon} I\{c < c_0\}. \tag{16}$$

In our setting, we assume that the reference point is derived from the distribution of consumption (wealth) returns, specifically, we obtain the benchmark value as $c_{0,t+1} = c_t r$ where $r$ is derived from the distribution of consumption growth and it is its $\tau$-quantile, $r = q_{r_c}(\tau)$, where $q_{r_c}$ is the quantile function of the distribution of consumption growth[9].

Because of the presence of the kink at the reference point, the original utility function of PT is not differentiable at the point of the kink. This feature makes the utility hard to incorporate into the intertemporal setting. So, instead of using the original utility function of Tversky and Kahneman (1979), we use the specification introduced in Hung and Wang (2005)

$$u(c) = (1 - e^{-\epsilon(c-c_0)}) I\{c \geq c_0\} - \delta(1 - e^{\frac{\epsilon}{\delta}(c-c_0)}) I\{c < c_0\} \tag{17}$$

where $\epsilon$ is risk aversion coefficient and $\delta$ is aversion to losses coefficient[10]. This leads to

[6] In the setting of the original prospect theory (Tversky and Kahneman, 1979) and cumulative prospect theory (Tversky and Kahneman, 1992), not only the utility function posses a kink in the reference point, but also the probabilities are non-linearly transformed mimicking the distortion of the objective probabilities into the decision weights. We do not aim to fully incorporate this theory into our model as the main objective is to introduce the loss aversion in the most simple way.

[7] This is just a tip of the iceberg, some of the recent applications of PT in asset pricing feature, for example, Yogo (2008), Barberis et al. (2016), or Wang et al. (2017)

[8] There are various ways how to define loss aversion. Since we assume differentiability of the utility function, we use common approach based on $u'(x) < u'(-x)$, $\forall x > 0$. On the other hand, Köbberling and Wakker (2005) define it as $\frac{u'(0^-)}{u'(0^+)} \geq 1$, which works well with non-differentiable utility functions.

[9] There is a large literature which deals with the question how to set the reference point, see, e.g., Kőszegi and Rabin (2006).

[10] This utility function is a special case of of exponential loss-averse utility from Köbberling and Wakker (2005) with index of loss aversion being equal to 1, and it posses risk aversion in our sense as long as $\delta > 1$. There is nothing particularly special about the proposed utility function. We may as well use another twice-differentiable loss-averse utility function, e.g., one from Bahamonde-Birke (2018).

consumption-dependent relative risk aversion of the form

$$\gamma_t = -\frac{c_t u''(c_t)}{u'(c_t)} = c_t \epsilon I\{c_t \geq c_{0,t}\} - c_t \frac{\epsilon}{\delta} I\{c_t < c_{0,t}\}. \tag{18}$$

The relation between consumption level and relative risk aversion of the asymmetric utility function is interesting. As we approach the reference point from the left, the relative risk aversion decreases and the investor is risk-seeker in this region. On the other hand, as we move to the right from the reference point, the risk aversion is increasing and the investor is risk-averse.

The asymmetric utility then leads to the following decomposition of the risk premium

$$\mathbb{E}_t(R) - r_t^f dt = \beta_t^1 \lambda_t^1 + \beta_t^2 \lambda_t^2 + \beta_t^1 \lambda_t^3 + \beta_t^2 \lambda_t^4 \tag{19}$$

where the betas are defined as follows

$$\begin{aligned}
\beta_t^1 &\equiv \frac{\rho_1 \sigma dt}{s_t \sigma_1 dt} I\{c_t \geq c_{0,t}\} \\
\beta_t^2 &\equiv \frac{\rho_1 \sigma dt}{s_t \sigma_1 dt} I\{c_t < c_{0,t}\} \\
\beta_t^3 &\equiv \frac{\rho_{2,} \sigma dt}{(1 - s_t) \sigma_2 dt} I\{c_t \geq c_{0,t}\} \\
\beta_t^4 &\equiv \frac{\rho_2 \sigma dt}{(1 - s_t) \sigma_2 dt} I\{c_t < c_{0,t}\},
\end{aligned}$$

and lambdas read as

$$\begin{aligned}
\lambda_t^1 &\equiv c_t \epsilon s_t^2 \sigma_1^2 dt \\
\lambda_t^2 &\equiv -c_t \frac{\epsilon}{\delta} s_t^2 \sigma_1^2 dt \\
\lambda_t^3 &\equiv c_t \epsilon (1 - s_t)^2 \sigma_2^2 dt \\
\lambda_t^4 &\equiv -c_t \frac{\epsilon}{\delta} (1 - s_t)^2 \sigma_2^2 dt.
\end{aligned}$$

From Equation 19, we see that the traditional beta is decomposed to a 4-beta specification decomposing market risk to a quantile spectral risk. If the consumption is below the reference point, the investor is a risk seeker and tries to do everything to get out of his inconvenient situation. On the other hand, if the investor is above the reference point, he posses a risk aversion and requires a higher expected return on the asset.

Intuitively, because the investor will require a negative risk premium in the bad states (assuming positive correlations between the Brownian motion of dividend and asset), the assets that strongly covary with the consumption in these states should yield a higher risk premium. Moreover, this premium is not captured by the conventional beta because it aggregates the dependence over all the possible states.

# 4 Pricing model for extreme risks across frequency domain

Quantile spectral betas defined in the previous sections will be the cornerstone of our empirical model. Using the theoretical motivation, we assume that QS betas for low threshold values will be significant determinants of risk. Similarly to Ang et al. (2006) and Lettau et al. (2014), we define relative QS betas which capture additional information not contained in the classical CAPM beta. This way we can test the significance of tail market risk and extreme volatility risks decomposed into the long- and short-term components in order to obtain their prices of risk separately.

Tail market risk (TR) represents dependence between extreme negative events of both market as well as asset return. It differs from downside risk used in Ang et al. (2006); Lettau et al. (2014) since downside betas are computed based on covariates of asset return with a market return being under some threshold value. In contrast, QS betas captures risk that both market as well as asset return will be extremely unfavorable. In other words, it captures joint probability that market as well as asset returns will be below some threshold level.

Extreme market volatility risk (EVR) captures unpleasant situations in which extremely high increments of market volatility are linked to the extremely low asset asset returns. We argue that both these risks are significant determinants of risk of an asset and thus should be priced in cross-section of asset returns.

Here, we remind the reader how we set the threshold values in the covariance between indicator measure of dependence. Values of $\tau_i$, percentage value for the quantiles for asset thresholds, are not explicitly fixed to quantile of their returns because we do not explicitly care about dependence between quantile values in the cross-section. We rather care about dependence in extreme market situations. Thus the threshold values for asset returns are given by values of quantile of market returns; these threshold values are same for all the assets, which corresponds to different quantiles for each asset. Formally, for each portfolio we obtain threshold values as a $\tau_i$ quantile of its distribution where $\tau_i = F_{r_i}\{q_m(\tau)\}$. Let's consider a model in which we set threshold value to be equal to 5% quantile of market return. Value of $\tau$ is equal to 5% but $\tau_i$ must be estimated. First, this 5% market quantile must be transformed using empirical cumulative distribution functions into probability that given asset return is below this value for each asset, and then the QS betas are computed as $\beta_i(\omega; \tau)$. This implies that $\tau_i$ differs across assets (for one asset 5% quantile of market return may correspond to 1% quantile of its distribution, for another asset it may correspond to 8% quantile of its distribution). Same logic is applied to both tail market risk betas and extreme volatility risk betas. By setting market return and portfolio threshold equal, we avoid problem of potential data-mining. Potentially better fit could be obtained by finding threshold values with the best model fit for a specific dataset, but may not be robust across datasets.

Regarding the frequency decomposition of the risks, we specify our models to include disaggregation of risk into two horizons - long and short. Long horizon is defined by corresponding frequencies of cycles of 1.5 year and longer, and short horizon by frequencies of cycles shorter than 1.5 year. Procedure how to obtain these betas is explained in Section 5.

In each of the models defined in the paper we control for CAPM beta as a baseline measure of risk. This ensures that if the QS betas are significant determinants of risk premium, they do not simply duplicate information contained in CAPM beta. Moreover, in case of tail market risk, we define relative betas that explicitly capture only the additional information over CAPM beta. Throughout the paper we impose the restriction that market price of risk is correctly priced implying that it is equal its average return.

## 4.1 Tail market risk

We expect the dependence between market return and asset return during extreme negative events will be priced across assets. The stronger the relationship, the higher the risk premium required by investors. Because we want to quantify risk which is not captured by CAPM beta, we propose to test significance of tail market risk via differences of the estimated QS beta and QS beta implied by the Gaussian white noise assumption. The notion comes from the fact that traditional market beta is decomposed to a quantile spectral beta. Hence if information captured by tails at different frequencies is valuable, it will not be subsumed by an aggregated beta which assumes constant risk across frequencies and tails. We call this difference *relative* QS betas. For a given frequency band $\Omega_j$ and given market $\tau$-quantile level, the relative beta is defined as follows

$$\beta_i^{\text{rel}}(\Omega_j;\tau) \equiv \beta_i(\Omega_j;\tau) - \beta_i^{\text{Gauss}}(\Omega_j,\tau). \tag{20}$$

Relative QS betas measure additional information not captured by classical CAPM beta. In case that CAPM beta captures all information, and returns are Gaussian, the relative QS beta will be zero at all frequencies and quantiles.

Our first model is a three-factor market model which contains only tail market risk, and is defined as

$$\mathbb{E}[r_{i,t+1}^e] = \sum_{j=1}^{2} \beta_i^{\text{rel}}(\Omega_j;\tau)\lambda^{\text{TR}}(\Omega_j;\tau) + \beta_i^{\text{CAPM}}\lambda^{\text{CAPM}}, \tag{21}$$

where $\beta_i^{\text{CAPM}}$ is an aggregate CAPM beta, $\lambda^{\text{CAPM}}$ is price of aggregate risk of market captured by the classical beta, and $\lambda^{\text{TR}}(\Omega_j,\tau)$ is price of tail risk (TR) for given quantile and given frequency band. We follow Lettau et al. (2014) and impose a restriction that the market risk is correctly priced, i.e. $\lambda^{\text{CAPM}}$ is equal to average market return. If asset returns do not posses features of deviations from assumptions mentioned above, then the relative betas will be equal to zero and thus all the information about dependency during extreme events is captured by CAPM betas. On the other hand, if there is a significant difference between information captured by CAPM beta and QS betas, then the difference will be nonzero and may be priced in cross-section of asset returns, which will be assessed based on significance of related prices of risk. This model directly relates to the model proposed in the Section 3.

## 4.2 Extreme volatility risk

Assets with high sensitivities to innovations in aggregate volatility have low average returns (Ang et al., 2006). Because of the fact that time of high volatility within the economy is perceived as a time with high uncertainty, investors are willing to pay more for the assets that yield high returns during these turmoils and thus positively covary with innovations in market volatility. This drives the prices of these assets up and decreases expected returns. This notion is formally anchored in the intertemporal pricing model, such as intertemporal CAPM model of Merton (1973) or Campbell (1993). According to these models, market volatility is stochastic and causes changes in the investment opportunity set by changing the expected market returns, or by changing the risk-return trade-off. Market volatility thus determines the systematic risk and should determine expected returns of individual assets or portfolios. Moreover, according to the tail-risk aversion utility framework employed in this paper, we assume that extreme events in the market volatility play significant role in the perception of systematic risk, and that the exposure to them affects the risk premium of the assets.

In addition, decomposition of volatility into short-run and long-run when determining asset premium was proven to be useful as well (Adrian and Rosenberg, 2008). Moreover, Bollerslev et al. (2016) incorporated notion of downside risk into concept of volatility risk and showed that stocks with high negative realized semivariance yield higher returns. Farago and Tédongap (2018) examine downside volatility risk in their five-factor model and obtain model with negative prices of risk of volatility downside factor yielding low returns for assets that positively covary with innovations of market volatility during disappointing events.

We assume that assets that yield highly negative returns during times of large innovations of volatility are less desirable for investors and thus holding these assets should be rewarded by higher risk premium. For simplicity reasons, we estimate market volatility using basic GARCH(1,1) model [11] and obtain estimates of squared volatility. Then the changes in squared volatility are calculated as

$$\Delta\sigma_t^2 \equiv \sigma_t^2 - \sigma_{t-1}^2.$$

Because of the nature of covariance between indicator functions, we work with negative differences of the volatility, $-\Delta\sigma_t^2$, then the high volatility increments correspond to low quantiles of distribution of the negative differences. We investigate whether dependence between extreme market volatility and tail events of assets is priced across assets. Threshold values for portfolio returns are obtained in the same manner as for tail market risk and are derived from distribution of market returns, $\tau_i = F_{r_i}\{q_{r_m}(\tau)\}$. For example, for model with $\tau = 0.05$, extreme market volatility beta is computed using threshold for innovations of market squared volatility as 5% quantile of its distribution of negative values (corresponding to 95% quantile of the original distribution), and threshold for portfolio returns is computed as 5% quantile of distribution of market returns.

Three-factor model containing solely extreme volatility risk (EVR) is defined as

$$\mathbb{E}[r_{i,t+1}^e] = \sum_{j=1}^{2} \beta_i^{\Delta\sigma^2}(\Omega_j;\tau)\lambda^{\mathrm{EV}}(\Omega_j;\tau) + \beta_i^{\mathrm{CAPM}}\lambda^{\mathrm{CAPM}},$$

where, as earlier, we also impose restriction that market risk is correctly priced, i.e. $\lambda^{\mathrm{CAPM}}$ is equal to average market return.

### 4.3 Full five-factor model

Finally, we combine the risks into a single five-factor model that includes both tail market risk and extreme volatility risk for both short- and long-run horizons, as well as market risk associated with classical CAPM beta. Model posses the following form

$$\begin{aligned}\mathbb{E}[r_{i,t+1}^e] = \sum_{j=1}^{2} \beta_i^{rel}(\Omega_j;\tau)\lambda^{\mathrm{TR}}(\Omega_j;\tau) + \beta_i^{\mathrm{CAPM}}\lambda^{\mathrm{CAPM}} \\ + \sum_{j=1}^{2} \beta_i^{\Delta\sigma^2}(\Omega_j;\tau)\lambda^{\mathrm{EV}}(\Omega_j;\tau)\end{aligned}$$

where we restrict $\lambda^{\mathrm{CAPM}}$ to be equal to the average market return. We remind that the market threshold is equal to portfolio threshold. Throughout the paper, we focus on results for $\tau$ equal to 5% and 10% (models denoted as QS05 and QS10). In addition, we report various results for 1%, 15%, and 25% quantiles. Moreover, root mean squared pricing error of the fitted models is reported for continuum of quantiles between 1% and 50% for completeness. The choice of 5% and 10% quantiles is natural and arises in many economic and finance applications. Probably the most prominent example is Value-at-Risk, which is a benchmark measure of risk widely used in practice and studied among academics.

[11] As a robustness check, we compute volatility as realized volatility from daily data

### 4.4 Three-factor model aggregating frequencies

As an intermediary step, we define model which contains both tail market risk and extreme market volatility risk but does not take into consideration frequency decomposition. It posses the following form

$$\mathbb{E}[r^e_{i,t+1}] = \beta_i^{rel}(\tau)\lambda^{\mathrm{TR}}(\tau) + \beta_i^{\mathrm{CAPM}}\lambda^{\mathrm{CAPM}} + \beta_i^{\Delta\sigma^2}(\tau)\lambda^{\mathrm{EV}}(\tau)$$

where we define quantile betas as in Equation 4. This measure of dependence is very similar to the one proposed by Han et al. (2016). The main difference is that our measure is only partly based on quantile-hit process (market return) and partly on threshold-hit process (asset return). To see that, compare Equation 3.

Relative beta in case of TR is defined as difference between quantile beta and beta defined under normality assumption

$$\beta_i^{rel}(\tau) \equiv \beta_i(\tau) - \beta_i^{\mathrm{Gauss}}(\tau) \tag{22}$$

where beta under normality assumption is the same as in Equation 9 since it does not depend on frequency. Threshold values are obtained in the same way as in case of Full 5-factor model.

## 5 Quantile spectral risk and the cross-section of expected returns

In this section, we estimate the models defined in the previous section and assess whether the extreme risks are priced in the cross-section of asset returns and whether we capture new features of priced risk not described by other competing models.

Estimation of QS betas (for both TR and EVR) relies on proper estimation of quantile cross-spectral densities using rank-based copula cross-periodograms, which are then smoothed in order to obtain consistency of the estimator. Technical details are provided in the Appendix B. Betas from the simplified model defined in Equation 22 are simply estimated using empirical distribution function of the market return distribution.

### 5.1 Fama-MacBeth regressions

To test our models, we employ procedure of Fama and MacBeth (1973). In the first stage, we estimate all required QS betas, relative QS betas, and CAPM betas for all portfolios. We define two non-overlapping horizons: short and long. Horizon is specified by the corresponding frequency band. We specify long horizon by frequencies with corresponding cycles 1.5 year and longer, and short horizon by frequencies with corresponding cycles below 1.5 year. QS betas for these horizons are obtained by averaging QS betas over these frequency bands

$$\begin{aligned}
\beta_i(\Omega_L;\tau) &\equiv \frac{1}{n_L}\sum_{j=1}^{n_L}\beta_i(\omega_j^L;\tau) \\
\beta_i(\Omega_S;\tau) &\equiv \frac{1}{n_S}\sum_{j=1}^{n_S}\beta_i(\omega_j^S;\tau)
\end{aligned} \tag{23}$$

where $\Omega_L$ ($\Omega_S$) is frequency band for long (short) horizon, and $\omega_j^L \in \Omega_L$ ($\omega_j^S \in \Omega_S$).

In the second stage, we use these betas as explanatory variables and regress average portfolio returns on them. We assess significance of a given risk by significance of the corresponding price of risk [12]. Thus, in the second stage in case of the Full 5-factor full model, we estimate model of the following form

$$\begin{aligned}\bar{r}_i^e = &\sum_{j=1}^{2} \widehat{\beta}_i^{\mathrm{rel}}(\Omega_j;\tau)\lambda^{\mathrm{TR}}(\Omega_j;\tau) + \widehat{\beta}_i^{\mathrm{CAPM}}\lambda^{\mathrm{CAPM}} \\ &+ \sum_{j=1}^{2} \widehat{\beta}_i^{\Delta\sigma^2}(\Omega_j;\tau)\lambda^{\mathrm{EV}}(\Omega_j;\tau) + e_i.\end{aligned} \tag{24}$$

The same estimation logic applies to other three-factor models.

As mentioned earlier, we estimate our models for various values of threshold value given by $\tau$ quantile of market return. In the scatter plots of actual and predicted returns, we focus on our simple model with $\tau = 0.05$ (Q05) and two versions of our full model where $\tau = 0.05$ (QS05) and $\tau = 0.10$ (QS10). We compare the results of our models with i) classical CAPM ii) downside risk model of Ang et al. (2006) (DR1) iii) downside risk model of Lettau et al. (2014) (DR2) iv) 3-factor model of Fama and French (1993) v) GDA3 and GDA5 models of Farago and Tédongap (2018). Performance of all models is assessed based on their root mean squared pricing error (RMSPE), which is widely used metric for assessing model fit in asset pricing literature. All the competing models are estimated for comparison purposes without any restrictions except that the market price of risk is correctly priced (equal to the average market return over the observed period) using OLS. Thus, GDA3 and GDA5 are despite their theoretical background estimated without setting any restriction to their coefficients and are also estimated in two stages.

## 5.2 Data

To illustrate the main findings, we use three popular datasets. First, we look at 30 Fama-French industry portfolios sampled monthly between July 1926 and November 2017 (1097 observations). These data satisfy the need of our model to posses long enough history in order to obtain reliable results. In Appendix E, we report also results for 25 Fama-French portfolios sorted on size and book-to-market over the same time span. In our empirical investigation, we measure aggregate wealth by the market portfolio[13]. Specifically, the excess market return is computed using value-weight average return on all CRSP stocks and Treasury bill rate from Ibbotson Associates[14].

Since most of the state of the art models do not perform very well on a daily sampling frequency, we also challenge our models with daily data. We employ the same datasets as in the main analysis but with daily frequency. The sampling period for daily data is between July 1926 and March 2019. The performance in this case will be also compared with above mentioned competing models.

Finally, we also use individual stock data from the CRSP database. These data are sampled monthly between January 1926 and December 2015.

[12] As shown in Shanken (1992), if the betas are estimated over the whole period, the second stage regression is $T$-consistent.

[13] This is not an unusual assumption in the asset pricing literature. The same approach is adopted in original versions of every competing model considered in our analysis to avoid problems with consumption data. Campbell (1993) gives a justification for this simplification in case of preferences of Epstein and Zin (1989).

[14] All the data were obtained from Kenneth French's online data library.

| | Tail market risk | | | | Extreme volatility risk | | | |
|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\lambda^{\text{TR}}_{\text{long}}$ | $\lambda^{\text{TR}}_{\text{short}}$ | $\lambda^{\text{CAPM}}$ | RMSPE | $\lambda^{\text{EV}}_{\text{long}}$ | $\lambda^{\text{EV}}_{\text{short}}$ | $\lambda^{\text{CAPM}}$ | RMSPE |
| 0.01 | 0.18 | 1.14 | 0.66 | 14.66 | 0.32 | -1.01 | 0.66 | 23.61 |
| | (3.50) | (7.32) | | | (7.85) | (-1.61) | | |
| 0.05 | 0.86 | 1.10 | 0.66 | 20.15 | 0.45 | -3.26 | 0.66 | 17.70 |
| | (4.29) | (1.82) | | | (11.72) | (-2.84) | | |
| 0.1 | 1.38 | 0.85 | 0.66 | 19.65 | 0.55 | -2.61 | 0.66 | 16.44 |
| | (5.19) | (1.14) | | | (14.62) | (-2.22) | | |
| 0.15 | 1.53 | 1.26 | 0.66 | 17.51 | 0.71 | -3.64 | 0.66 | 14.41 |
| | (8.45) | (1.58) | | | (14.97) | (-3.35) | | |
| 0.25 | 2.36 | 1.24 | 0.66 | 21.24 | 0.93 | -4.06 | 0.66 | 16.64 |
| | (6.77) | (1.14) | | | (14.14) | (-2.91) | | |

**Table 1:** Estimated coefficients. Prices of risk of two versions of three-factor model estimated on monthly data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and November 2017. Models are estimated for various values of thresholds. Market price of risk is imposed to be equal to the average market return.

## 5.3 Estimation results

### 5.3.1 Three-factor models

We report estimation results of the three-factor models in the Table 1. To take into account multiple hypothesis testing, we follow Harvey et al. (2016) and report $t$-statistics of estimated parameters (in parenthesis). Regarding the TR model, beta for short-horizon is more significant for $\tau$ being equal to 0.01, in the rest of the cases, beta for long horizon is more significant. In case of the EVR model, beta for the long-horizon is more significant determinant of risk premium for all the values of $\tau$ in comparison to short-horizon beta. We can see that the TR model outperforms the EVR model for $\tau$ being equal to 0.01.

### 5.3.2 Full model

As a preliminary investigation, we conduct an analysis in which we examine tail risk and extreme volatility risk without taking into consideration the frequency aspect. To do that, we employ our simple model. Estimated coefficients can be found in left panel of Table 2. We can observe that TR is significantly priced across low quantiles with expected positive sign. Extreme volatility risk is significantly priced for 10%, 15%, and 25% quantiles suggesting that investors price dependence between assets and market volatility, but focus on more probable market situations. RMSPE of the model for various market threshold defined as $\tau$ quantile of market return is depicted in left panel of Figure 2. We can deduce that better fit is obtained for lower values of thresholds and for very low $\tau$ it outperforms the best performing GDA5 model. For higher values of $\tau$, RMSPE of our simple model exceeds RMSPE of GDA5 model suggesting that indeed extreme risks of the assets are priced factor.

Estimated parameters of the full model can be found in the right panel of Table 2. We observe that significant determinants of the risk are short tail risk and long extreme volatility risk, both significantly priced across portfolios with expected signs. Tail risk is more significant for lower values of $\tau$ meaning that dependence between market return and portfolio return during extremely negative events is a significant determinant of risk premium. On the other hand, long-run extreme volatility risk is significantly priced across all values of $\tau$, but becomes more prominent for higher values of the quantile. We can deduce that price of long-run risk of Bansal and Yaron (2004) is hidden in this coefficient. Coefficients of the prices of risk for long tail risk

| $\tau$ | Simple model | | | | Full model | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\lambda^{\mathrm{TR}}$ | $\lambda^{\mathrm{EV}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE | $\lambda^{\mathrm{TR}}_{\mathrm{long}}$ | $\lambda^{\mathrm{TR}}_{\mathrm{short}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{long}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{short}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE |
| 0.01 | 1.56 | 0.35 | 0.66 | 15.57 | -0.12 | 1.14 | 0.22 | 0.02 | 0.66 | 13.74 |
| | (10.96) | (0.80) | | | (-0.71) | (7.01) | (1.88) | (0.06) | | |
| 0.05 | 2.41 | 0.48 | 0.66 | 20.78 | -0.45 | 1.27 | 0.50 | -2.93 | 0.66 | 15.87 |
| | (4.72) | (0.55) | | | (-0.96) | (2.49) | (2.86) | (-2.71) | | |
| 0.1 | 2.07 | 1.61 | 0.66 | 19.69 | -0.15 | 0.81 | 0.51 | -2.23 | 0.66 | 15.94 |
| | (3.48) | (2.34) | | | (-0.29) | (1.25) | (3.52) | (-1.82) | | |
| 0.15 | 3.67 | 1.58 | 0.66 | 19.85 | 0.61 | 0.72 | 0.46 | -3.51 | 0.66 | 13.30 |
| | (4.80) | (1.88) | | | (1.60) | (1.13) | (3.50) | (-3.35) | | |
| 0.25 | 4.50 | 3.26 | 0.66 | 25.44 | 0.69 | 0.71 | 0.68 | -3.86 | 0.66 | 15.61 |
| | (4.15) | (2.34) | | | (1.35) | (0.85) | (4.37) | (-2.83) | | |

**Table 2:** Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and November 2017. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return.

and short extreme volatility risk posses negative sign, which may seem counterintuitive. This may suggests that investors are extremely averse to long-run dependence between extremely negative returns and high volatility but at the same time exposure to the extreme volatility risk in the short run is desirable as the prices will adjust to the market turmoil quickly. Tail risk in the long run for lower quantiles is also negative but the coefficients are not significant.

One potential explanation for the results that the long-run TR is not significantly priced is that only a small fraction of the market return is due to the long-term component in comparison to short-term risk premium, and thus the risk premium for this risk is only small. Moreover, the long-term aspect of the risk may by fully captured by the extreme volatility risk. Variance is much more persistent than the market return (high portion of variance due to the long-term part) and thus the investors fear the variability in long-term variance much more than the variance in the short term.

In Figure 1, we compare performance of our QS models, QS05 ($\tau = 0.05$) and QS10 ($\tau = 0.10$), with various other models. It is notable that CAPM, and DR1 model completely fail to price the portfolios, better fit and lower RMSPE is obtained by GDA3 and GDA5 models. Finally, the best fit is provided by our QS models since returns lie closer to the 45 degree line. Right panel of Figure 2 depicts performance of the QS model against market thresholds given by $\tau$ quantile of market distribution. We observe better performance of our model in comparison to GDA5 model for all threshold values below 30% market quantile, and generally very good performance for low values of threshold suggesting that extreme risks are significant determinants of risk premium.

Moreover, to compare our model with a model that is not based on a specific measure of market (volatility) risk, we estimate 3-factor model of Fama and French (1993). Same as in the case of other models, we restrict the risk premium of market risk to be equal to the average market return, and obtain RMSPE of 23.72[15]. This shows the overall strength of our models to deliver quality results.

[15]Without any restriction on risk premium, their model produces RMSPE of 18.01, which is also considerably higher than RMSPE of our models.

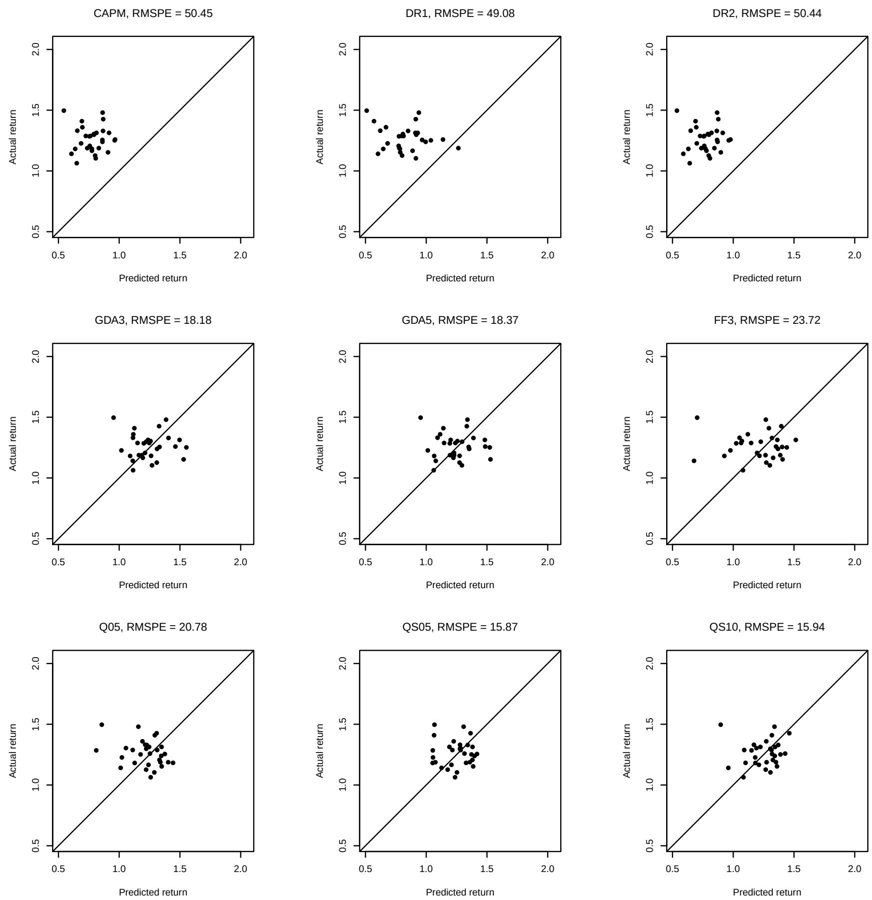


**Figure 1:** Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French equal-weight industry portfolios.

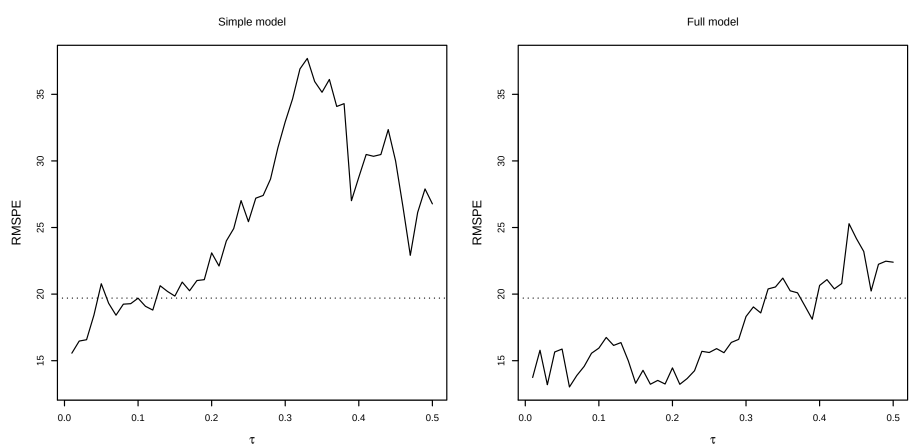


**Figure 2:** RMSPE for simple and full model estimated on monthly data of 30 Fama-French equal-weight industry portfolios for various values of threshold given by $\tau$ quantile of market returns. Horizontal line represents RMSPE of GDA5 model.

| $\tau$ | Tail market risk $\lambda^{TR}$ | $\lambda^{TR}_{long}$ | $\lambda^{TR}_{short}$ | $\lambda^{CAPM}$ | RMSPE | Extreme volatility risk $\lambda^{EV}$ | $\lambda^{EV}_{long}$ | $\lambda^{EV}_{short}$ | $\lambda^{CAPM}$ | RMSPE |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.31 | 0.15 | 0.90 | 0.66 | 14.62 | 3.85 | -0.01 | -3.58 | 0.66 | 21.35 |
| | (0.41) | (2.05) | (1.45) | | | (2.45) | (-0.05) | (-2.99) | | |
| 0.05 | 0.53 | 0.74 | 0.73 | 0.66 | 20.14 | 0.17 | 0.43 | -3.41 | 0.66 | 17.70 |
| | (0.18) | (1.05) | (0.35) | | | (0.04) | (0.80) | (-0.78) | | |
| 0.1 | 3.09 | 0.71 | -1.64 | 0.66 | 19.34 | -3.75 | 0.98 | 0.73 | 0.66 | 16.18 |
| | (0.93) | (0.92) | (-0.59) | | | (-0.94) | (2.15) | (0.20) | | |
| 0.15 | 8.03 | 0.14 | -5.72 | 0.66 | 16.12 | -8.88 | 1.38 | 4.47 | 0.66 | 13.30 |
| | (2.20) | (0.21) | (-1.75) | | | (-2.17) | (4.40) | (1.15) | | |
| 0.25 | 6.42 | 1.58 | -4.43 | 0.66 | 20.63 | -1.09 | 1.00 | -3.06 | 0.66 | 16.63 |
| | (1.27) | (2.24) | (-0.97) | | | (-0.19) | (2.80) | (-0.55) | | |

**Table 3:** Estimated coefficients from the horse race estimation. Prices of risk of simple 3-factor models also including the simple betas for the respective risks estimated on monthly data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and November 2017. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return.

## 5.4 Disentangling model performance

We will answer the question whether the performance of our model is driven by the quantile decomposition of the risk only, or the frequency decomposition brings a significant improvement over the simple specification. To do that, we employ horse race between betas from the simple model and betas from the full model, and assess the significance of the prices of risk for given betas based on their $t$-statistics. We will run the regression separately for the tail market risk betas and extreme volatility risk betas.

The results can be seen in Table 3 and they clearly indicate that the frequency decomposition of risk is a valuable dimension to explore. First, for the TR model we observe that, especially for the lowest quantile, i.e. $\tau = 0.01$, frequency decomposed measures outperform the simple measure of tail market risk. For the models given by the higher quantiles, we observe ambiguous results and cannot clearly decide whether the performance is more driven by the quantile definition of risk only. Moreover, the values of the coefficients (both simple and full model) vary significantly probably because of the correlation between these measures. In that case, we argue that the frequency decomposition is valuable as it is not decisively outperformed by the simple quantile measure. Moreover, the best performance of the model is achieved for $\tau = 0.01$ and in that case the long- and short-term betas drive out the simple beta.

Second, the results of the EVR model are less indecisive. For the low values of the quantiles, the decomposition into horizons is outperformed by the simple measure of extreme volatility risk. On the other hand, with increasing value of quantile, we can see that disentangling the risk into long and short horizon brings a valuable information and moreover, the performance improves in comparison to the low values of $\tau$.

## 5.5 GDA and QS measures of risk

In this subsection, we compare the performance of our model with the GDA5 model of Farago and Tédongap (2018). To do that, we construct horse race regressions between their measures of risk and ours. We compare risk measures associated with market return and market volatility increments separately. The aim of this analysis is to decide which measures of risk better capture the notion of extreme risks associated with risk premium.

The results are depicted in Table 4. In case of tail market risk, we see that GDA5 measures of risk ($\lambda_D$ and $\lambda_{WD}$) do not drive out our measures of risk for any value of $\tau$. This clearly suggests that our measures are better in capturing the asymmetric risk priced in the cross-section of assets.

In case of volatility risk, we see from Table 4 that the situation is pretty much the same as in the case of tail market risk. Especially, the price of risk for long-term EVR betas stays strong over all values of quantile, and GDA5 measures of volatility risk remain insignificant in all of the cases.

All the results suggest that our model brings an improvement in terms of identifying form of asymmetric risk which is priced in the cross-section of asset returns.

| | Tail market risk | | | | | | Extreme volatility risk | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\lambda_D$ | $\lambda_{WD}$ | $\lambda^{\mathrm{TR}}_{\mathrm{long}}$ | $\lambda^{\mathrm{TR}}_{\mathrm{short}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE | $\lambda_X$ | $\lambda_{XD}$ | $\lambda^{\mathrm{EV}}_{\mathrm{long}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{short}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE |
| 0.01 | -0.03 | 0.04 | -0.14 | 0.89 | 0.66 | 12.42 | 2.73 | -1.12 | 0.17 | -0.84 | 0.66 | 22.26 |
| | (-0.69) | (0.12) | (-0.97) | (4.66) | | | (1.21) | (-0.57) | (1.85) | (-1.30) | | |
| 0.05 | -0.02 | 0.14 | -0.14 | 1.25 | 0.66 | 17.08 | -1.73 | 1.36 | 0.48 | -3.72 | 0.66 | 17.44 |
| | (-0.36) | (0.36) | (-0.39) | (2.28) | | | (-0.87) | (0.86) | (4.54) | (-2.83) | | |
| 0.1 | 0.01 | 0.28 | 0.33 | 0.98 | 0.66 | 17.28 | -1.98 | 1.34 | 0.63 | -3.22 | 0.66 | 16.07 |
| | (0.19) | (0.72) | (0.71) | (1.41) | | | (-1.07) | (0.89) | (5.77) | (-2.43) | | |
| 0.15 | -0.02 | -0.03 | 0.93 | 1.01 | 0.66 | 16.17 | 0.07 | -0.51 | 0.78 | -3.69 | 0.66 | 14.21 |
| | (-0.45) | (-0.07) | (2.40) | (1.30) | | | (0.05) | (-0.42) | (7.02) | (-3.32) | | |
| 0.25 | -0.03 | 0.04 | 1.02 | 0.66 | 0.66 | 16.90 | 1.56 | -1.10 | 0.85 | -4.14 | 0.66 | 16.34 |
| | (-0.44) | (0.10) | (2.09) | (0.72) | | | (0.96) | (-0.79) | (5.42) | (-2.89) | | |

**Table 4:** Estimated coefficients from the horse race estimation between QS measures of risk and GDA5 measures of risk. Prices of risk of 3-factor models including respective GDA5 measures of risk estimated on monthly data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and November 2017. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return.

### 5.6 Robustness checks

As a robustness check, we first report results based on 30 Fama-French industry portfolio data which are value-weight. Results are summarized in Appendix C.1. We report estimated coefficients for both simple and full model, RMSPE for continuum of $\tau$ and comparison with competing models. We also conduct the same analysis with volatility being computed from daily data as a realized volatility for each month in the sample. It is obvious from estimated simple models that both tail market risk and extreme volatility risk are priced in cross-section. Estimated full models suggest that short tail risk is the driving force of aggregated tail risk, and although coefficients for long extreme volatility risk are not significant, they posses the right sign and are numerically close to their counterparts computed on volatility from GARCH model. We argue that this is due to highly non-smoothed nature of the volatility computed as a sum over respective months. On the other hand, EVR is consistently priced using the Simple 3-factor model. This seems natural as the realized volatility poses non-smoothed nature and the frequency decomposition is not so effective as in the case of smoothed volatility estimates as in the case of GARCH model.

In Appendix E, we perform the same analysis on 25 Fama-French portfolios sorted on size and book-to-market. We report results based on both equal and value-weight portfolios, and volatility is computed using GARCH model and as realized volatility from daily data. In the case of models with volatility computed from GARCH model, our model performs comparable to GDA5 model but slightly worse, but outperforms all the other competing models, and moreover, all the features observed in the case of 30 industry portfolios are present in this case, also, with values of the coefficients being similar. In the case of volatility computed from daily data, our model outperforms all the competing models including GDA5 model.

| $\tau$ | Simple model $\lambda^{TR}$ | $\lambda^{EV}$ | $\lambda^{CAPM}$ | RMSPE | Full model $\lambda^{TR}_{long}$ | $\lambda^{TR}_{short}$ | $\lambda^{EV}_{long}$ | $\lambda^{EV}_{short}$ | $\lambda^{CAPM}$ | RMSPE |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 1.31 | 0.31 | 0.65 | 35.90 | 0.08 | 1.03 | 0.10 | 0.07 | 0.65 | 34.28 |
| | (12.79) | (1.01) | | | (0.55) | (6.92) | (0.88) | (0.24) | | |
| 0.05 | 2.86 | 1.22 | 0.65 | 33.41 | -0.00 | 1.60 | 0.45 | 0.48 | 0.65 | 33.20 |
| | (10.21) | (2.66) | | | (-0.01) | (4.49) | (4.26) | (0.98) | | |
| 0.1 | 3.13 | 3.67 | 0.65 | 34.27 | -0.13 | 1.11 | 0.72 | 1.35 | 0.65 | 32.09 |
| | (6.35) | (5.92) | | | (-0.54) | (2.28) | (8.13) | (1.83) | | |
| 0.15 | 4.17 | 4.68 | 0.65 | 35.33 | 0.14 | 1.50 | 0.78 | 1.15 | 0.65 | 31.42 |
| | (7.67) | (7.07) | | | (0.65) | (2.93) | (7.79) | (1.58) | | |
| 0.25 | 3.76 | 8.82 | 0.65 | 43.53 | -0.47 | 1.18 | 1.58 | 0.18 | 0.65 | 30.66 |
| | (5.22) | (8.91) | | | (-2.00) | (2.18) | (14.77) | (0.21) | | |

**Table 5:** Estimated coefficients. Prices of risk of Simple 3-factor and Full 5-factor model estimated on individual stocks from CRSP database. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return.

As an another robustness check for the TR betas, in the first stage regression, we standardize the returns of both market and portfolio returns using estimated volatility and estimate the TR betas using these transformed series, and the second stage regression remains the same. Volatility is estimated for each time series separately using GARCH(1,1) model for simplicity reasons. This robustness check aims to show that the betas do not solely capture the common trend in volatility present in both market and portfolio returns. Results of this analysis are captured in Table 14. We observe that, especially for the long-term TR betas, the coefficients remain significant even after this standardization procedure.

## 5.7 Individual stocks performance

We estimate both simple model and full model on individual stocks obtained from Center for Research in Security Prices database (CRSP). Our version of the database covers period between January 1926 and December 2015. For the excess market return we work with returns computed using value-weight average return on all CRSP stocks and Treasury bill rate from Ibbotson Associates. We select stocks with history longer than 70 years, which leads to the total number of stocks included in our analysis equal to 147. The data are sampled with a monthly frequency. The results are summarized in the Table 5. We observe that the results are very similar to those obtained in the analysis of 30 industry portfolios discussed earlier. For the simple model, we observe that both tail risk and extreme volatility risk are priced in the cross-section. Tail risk is more significant for low threshold values and extreme volatility risk increases with the threshold. In the case of full model, we see that the significance of the tail risk is mainly driven by its short-term component. On the other hand, the extreme volatility risk is contained in its long-term part. Overall, the RMSPE slightly decreases with the threshold values suggesting that for individual stocks the extreme volatility risk is significant determinant of risk premium. Comparison with competing models captures Figure 3. The best performance provide QS10 model, and even our simple 3-factor model can outperform GDA5 model.

## 5.8 Estimation on daily data

We estimate our simple and full models using daily data, too. First, we estimate our models on 30 Fama-French equal-weight industry portfolios and compare the results with the competing models. We know that the performance of asset pricing models, in most of the cases, is substantially worse when working with data with sampling frequency higher than one month and

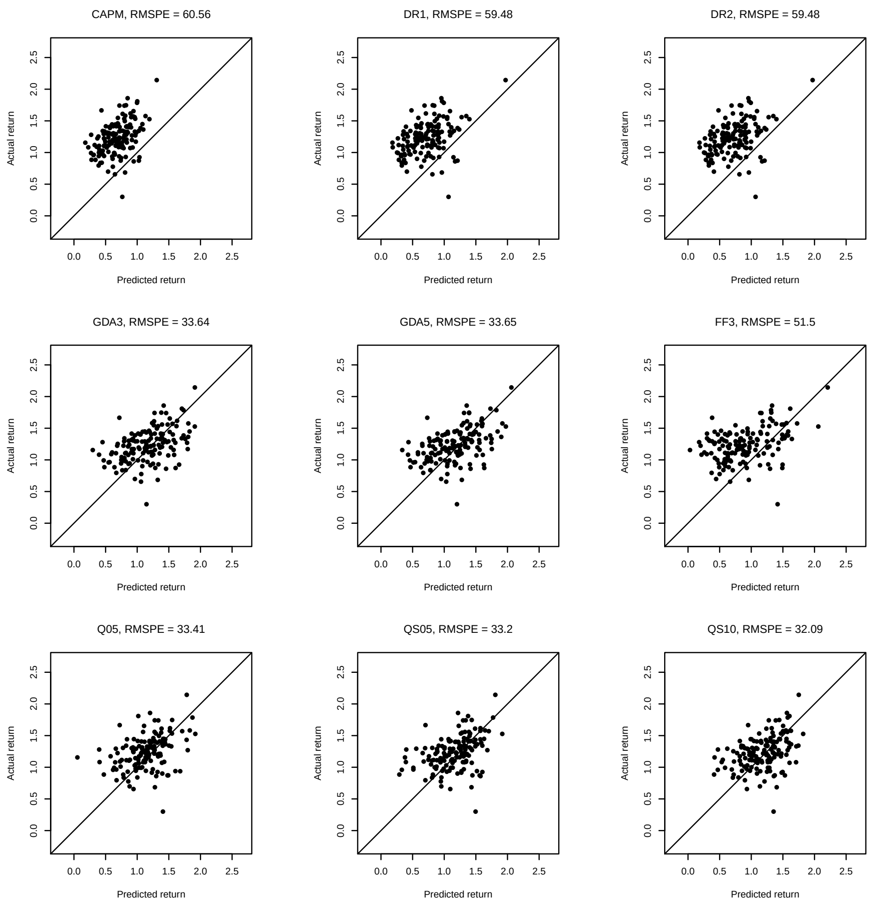


**Figure 3:** Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of individual stocks from CRSP.

|  | Simple model | | | | Full model | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\lambda^{\mathrm{TR}}$ | $\lambda^{\mathrm{EV}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE | $\lambda^{\mathrm{TR}}_{\mathrm{long}}$ | $\lambda^{\mathrm{TR}}_{\mathrm{short}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{long}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{short}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE |
| 0.01 | 0.11 | 0.24 | 0.03 | 1.02 | -0.02 | 0.16 | 0.03 | 0.26 | 0.03 | 1.02 |
|  | (3.50) | (2.66) |  |  | (-0.81) | (3.14) | (2.10) | (2.19) |  |  |
| 0.05 | 0.16 | 0.28 | 0.03 | 1.01 | -0.01 | 0.19 | 0.03 | 0.28 | 0.03 | 1.02 |
|  | (3.84) | (4.05) |  |  | (-0.27) | (2.23) | (2.13) | (1.32) |  |  |
| 0.1 | 0.20 | 0.33 | 0.03 | 1.07 | -0.02 | 0.24 | 0.03 | 0.19 | 0.03 | 1.04 |
|  | (4.34) | (4.75) |  |  | (-0.35) | (3.00) | (2.61) | (0.90) |  |  |
| 0.15 | 0.22 | 0.46 | 0.03 | 1.25 | -0.02 | 0.27 | 0.05 | 0.02 | 0.03 | 1.15 |
|  | (3.85) | (5.43) |  |  | (-0.39) | (3.46) | (3.75) | (0.09) |  |  |
| 0.25 | 0.27 | 1.01 | 0.03 | 1.73 | -0.11 | 0.40 | 0.06 | 0.59 | 0.03 | 1.37 |
|  | (3.37) | (7.50) |  |  | (-1.58) | (4.19) | (4.21) | (1.39) |  |  |

**Table 6:** Estimated coefficients. Prices of risk of Simple 3-factor and Full 5-factor model estimated on daily data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and March 2019. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return.

some of them are even useless in this case. We want to show that this is not the case of our models and that the models perform better than the other models

Estimated parameters of our simple and full model are summarized in Table 6. We can see that the coefficients for simple model are significant through all the quantiles but the best performance is achieved for low values of $\tau$. The comparison between our models and the other is depicted in Figure 4 and based on RMSPE, our model predicts the returns the best among all the tested models. Performance of the model in relation to $\tau$ is captured in Figure 5. We can see that our models achieve better performance that the GDA5 model for low values of $\tau$.

In the Appendix F, we report also estimates of the models on daily data of 25 Fama-French equal-weight portfolios sorted on size and book-to-market. We choose this dataset for the analysis because our models in the robustness check procedure perform relatively the worst on this dataset when compared to the GDA5 model. So, to see whether this also holds for the daily data, we compare all the competing models on this dataset. The estimated coefficients of the simple 3-factor and full 5-factor model are reported in Table 15. From the comparisons between the competing models depicted in Figures 21 and 20, we can see that our models provide a significantly better fit to the daily data, and the full 5-factor model achieves better performance for all values of $\tau$.

To summarize, we can see that the both in terms of coefficient significancy and relative model performance, our models can provide a good fit to the daily data, which is quite uncommon among asset pricing models. This results only confirms our hypothesis that the extreme risks are priced in the cross-section of asset returns.

# 6 Conclusion

We have shown that extreme risks at different horizons are priced in cross-section of asset returns. In the paper, we argue it is important to distinguish between tail market risk and extreme volatility risk. Tail market risk is characterized by the dependence between highly negative market and asset events. Extreme volatility risk is defined as co-occurrence of extremely high increases of market volatility and highly negative asset returns. Negative events are derived from distribution of market returns and its respective quantile is used for determining threshold values for computing quantile spectral betas. We define two empirical models for testing associated risk premium. Simple model, which does not take into consideration

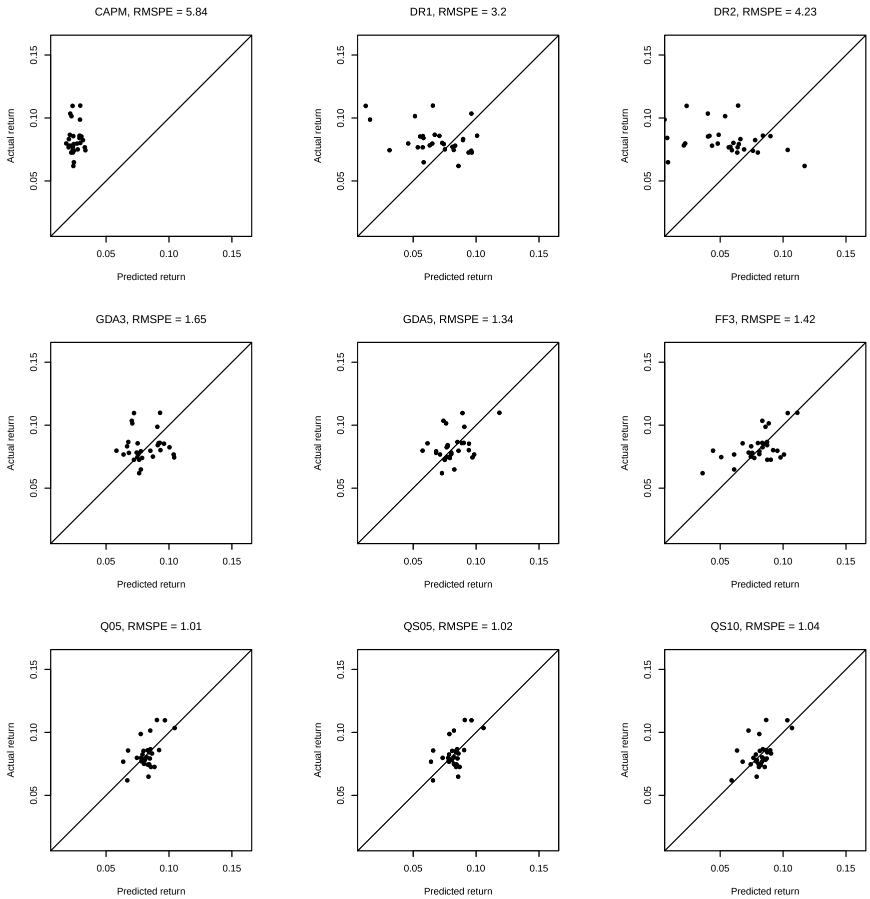


**Figure 4:** Predicted returns. Plots of predicted versus actual returns for competing models estimated on daily data of 30 Fama-French equal-weight industry portfolios.

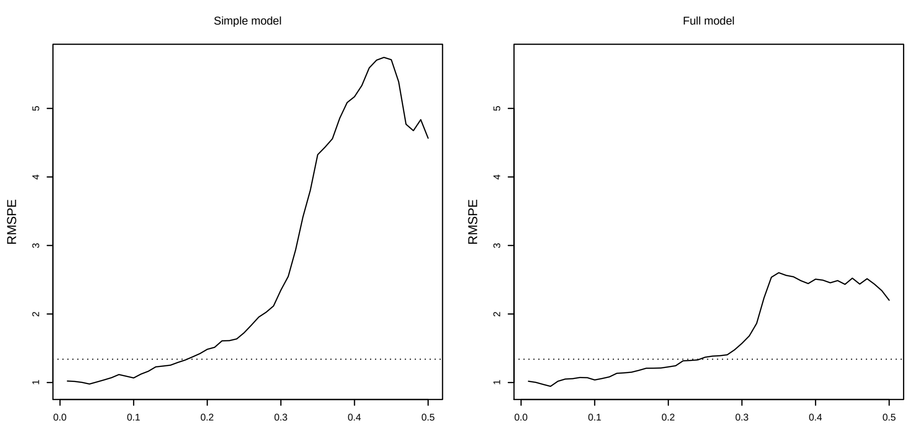


**Figure 5:** RMSPE for simple and full model estimated on daily data of 30 Fama-French equal-weight industry portfolios. Horizontal line represents RMSPE of GDA5 model.

frequency aspect, confirms that investors require premium for bearing both tail market risk and extreme volatility risk. Full model further identifies that premium for tail market risk is mostly featured in its short-term component, and premium for extreme volatility risk is mostly associated with its long-term component.

In order to consistently estimate the model, data with long enough history has to be employed. But if the data are available, our model is able to outperform competing models and its performance is best for low threshold values suggesting that investors require risk premium for holding assets susceptible to extreme risks. Moreover, our models can perform very well even on the daily data, which is not common for asset pricing models.

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# A Asset pricing model with horizon specific risk

## A.1 General result

In case of general form of curvature parameter $\gamma_t$, we obtain the following pricing equation

$$\begin{aligned}
\mathbb{E}_t(R) - r_t^f dt &= \gamma_t\Big[\mathbb{C}ov_t\Big(s_t\sigma_1 dZ_1, dZ\Big) + \mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, dZ\Big)\Big] \\
&= \frac{\mathbb{C}ov_t\Big(s_t\sigma_1 dZ_1, dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)}\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)\gamma_t \\
&+ \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, dZ\Big)}{\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)}\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)\gamma_t \\
&= \beta_t^1\lambda_t^1 + \beta_t^2\lambda_t^2
\end{aligned}$$

where we define the $\beta$'s and $\lambda$'s for long and short horizon risk, respectively, as

$$\begin{aligned}
\beta_t^1 \equiv \frac{\mathbb{C}ov_t\Big(s_t\sigma_1 dZ_1, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)}, \quad \beta_t^2 \equiv \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, \sigma dZ\Big)}{\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)} \\
\lambda_t^1 \equiv \gamma_t\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big), \quad \lambda_t^2 \equiv \gamma_t\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big).
\end{aligned} \tag{25}$$

## A.2 Power utility

Here, we derive the formulas for 2-factor asset pricing model including horizon specific risk in case that the representative investor posses power utility. In this case, the curvature parameter is constant and equal to $\xi$.

$$\begin{aligned}
\mathbb{E}_t(R) - r_t^f dt &= \frac{\mathbb{C}ov_t\Big(s_t\sigma_1 dZ_1, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)}\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)\xi \\
&+ \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, \sigma dZ\Big)}{\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)}\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)\xi \\
&= \beta_t^1\lambda_t^1 + \beta_t^2\lambda_t^2
\end{aligned}$$

where the betas (quantities of risk) for long and short horizon risk are defined as

$$\begin{aligned}
\beta_t^1 &\equiv \frac{\mathbb{C}ov_t\Big(s_t\sigma_1 dZ_1, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)} = \frac{\rho_1\sigma dt}{s_t\sigma_1 dt}, \\
\beta_t^2 &\equiv \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, \sigma dZ\Big)}{\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)} = \frac{\rho_2\sigma dt}{(1-s_t)\sigma_2 dt},
\end{aligned}$$

and lambdas (prices of risk) read as

$$\lambda_t^1 \equiv \xi \mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big) = \xi s_t^2\sigma_1^2 dt,$$
$$\lambda_t^2 \equiv \xi \mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big) = \xi(1-s_t)^2\sigma_2^2 dt$$

### A.3 Asymmetric utility

Asymmetric utility leads to a model which can be decomposed into 4 factors. 2 factors consist of risk below the reference point (for long and short horizon) and 2 factors stand for risk above the reference point. More specifically, knowing that the relative risk aversion reads as

$$\gamma_t = -\frac{c_t u''(c_t)}{u'(c_t)} = c_t\epsilon I\{c_t \geq c_{0,t}\} - c_t\frac{\epsilon}{\delta} I\{c_t < c_{0,t}\},$$

the risk premium can be rewritten in the following way

$$\begin{aligned}
\mathbb{E}_t(R) - r_t^f &= \gamma_t \mathbb{C}ov_t\Big(\frac{dc_t}{c_t}, \frac{dp_t}{p_t}\Big) \\
&= c_t\epsilon I\{c_t \geq c_{0,t}\}\mathbb{C}ov_t\Big(\frac{dc_t}{c_t}, \frac{dp_t}{p_t}\Big) - c_t\frac{\epsilon}{\delta} I\{c_t < c_{0,t}\}\mathbb{C}ov_t\Big(\frac{dc_t}{c_t}, \frac{dp_t}{p_t}\Big) \\
&= \beta_t^1\lambda_t^1 + \beta_t^2\lambda_t^2 + \beta_t^1\lambda_t^3 + \beta_t^2\lambda_t^4
\end{aligned}$$

where the quantities of risk posses the following form

$$\beta_t^1 \equiv \frac{\mathbb{C}ov_t\Big((s_t\sigma_1 dZ_1, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)} I\{c_t \geq c_{0,t}\} = \frac{\rho_1\sigma dt}{s_t\sigma_1 dt} I\{c_t \geq c_{0,t}\}$$

$$\beta_t^2 \equiv \frac{\mathbb{C}ov_t\Big((s_t\sigma_1 dZ_1, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)} I\{c_t < c_{0,t}\} = \frac{\rho_1\sigma dt}{s_t\sigma_1 dt} I\{c_t < c_{0,t}\}$$

$$\beta_t^3 \equiv \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, \sigma dZ\Big)}{\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)} I\{c_t \geq c_{0,t}\} = \frac{\rho_{2,}\sigma dt}{(1-s_t)\sigma_2 dt} I\{c_t \geq c_{0,t}\}$$

$$\beta_t^4 \equiv \frac{\mathbb{C}ov_t\Big((1-s_t)\sigma_2 dZ_2, \sigma dZ\Big)}{\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_2\Big)} I\{c_t < c_{0,t}\} = \frac{\rho_2\sigma dt}{(1-s_t)\sigma_2 dt} I\{c_t < c_0\},$$

and prices of risk are

$$\begin{aligned}
\lambda_t^1 &\equiv \mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)c_t\epsilon = c_t\epsilon s_t^2\sigma_1^2 dt \\
\lambda_t^2 &\equiv -\mathbb{V}ar_t\Big(s_t\sigma_1 dZ_1\Big)c_t\frac{\epsilon}{\delta} = -c_t\frac{\epsilon}{\delta}s_t^2\sigma_1^2 dt \\
\lambda_t^3 &\equiv \mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)c_t\epsilon = c_t\epsilon(1-s_t)^2\sigma_2^2 dt \\
\lambda_t^4 &\equiv -\mathbb{V}ar_t\Big((1-s_t)\sigma_2 dZ_2\Big)c_t\frac{\epsilon}{\delta} = -c_t\frac{\epsilon}{\delta}(1-s_t)^2\sigma_2^2 dt.
\end{aligned}$$

# B Estimation of quantile spectral betas

Estimation of QS betas defined in our paper is based on the smoothed quantile cross-periodograms studied in Baruník and Kley (2019). For a strictly stationary time series $X_{0,j},\dots,X_{n-1,j}$, we define $I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} = I\{R_{n;t,j} \leq n\tau\}$ where $\hat{F}_{n,j}(x) \equiv n^{-1}\sum_{t=0}^{n-1} I\{X_{t,j} \leq x\}$ is the empirical distribution function of $X_{t,j}$ and $R_{n;t,j}$ denotes the rank of $X_{t,j}$ among $X_{0,j},\dots,X_{n-1,j}$. We have seen that the cornerstone of quantile spectral beta is quantile cross-spectral density defined in Equation 5. Its population counterpart is called rank-based copula cross-periodogram, CCR-periodogram, and is defined as

$$I_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2) \equiv \frac{1}{2\pi n} d_{n,R}^{j_1}(\omega;\tau_1) d_{n,R}^{j_2}(-\omega;\tau_2) \tag{26}$$

where

$$d_{n,R}^{j}(\omega;\tau) \equiv \sum_{t=0}^{n-1} I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} e^{-i\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t,j} \leq n\tau\} e^{-i\omega t}, \quad \tau \in [0,1]. \tag{27}$$

As discussed in Baruník and Kley (2019), CCR-periodogram is not a consistent estimator of quantile cross-spectral density. Consistency can be achieved by smoothing CCR-periodogram across frequencies. Following Baruník and Kley (2019), we employ the following

$$\hat{G}_{n,R}^{j_1,j_1}(\omega;\tau_1,\tau_2) \equiv \frac{2\pi}{n} \sum_{s=0}^{n-1} W_n(\omega - 2\pi s/n) I_{n,R}^{j_1,j_2}(2\pi s/n,\tau_1,\tau_2) \tag{28}$$

where $W_n$ is defined in Section 3 of Baruník and Kley (2019). Estimator of quantile spectral beta is defined as

$$\hat{\beta}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2) \equiv \frac{\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)}{\hat{G}_{n,R}^{j_2}(\omega;\tau_2)}. \tag{29}$$

Consistency of the estimator can be proven using exactly same logic as in Theorem 3.4 in Baruník and Kley (2019) by replacing quantile coherency with quantile spectral beta.

# C Robustness checks

## C.1 Realized volatility

| | Simple model | | | | Full model | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\lambda^{\mathrm{TR}}$ | $\lambda^{\mathrm{EV}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE | $\lambda^{\mathrm{TR}}_{\mathrm{long}}$ | $\lambda^{\mathrm{TR}}_{\mathrm{short}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{long}}$ | $\lambda^{\mathrm{EV}}_{\mathrm{short}}$ | $\lambda^{\mathrm{CAPM}}$ | RMSPE |
| 0.01 | 1.00 | 0.45 | 0.66 | 13.14 | 0.04 | 0.80 | -0.03 | 0.64 | 0.66 | 12.70 |
| | (5.20) | (3.72) | | | (0.37) | (4.33) | (-0.39) | (2.83) | | |
| 0.05 | 1.23 | 0.57 | 0.66 | 18.65 | 0.17 | 0.97 | 0.05 | 0.66 | 0.66 | 17.68 |
| | (2.12) | (2.65) | | | (0.38) | (1.57) | (0.16) | (1.30) | | |
| 0.1 | 0.72 | 0.91 | 0.66 | 16.28 | 0.24 | 0.82 | 0.18 | 0.52 | 0.66 | 17.00 |
| | (1.18) | (4.56) | | | (0.43) | (1.17) | (0.43) | (0.83) | | |
| 0.15 | 1.59 | 0.91 | 0.66 | 15.80 | 1.09 | 0.22 | -0.64 | 1.77 | 0.66 | 14.64 |
| | (2.05) | (4.66) | | | (2.45) | (0.27) | (-1.52) | (2.44) | | |
| 0.25 | 1.33 | 1.43 | 0.66 | 15.83 | 0.41 | 0.81 | 0.38 | 0.63 | 0.66 | 15.36 |
| | (1.65) | (7.61) | | | (0.83) | (0.96) | (1.41) | (1.40) | | |

**Table 7:** Estimated coefficients. Prices of risk of simple 3-factor and full 5-factor model estimated on monthly data of 30 Fama-French equal-weight industry portfolios sampled between July 1926 and November 2017. Model is estimated for various values of thresholds given by $\tau$. Market price of risk is imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

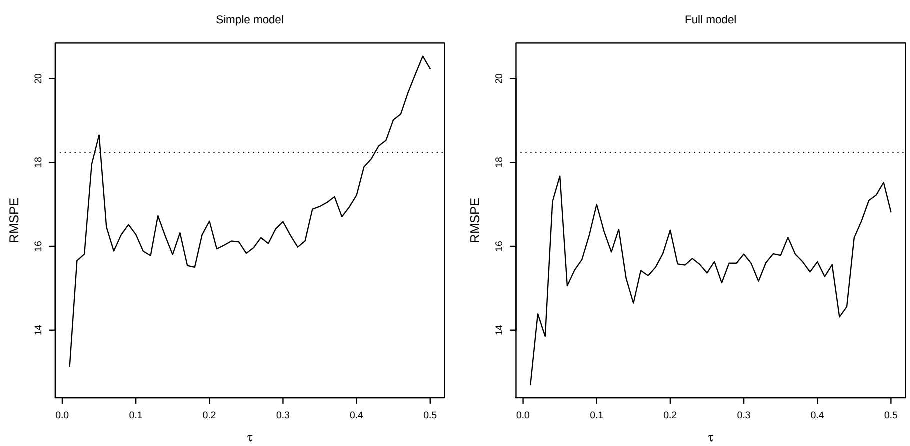


**Figure 6:** RMSPE for simple and full model estimated on monthly data of 30 Fama-French equal-weight industry portfolios for various values of threshold given by $\tau$ quantile of market returns. Horizontal line represents RMSPE of GDA5 model. Volatility is computed as realized volatility from daily data.

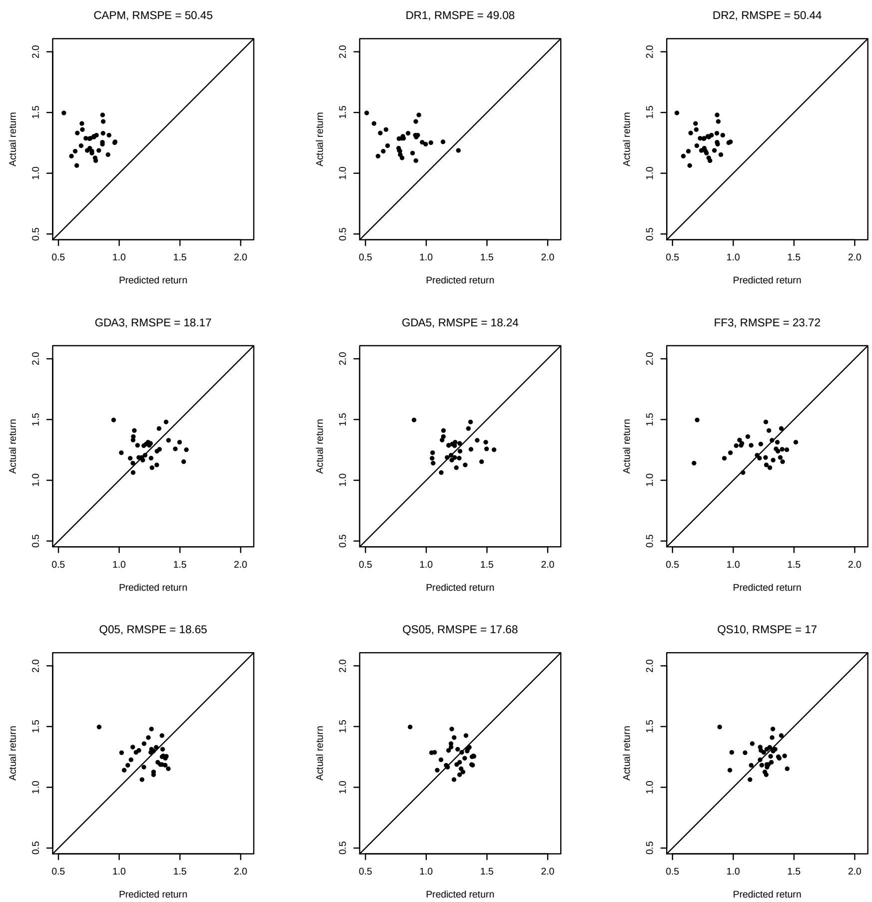


**Figure 7:** Predicted returns. Plots of predicted versus actual returns for competing models estimated on monthly data of 30 Fama-French equal-weight industry portfolios. Volatility is computed as realized volatility from daily data.