Portfolio Pretesting with Machine Learning*

Ekaterina Kazak† University of Konstanz, GSDS

Winfried Pohlmeier‡ University of Konstanz, CoFE, RCEA

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Abstract

The general idea of the pretest estimator is to develop an optimal decision making rule to switch between the two competing estimators, i.e. a procedure based on the test outcome for a given significance level to be chosen. In some circumstances it might be reasonable to select a lower significance level (higher probability of a Type I error) than a conventional one in order to increase the probability of rejecting the benchmark strategy. Particularly for the problem of deciding between the two portfolio strategies in the presence of low power of the existing tests this implies assigning to the alternative a higher probability to be pursued if it is truly superior.

In this paper we develop a data driven approach for choosing an optimal significance level for the pretest estimators. We show that the bagged pretest estimator performs exceptionally well, especially when combined with adaptive smoothing. The resulting strategy allows for a flexible and smooth switch between the underlying strategies and is shown to outperform the corresponding stand-alone strategies.

Keywords: pretest estimation, bagging, portfolio allocation

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†Department of Economics, Universitätsstraße 1, D-78462 Konstanz, Germany. Phone: +49-7531-88-2219, fax: -4450, email: Ekaterina.Kazak@uni-konstanz.de.

‡Department of Economics, Universitätsstraße 1, D-78462 Konstanz, Germany. Phone: +49-7531-88-2660, fax: -4450, email: Winfried.Pohlmeier@uni-konstanz.de.

1 Introduction

Estimation risk is a well-known issue in empirical portfolio modeling. For a given performance measure, estimation risk may cause a theoretically superior portfolio strategy to be inferior compared to simple alternatives when it comes to a comparison of the performance measures based on their estimated counterparts. The most prominent example is the equally weighted $(1/N)$ portfolio strategy, for which the null hypothesis of equal out-of-sample performance compared to a more sophisticated, theory based strategy often cannot be rejected at conventional significance levels (DeMiguel et al. (2009)).

A lot of effort in the recent literature has been devoted to stabilizing the portfolio weight estimates. Among others Jagannathan and Ma (2003) propose to impose a norm-constraint directly to the portfolio optimization for stabilizing the weight estimates in small samples; Ledoit and Wolf (2003), Ledoit and Wolf (2004), Ledoit and Wolf (2014) wrote a series of papers focusing on the improved covariance matrix estimation, which is a key ingredient in the portfolio optimization; Kourtis et al. (2012) proposed a shrinkage approach for the inverse of the covariance matrix, which can be directly used in the most of portfolio weight estimates; DeMiguel et al. (2014) introduced a VAR model to capture the serial dependence in stock returns resulting in better out-of-sample portfolio performance.

Despite all the effort no general statement can be made whether any of the approaches can outperform the equally weighted portfolio uniformly over all parameter constellations and datasets. We address the problem from a different perspective. Instead of working with the weight estimation directly we develop a flexible algorithm which optimally combines a fixed given set of weight estimates in a data-driven way with respect to a certain portfolio performance measure. In particular, we use the pretest estimator as a statistical tool to choose an optimal strategy with respect to the out-of-sample Certainty Equivalent net of the transaction costs. The pretest estimator is based on the simple t-test commonly used in the literature (Ledoit and Wolf (2008), DeMiguel et al. (2009)), where the investor decides on how to invest his wealth for the next period based on the test outcome. Similar to shrinkage strategies which combine a given portfolio strategy with the equally weighted portfolio by some optimality criterion (DeMiguel et al. (2009), Frahm and Memmel (2010)), our pretest estimator uses the information about all underlying strategies through the outcome of the performance test. However, contrary to the

shrinkage approaches, our pretest strategy can be continuously updated to incorporate the most recent information in the rolling window forecasting set-up. In the previous work Kazak and Pohlmeier (2017) show that the existing portfolio performance tests are correctly sized, but for realistic scenarios have a very low power. This implies that the pretest estimator cannot be used directly and needs to be adjusted. Our first contribution is a novel approach of optimizing the significance level for the pretest estimator.

We propose a fully data-driven and time-adaptive significance level choice, which optimizes a trade-off between Type I and Type II error with respect to Certainty Equivalent. The second contribution is introducing machine learning in the pretest estimation. To the best of our knowledge this paper is the first one combining bagging with pretest estimation in the portfolio context. We modify the classical pretest estimator replacing the indicator functions with the bootstrapped probabilities, which helps to stabilize the pretest estimator and reduces portfolio turnover. In an extensive empirical study we show that our proposed bagged pretest estimator outperforms the underlying weight estimation strategies and other competitors and is robust to different parameter constellations.

This paper is organized as follows. In Section 2 we use a simple motivating example which illustrates the problem of an optimal strategy choice and propose the novel bagged pretest estimator. Section 3 provides the reader with an empirical illustration of the proposed method. Section 4 summarizes the main findings and gives an outlook on future research.

2 Pretest Estimator

Consider a standard portfolio choice set-up with N risky assets. Let r_t be an excess return vector at time t with mean vector $\mathrm{E}[r_t] = \mu$ and variance-covariance matrix $\mathrm{V}[r_t] = \Sigma$. Moreover, let $\omega(s) = \omega_{(s)}(\mu, \Sigma)$ be the $N \times 1$ vector of portfolio weights for strategy s, e.g. $\omega(g) = \frac{\Sigma^{-1}\iota}{\iota\Sigma^{-1}\iota}$ for the global minimum variance portfolio (GMVP) minimizing the portfolio variance, $\omega(e) = \frac{1}{N}\iota$ for the equally weighted portfolio and $\omega(tn) = \frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\mu}$ for the tangency portfolio, maximizing the Sharpe Ratio. For strategy s the portfolio return at time t is given by $r_t^p(s) = \omega(s)'r_t$ with mean $\mu_p(s) = \mathrm{E}[r_t^p(s)] = \omega(s)'\mu$ and variance $\sigma_p^2(s) = \mathrm{V}[r_t^p(s)] = \omega(s)'\Sigma\omega(s)$.

Consider Certainty Equivalent (CE) as a portfolio performance measure, which is given by $CE(\omega(s)) = \mu_p(s) - \frac{\gamma}{2}\sigma_p^2(s)$ with γ being a risk aversion coefficient of the investor. The analysis below can also be extended to the Sharpe Ratio or other popular portfolio evaluation criteria, however we mainly consider CE due to its utility interpretation. In particular, CE is defined as a return which makes the investor indifferent between investing into the risky portfolio or receiving the certainty equivalent return $U(CE(\omega(s))) = \mathrm{E}[U(r^p(s))]$, with $U(\cdot)$ being the utility function of the investor (Merton and Samuelson, 1992). In the following strategy s is said to outperform strategy $\tilde{s}$ if the difference in certainty equivalents is non-negative $\Delta_0(s, \tilde{s}) = CE(\omega(s)) - CE(\omega(\tilde{s})) \geq 0$.

Given the population parameters (μ, Σ) the dominant strategy according to a chosen portfolio evaluation criteria is known, however in empirical applications the first two moments of the return process have to be estimated and the estimation risk has to be taken into account. Furthermore, investment decision is a dynamic process, therefore the financial or forecasting risk has to be accounted for as well. In empirical work the actual performance of the competing portfolio allocation strategy is often evaluated based on the out-of-sample Certainty Equivalent, which takes into account both estimation and forecasting risk (Kazak and Pohlmeier, 2017). In the following we consider a typical rolling window set-up, where for the period $t+1$ the out-of-sample portfolio return $\hat{r}_{t+1}^p(s)$ is based on a one-step forecast of the portfolio weights $\hat{\omega}_{t+1|t}(s)$ with period $\{t-T, \ldots, t\}$ as an estimation window. We adopt the standard assumption for static models that the last available estimate $\hat{\omega}_t(s)$ is used to compute the out-of-sample return for the next period: $\hat{r}_{t+1}^p(s) = \hat{\omega}_{t+1|t}(s)'r_{t+1} = \hat{\omega}_t(s)'r_{t+1}$. Estimation window is shifted one period ahead H times resulting in the $H \times 1$ vector of the out-of-sample portfolio returns

$\hat{r}^p(s)$. Different portfolio strategies are then evaluated based on the out-of-sample Certainty Equivalent $\widehat{CE}_{op}(\hat{\omega}(s))$ given by:

$$\begin{aligned}\widehat{CE}_{op}(\hat{\omega}(s)) =& \hat{\mu}_{op}(s) - \frac{\gamma}{2}\hat{\sigma}^2_{op}(s), \\ \text{where:} \qquad \hat{\mu}_{op}(s) =& \frac{1}{H}\sum_{h=1}^{H}\hat{r}^p_{t+h}(s) = \frac{1}{H}\sum_{h=1}^{H}\hat{\omega}_{t+h-1}(s)'r_{t+h}, \\ \hat{\sigma}^2_{op}(s) =& \frac{1}{H-1}\sum_{h=1}^{H}\left(\hat{r}^p_{t+h}(s) - \hat{\mu}_{op}(s)\right)^2.\end{aligned} \tag{1}$$

The driving force of the portfolio performance based on the out-of-sample CE is the estimation noise, i.e. theoretically superior strategies do not usually perform well in practice as the estimation error dominates the theoretical gain. Depending on the size of the portfolio N, length of the estimation window T portfolio performance varies dramatically.

2.1 Motivating Example

As an illustrative example consider an investor who chooses among three different strategies and wants to rebalance his portfolio monthly. Based on the monthly excess returns of 100 industry portfolios from K.R.French database[1] he estimates the weights of the following strategies:

1. GMVP based on the plug-in covariance matrix estimator

$$\hat{\omega}(g) = \frac{\hat{\Sigma}^{-1}\iota}{\iota\hat{\Sigma}^{-1}\iota}, \tag{2}$$

   where $\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(r_t - \bar{r})(r_t - \bar{r})'$ and ι is an $N \times 1$ vector of ones.

2. Tangency portfolio with a shrinked covariance matrix

$$\hat{\omega}(tn) = \frac{\hat{\Sigma}^{-1}(\lambda)\hat{\mu}}{\iota\hat{\Sigma}^{-1}(\lambda)\hat{\mu}}, \tag{3}$$

   where $\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t$ and $\hat{\Sigma}(\lambda) = \hat{\Sigma} + \lambda I_N$ with $\lambda = 0.05N$ and I_N - identity matrix of size N.

3. Equally weighted portfolio

$$\hat{\omega}(e) = \omega(e) = \frac{1}{N}\iota. \tag{4}$$

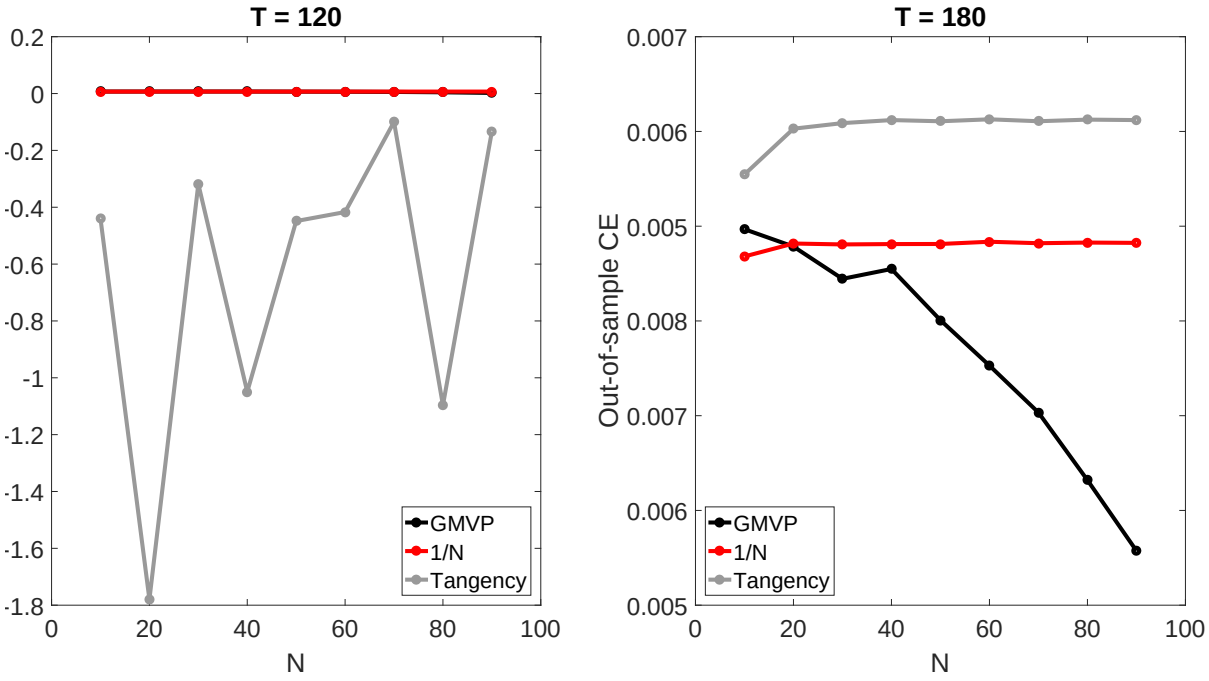
[1]The data is taken from K.R.French website and contains monthly excess returns from 01/1953 till 12/2015.

For the portfolio evaluation the length of the out-of-sample period is set to $H = 500$ and the risk aversion parameter to $\gamma = 1$. Transaction costs at period t which an investor has to pay every month after portfolio rabalancing are computed as follows:

$$TC_t(s) = c \cdot \sum_{j=1}^{N} |\hat{\omega}_{j,t+1}(s) - \hat{\omega}_{j,t^+}(s)|, \tag{5}$$

where $TC_t(s)$ denotes transaction costs at period t, $\hat{\omega}_{j,t^+}$ - portfolio weight before rebalancing at $t+1$ and c - cost per transaction (5 basis points, DeMiguel et al. (2009)). The out-of-sample CE is then computed based on the net portfolio returns $\hat{r}_t^{p,net}(s) = \hat{r}_t^p(s) - TC_t(s)$ in the very similar way to (1). For a more general comparison for a given pair of (N, T) we randomly draw N assets from the available asset space and compute the out-of-sample CE for each of the 500 random draws.

Figure 1: Out-of-sample CE for underlying strategies



Lines on the plots represent the average out-of-sample CE computed on net portfolio returns over a 500 randomly drawn portfolios of size N (x-axis). For each randomly drawn portfolio the out-of-sample CE is computed over an out-of-sample evaluation horizon of $H = 500$ observations, risk aversion parameter $\gamma = 1$. The left panel depicts the average out-of-sample CE with the weights of GMVP (in black) from (2), tangency portfolio (in gray) from (3), and equally weighted portfolio (in red) from (4) computed over an estimation window length of $T = 120$ observations. The right panel plots the average out-of-sample CE for $T = 180$.

The right panel of Figure 1 depicts the average out-of-sample CE for a grid of portfolio sizes

and the in-sample estimation window length T of 15 years (180 months). The equally weighted portfolio (in red) does not require weight estimation and is therefore stable across the portfolio sizes N. The out-of-sample CE of GMVP (black line) is deteriorating with the increase in N as the plug-in covariance matrix estimation adds more and more noise. Tangency portfolio (gray line) with shrunken covariance matrix performs very well for all N. However, on the left panel with a smaller estimation window length $T = 120$ the performance of the tangency portfolio worsens dramatically (note the difference in scaling of the y-axes). Tangency portfolio is known to produce extremely unstable weight estimates for the smaller estimation windows, while the performance of the GMVP and the equally weighted portfolio is not that sensitive. Moreover, in some circumstances it is better to keep the estimation window length smaller, e.g. due to structural breaks in the financial markets only the recent information should be included in the weight estimation.

All in all, no general statement can be done with respect to the optimal strategy for a given portfolio space and the estimation window length. In this paper we develop a data driven procedure for an optimal strategy choice, which outperforms the underlying weight estimation strategy regardless of the parameter constellations.

2.2 Machine Learning and Pretest

The pretest estimator is a statistical tool which can help the investor to decide between different strategies in a data-driven way. For a moment assume that the investor has to decide between two strategies s and $\tilde{s}$. The difference in the out-of-sample CE's between the two strategies is defined as

$$\begin{aligned}
\Delta_{op}(s, \tilde{s}) =& CE_{op}(\hat{\omega}_t(s)) - CE_{op}(\hat{\omega}_t(\tilde{s})), \\
CE_{op}(\hat{\omega}_t(s)) =& \mu_{op}(s) - \frac{\gamma}{2}\sigma^2_{op}(s), \quad \text{with} \\
\mu_{op}(s) =& \mathrm{E}\left[\hat{r}^p_{t+1}(s)\right] = \mathrm{E}\left[\hat{\omega}_t(s)\right]' \mu, \\
\sigma^2_{op}(s) =& \mathrm{V}\left[\hat{r}^p_{t+1}(s)\right] = \mathrm{E}\left[\hat{\omega}_t(s)' \Sigma\, \hat{\omega}_t(s)\right] + \mu' \mathrm{V}\left[\hat{\omega}_t(s)\right] \mu.
\end{aligned}$$

The goal is to chose either strategy s or strategy $\tilde{s}$ depending on the test outcome. Null and alternative hypotheses take the usual one-sided form:

$$H_0: \quad \Delta_{op}(s,\tilde{s}) \leq 0 \qquad \text{and} \qquad H_1: \quad \Delta_{op}(s,\tilde{s}) > 0. \tag{6}$$

Let the pretest estimator of the portfolio weights forecasts for $t+1$ be such that it depends either on strategy s in case the null is rejected or on $\tilde{s}$ otherwise:

$$\omega_t(s,\tilde{s},\alpha) = \mathbb{1}\left(\hat{\Delta}_{op}(s,\tilde{s}) > \Delta^*(\alpha)\right)\left(\omega_t(s) - \omega_t(\tilde{s})\right) + \omega_t(\tilde{s}), \tag{7}$$

with the estimated CE difference $\hat{\Delta}_{op}(s,\tilde{s}) = \widehat{CE}_{op}(s) - \widehat{CE}_{op}(\tilde{s})$ and the critical value $\Delta^*(\alpha)$ for significance level α. In other words the pretest estimator chooses the strategy s if it is significantly better than the alternative and the sensitivity of the pretest estimator depends on α: the lower is the chosen nominal level, the stricter is the pretest rule and the greater should be the difference $\hat{\Delta}_{op}(s,\tilde{s})$ for choosing the strategy s over $\tilde{s}$.

There are two main difficulties arising from applying the pretest estimator defined in (7) in practice. First of all, at time period t the investor does not know the out-of-sample CE difference $\hat{\Delta}_{op}(s,\tilde{s})$ and therefore the test decision from (6) is unknown. Secondly, in empirical applications an investor has to decide on α, which influences the performance of the pretest estimator, e.g. for $\alpha = 50\%$ the CE of strategy s has to be just slightly greater than the CE of $\tilde{s}$ in order to be chosen, whereas for the commonly used levels of significance of 1% and 5% the difference $\hat{\Delta}_{op}(s,\tilde{s})$ has to be fairly large for the null rejection. Kazak and Pohlmeier (2017) show that in realistic scenarios the empirical power of the portfolio performance tests is very low, which implies that even if the strategy s is truly superior, the pretest estimator is not able to choose the dominating strategy. Therefore going for the conservative α-level is not a reasonable choice, as in the presence of low power it will force the pretest estimator to choose $\tilde{s}$ even in cases when s is dominating. In particular, the problem of the low power calls for an optimal trade-off between Type I and Type II error. In this paper we propose a feasible pretest estimator with a data-driven and time adaptive significance level choice.

We solve the first issue of unfeasible testing by constructing the pretest weight estimator based on the in-sample CE difference which is available at time t. First, at time t the weights

$\hat{\omega}_t(s)$ are estimated based on the sample $\{t-T,\ldots,t\}$. The estimated within sample CE for the strategy s is computed as

$$\widehat{CE}_{in,t}(s) = \widehat{CE}_{in}(s|t-T,...,t) = \hat{\omega}_t(s)'\bar{r}_t - \frac{\gamma}{2}\hat{\omega}_t(s)'\hat{\Sigma}_t\hat{\omega}_t(s), \tag{8}$$

where $\bar{r}_t$ denotes the sample mean and $\hat{\Sigma}_t$ the sample covariance matrix of the returns based on the estimation window $\{t-T,\ldots,t\}$. The in-sample test statistic deciding between s and the benchmark $\tilde{s}$ is defined as

$$t_{in,t}(s,\tilde{s}) = \frac{\widehat{CE}_{in,t}(s) - \widehat{CE}_{in,t}(\tilde{s})}{S.E.\left[\widehat{CE}_{in,t}(s) - \widehat{CE}_{in,t}(\tilde{s})\right]},$$

where the standard error for the CE difference is computed via Delta method (DeMiguel et al., 2009). For the multivariate comparison where the investor chooses between M alternatives $s_1,...,s_M$ and a benchmark strategy $\tilde{s}$ and for a given significance level α the pretest estimator based on the in-sample CE is defined as

$$\begin{aligned}
\mathbb{1}(s_i,\alpha) &= \mathbb{1}\left[\max\left[t_{in,t}(s_1,\tilde{s}),...,t_{in,t}(s_M,\tilde{s}),t^*(\alpha)\right] = t_{in,t}(s_i,\tilde{s})\right], \quad i=1,...,M,\\
\hat{\omega}_{in,t}(\mathcal{S},\alpha) &= \hat{\omega}_{in,t}(s_1,..,s_M,\tilde{s},\alpha) = \sum_{i=1}^{M}\mathbb{1}(s_i,\alpha)\hat{\omega}_t(s_i) + \left(1-\sum_{i=1}^{M}\mathbb{1}(s_i,\alpha)\right)\hat{\omega}_t(\tilde{s}),
\end{aligned} \tag{9}$$

where $\mathcal{S}=\{s_1,..,s_M,\tilde{s}\}$, $t^*(\alpha)$ is the corresponding critical value for the nominal level α: the $(1-\alpha)$ quantile of the standard normal distribution. In other words, the pretest estimator chooses the strategy with the largest standardized difference from the benchmark $\tilde{s}$ and which at the same time crosses the threshold $t^*(\alpha)$.

We now address the problem of the significance level choice. Statistically speaking, increasing the nominal level in order to improve on the power of the test is not meaningful. In empirical finance however, there is a well-defined portfolio performance measure, such that the trade-off between the Type I and Type II error of the pretest estimator can be optimized with respect to it. We propose to choose a significance level α, which maximizes the in-sample CE difference. In particular, at each period t the pretest weight estimates are computed according to (9) on the grid of (0,1) α-values of length J. For every $\hat{\omega}_{in,t}(\mathcal{S},\alpha_j),\ \ j=1,..,J$ the in-sample CE of the pretest estimator is computed similarly to (8):

$$\widehat{CE}_{in,t}(\mathcal{S},\alpha_j) = \hat{\omega}_{in,t}(\mathcal{S},\alpha_j)'\bar{r}_t - \frac{\gamma}{2}\hat{\omega}_{in,t}(\mathcal{S},\alpha_j)'\hat{\Sigma}_t\hat{\omega}_{in,t}(\mathcal{S},\alpha_j). \tag{10}$$

Finally, the in-sample CE optimizing significance level α^*_{t+1} is chosen for the test, determining the strategy for the next period $t+1$:

$$\alpha^*_{in,t+1} = \arg\max_{\alpha} \widehat{CE}_{in,t}(\mathcal{S},\alpha_j). \tag{11}$$

The above procedure is repeated with every shift of the estimation window. In practice this results in a very unstable series of significance level choices $\{\alpha^*_{in,t+1}, ...\alpha^*_{in,t+H}\}$, as the choice of $\alpha^*_{in,t+1}$ is data driven and also depends on the instable estimates of the portfolio weights. On the other hand, the sequence of α^*_{in}'s along the rolling estimation window takes into account changes of the return process across time, e.g. volatility regimes. In order to mitigate the instability problem we suggest to adaptively smooth the α^*_{in}-series according to

$$\alpha^s_{t+1} = (1-\lambda)\alpha^*_{in,t+1} + \lambda\alpha^s_t, \tag{12}$$

where the tuning parameter λ is chosen to control the degree of smoothness. The adaptive smoothing takes into account not only the latest optimal choice $\alpha^*_{in,t+1}$ but also the previous estimates with geometrically decaying weights. The smoothing parameter λ is chosen via a grid search in the similar fashion as the $\alpha^*_{in,t+1}$: for a given couple $(\alpha^*_{in,t+1}, \alpha^s_t)$ the in-sample CE is computed on the grid of λ's and the optimal λ is the one maximizing the in-sample CE of the pretest estimator $\widehat{CE}_{in,t}(\mathcal{S},\alpha^s_{t+1})$.

Another feasible pretest estimator may be obtained by performing a pseudo-out-of-sample exercise which is commonly used for parameter training in machine learning. The goal of the pretest estimator is to choose an optimal strategy in a data-driven way which results in the highest out-of-sample CE. Choosing the strategy based on the in-sample comparison does not necessarily provide a good out-of-sample choice: the pretest estimator as defined in (9) does not take into account transaction costs and the forecasting risk. A feasible out-of-sample pretest estimator might be obtained by dividing the within-sample period into two parts, where the first part is used for the weight estimation and the second part is used for the pseudo-out-of-sample return computation. The optimal strategy is then defined as the one having the largest

pseudo-out-of-sample CE net of the transaction costs. In particular, the weights for the strategy s are computed based on the sample of length $T/2$: $\{t-T,..,t-T/2\}$. The out-of-sample portfolio returns are computed in the rolling window of length $T/2$ and the transaction costs are subtracted from the out-of-sample returns at each time point t as in (5). The resulting pseudo-out-of-sample CE is similar to (1):

$$\widehat{CE}^{*}_{op,t}(\hat{\omega}(s)) = \hat{\mu}^{*}_{op,t}(s) - \frac{\gamma}{2}\hat{\sigma}^{*2}_{op,t}(s), \tag{13}$$

$$\text{where:} \quad \hat{\mu}^{*}_{op,t}(s) = \frac{1}{T/2}\sum_{h=-T/2+1}^{-T} \hat{r}^{p}_{t+h}(s) = \frac{1}{T/2}\sum_{h=-T/2+1}^{-T} \hat{\omega}_{t+h-1}(s)' r_{t+h},$$

$$\hat{\sigma}^{*2}_{op,t}(s) = \frac{1}{T/2-1}\sum_{h=-T/2+1}^{-T} \left(\hat{r}^{p}_{t+h}(s) - \hat{\mu}_{op,t}(s)\right)^2.$$

The out-of-sample pretest estimates of portfolio weights are computed based on the difference in the pseudo-out-of-sample CE:

$$t_{op,t}(s_i,\tilde{s}) = \frac{\widehat{CE}^{*}_{op,t}(s) - \widehat{CE}^{*}_{op,t}(\tilde{s})}{S.E.\left[\widehat{CE}^{*}_{op,t}(s) - \widehat{CE}^{*}_{op,t}(\tilde{s})\right]}, \tag{14}$$

$$\mathbb{1}_{op}(s_i,\alpha) = \mathbb{1}\left[\max\left[t_{op,t}(s_1,\tilde{s}),...,t_{op,t}(s_M,\tilde{s}),t^{*}(\alpha)\right] = t_{op,t}(s_i,\tilde{s})\right], \quad i=1,...,M, \tag{15}$$

$$\hat{\omega}_{op,t}(\mathcal{S},\alpha) = \hat{\omega}_{op,t}(s_1,..,s_M,\tilde{s},\alpha) = \sum_{i=1}^{M}\mathbb{1}_{op}(s_i,\alpha)\hat{\omega}_t(s_i) + \left(1-\sum_{i=1}^{M}\mathbb{1}_{op}(s_i,\alpha)\right)\hat{\omega}_t(\tilde{s}). \tag{16}$$

The pseudo-out-of-sample CE from (13) is used only for indicator functions computations, and the out-of-sample pretest estimator chooses the weight estimates computed on the whole in-sample estimation window. The optimal significance level choice for the out-of-sample pretest estimator can be done in the very same way as in (12) using a grid search. Note that the out-of-sample portfolio returns used in the pseudo-out-of-sample CE are net of the transaction costs and therefore the pretest estimator takes into account the amount of rebalancing or turnover. Also note, that the estimation window length used in (13) is twice shorter than the initial estimation window, thus adding more estimation noise to the problem and making it very unstable. Moreover, the pretest estimator can be seen as a sharp thresholding choosing only one strategy for investing in the next period. In the rolling window scenario this may result in a huge amount of turnover if the pretest estimator switches between different strategies every period.

As a solution we propose to stabilize the pretest estimator using bagging. In order to do so,

the in-sample estimation window is bootstrapped B times smoothing the indicator functions of the pretest estimator by the bootstrapped probabilities. The computation of the bagged out-of-sample pretest estimator can be summarized in a following algorithm[2]:

Bagging the pretest estimator

1. At the period t define the estimation window of length T: $\{r_{t-T}, ..., r_t\}$.
2. Divide the sample into two parts and compute the pseudo-out-of-sample CE net of the transaction costs according to (13) and the out-of-sample test statistic according to (14) for each pair of strategies $(s_i, \tilde{s})$, $i = 1, ..., M$.
3. For a grid of J α values compute the out-of-sample pretest weight estimates according to (16) and choose $\alpha^*_{op,t+1}$ which results in the largest out-of-sample CE on the grid.
4. For a grid of λ values compute the smoothed $\alpha^s_{op,t+1}$ using the previous optimal significance level and choose the one maximizing the out-of-sample CE of the pretest estimator.

   For every bootstrap iteration $b = 1, ..., B$ compute (5) and (6):

5. Randomly sample the rows of the in-sample $T \times N$ data with replacement and repeat step (2) for the out-of-sample test statistic computation.
6. For a significance level chosen in (4) compute the indicator functions $\mathbb{1}^b_{op}(s_i, \alpha^s_{op,t+1})$ for every strategy i and bootstrap iteration b.
7. The bagged probability of the strategy s_i is then defined as $\hat{p}(s_i, \alpha^s_{op,t+1}) = \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}^b_{op}(s_i, \alpha^s_{op,t+1})$ using (15).
8. Finally the bagged out-of-sample pretest weight estimator for the period $t+1$ is defined as an average of the weights estimated on the whole sample weighted by the bootstrap probabilities:

$$\hat{\omega}^B_{op,t}(\mathcal{S}, \alpha^s_{op,t+1}) = \sum_{i=1}^{M+1} \hat{p}(s_i, \alpha^s_{op,t+1})\hat{\omega}_t(s_i). \tag{17}$$

The proposed out-of-sample pretest bagged estimator is a novel approach providing the investor with a fully data driven way of an optimal portfolio allocation strategy choice according to a specific performance measure. For instance, if the investor is looking for a strategy with the highest out-of-sample Sharpe Ratio, the proposed bagging algorithm can be easily adapted. Moreover, the algorithm takes into account the amount of transaction costs and is time adaptive through the significance level choice. Using bagging helps to stabilize the problem and reduce

[2]Note, that monthly returns do not posses any significant SACF or SPACF patterns, therefore the i.i.d. bootstrap of Efron (1992) is appropriate to use. For the returns of higher frequencies one should use the moving block bootstrap by Politis and Romano (1992).

transaction costs, as the bootstrapped probabilities smooth the transition between the strategies along the rolling window.

As a competitor for the bagged pretest estimator we consider a sequential relative performance weighting inspired by the approach of Shan and Yang (2009) used in the forecast combinations. The idea behind this approach is very similar to boosting. First, at each period t the relative performance of different strategies is measured by the exponential function of the in-sample CE. Then the time-adaptive weight $d_{t,i}$ for the strategy i is computed according to (18), where the initial values for the weights are fixed to $d_{0,i} = \frac{1}{M+1}$. The resulting weight estimator is denoted as $\hat{\omega}_t^{SP}(\mathcal{S})$:

$$d_{t,i} = \frac{d_{t-1,i} \exp(\widehat{CE}_{in,t}(s_i))}{\sum_{j=1}^{M+1} d_{t-1,j} \exp(\widehat{CE}_{in,t}(s_j))}, \tag{18}$$

$$\hat{\omega}_t^{SP}(\mathcal{S}) = \sum_{j=1}^{M+1} d_{t,i} \hat{\omega}_t(s_i), \quad i = 1, ..., M+1 \tag{19}$$

where $\mathcal{S} = \{s_1, .., s_M, \tilde{s}\}$ is a space of the underlying strategies and $d_{t,i}$ are the time-adaptive coefficients used to weight the relative performance of the underlying strategies. This approach is computationally very easy and is adapting the weights with the every shift of the estimation window. It is also a smooth combination of the underlying strategies, potentially requiring less rebalancing and reducing transaction costs.

3 Empirical Evidence

In this section we present the empirical application of the pretest estimators discussed in the previous section. We continue the example from Section 2.1 and consider the GMVP, tangency and the equally weighted portfolios as the underlying competing strategies. We take the data on monthly excess returns from K.R.French database[3] of a 100 industry portfolio. In the analysis below for a given portfolio size N we randomly draw N out of 100 unique assets and report the average portfolio performance over 500 random draws. For bagging the number of bootstrap iterations is set to $B = 200$, the out-of-sample evaluation window is fixed to $H = 500$ observations and the risk aversion parameter is set to $\gamma = 1$.

Table 1: Out-of-sample CE for $T = 120$.

	N = 10	N = 20	N = 30	N = 40	N = 50	N = 60	N = 70	N = 80	N = 90
GMVP	0.0069	0.0070	0.0070	0.0072	0.0069	0.0064	0.0055	0.0045	0.0023
Tangency	-0.0545	-0.8500	-0.6386	-0.1405	-1.0088	-1.2433	-0.1927	-43.9858	-0.2068
1/N	0.0068	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069
$\alpha = 5\%$	**0.0073**	0.0073	0.0041	0.0071	0.0071	0.0067	-0.0085	-0.0130	0.0008
α^s_{in}	0.0070	0.0070	-0.0087	0.0044	-0.4950	-0.2427	-0.0104	-0.3138	0.0000
α^s_{in} B	0.0072	0.0073	0.0073	0.0061	0.0072	0.0072	0.0071	0.0069	-0.0017
α^s_{op}	0.0041	-0.1294	-0.0190	-0.0002	-0.0483	-0.1552	-0.0124	-0.4365	-0.0121
α^s_{op} B	**0.0073**	**0.0074**	**0.0074**	**0.0075**	0.0074	**0.0075**	**0.0074**	**0.0074**	**0.0074**
Seq. Perf.	**0.0073**	**0.0074**	**0.0074**	0.0067	**0.0076**	**0.0075**	0.0073	0.0072	0.0066

Figures in the table correspond to the average out-of-sample CE computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an evaluation horizon of $H = 500$ observations, risk aversion parameter $\gamma = 1$, in-sample estimation window length $T = 120$. $\alpha = 5\%$ denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, α^s_{in} allows for a flexible significance level choice according to (12) and α^s_{in} B is the bagged version of α^s_{in}. α^s_{op} and α^s_{op} B denote the out-of-sample significance level choice and the one combined with bagging as in (17). Seq. Perf. corresponds to (19). Numbers in bold correspond to the largest CE for a given portfolio size N.

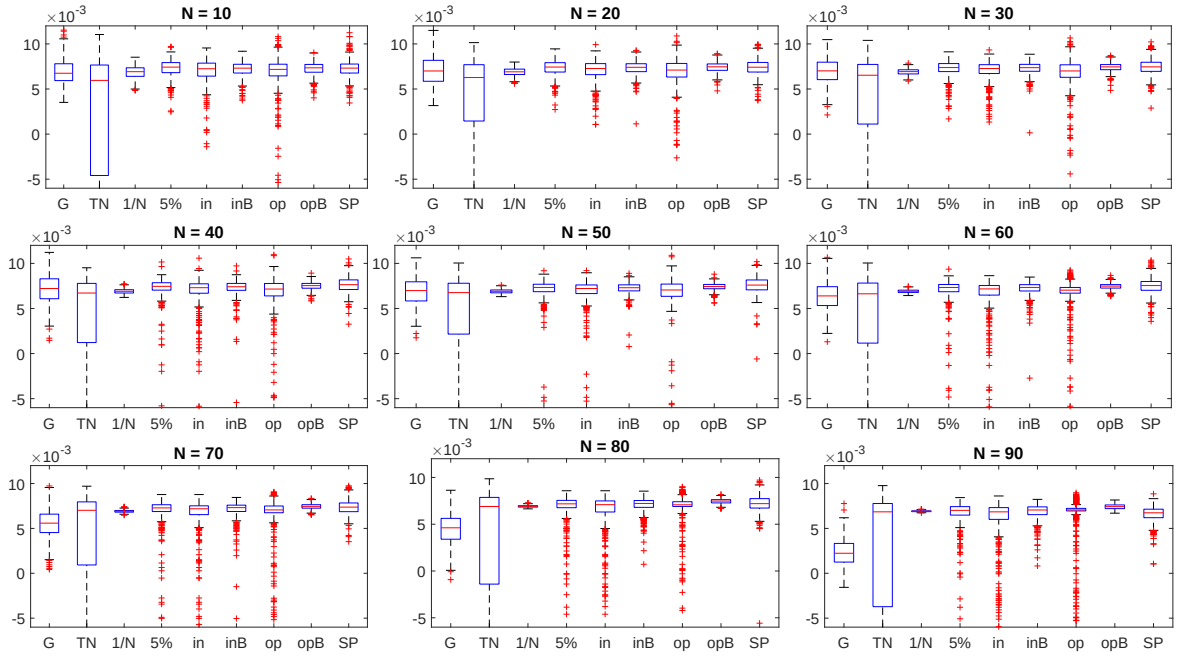
Table 1 reports the average out-of-sample Certainty Equivalent net of transaction costs for the underlying strategies (first block), pretest estimators (second block) and sequential performance weighting (last row) for in-sample estimation window length of 10 years (120 monthly observations) with different portfolio sizes N in the corresponding columns. As before, with the increase in N the out-of-sample CE of the GMVP decreases, CE of the tangency portfolio is negative resulting from the extreme weight estimates and the CE of the equally weighted portfolio is stable across N. The forth row corresponds to the CE of the pretest

[3]The data is taken from K.R.French website and contains monthly excess returns from 01/1953 till 12/2015.

estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%. Here the pretest estimator chooses the strategy which standardized in-sample CE difference from the $1/N$ is greater than the 95% quantile of the standard normal distribution. This pretest estimator is working quite well for smaller asset spaces, however for $N \geq 30$ its performance deteriorates and the CE of the pretest estimator is no longer greater than the values of the underlying strategies, potentially suffering from the low power property of the portfolio performance tests. Allowing for a flexible significance level choice according to (12) based on the in-sample CE is quite unstable across N and is not performing very well. However, combining this pretest estimator with bagging ($\alpha^s_{in}B$) outperforms the underlying strategies up to $N = 80$ having a greater out-of-sample CE than the best of the stand-alone strategies. The next row (α^s_{op}) reports the average out-of-sample CE for the pretest estimator with adaptively smoothed out-of-sample significance level choice, which takes into account transaction costs, but suffers from the short evaluation period of $T/2$. Out-of-sample pretest estimator is very unstable across N and is not outperforming the underlying strategies, however it always outperforms the worst underlying strategy. Notably, the bagged out-of-sample pretest estimator ($\alpha^s_{op}B$) performs extremely well, producing the largest out-of-sample CE for all portfolio sizes N always greater than the best underlying strategy. The sequential performance weighting results in the average out-of-sample CE which is as good as the bagged out-of-sample pretest estimator, except for $N = 90$, where it is not able to beat the equally weighted portfolio benchmark.

The proposed bagged out-of-sample pretest estimator is performing very well even in the presence of the estimation noise and is preferred to the sequential relative performance weighting. Figure 2 depicts the boxplots of the out-of-sample CE's of the considered strategies across the 500 random draws of portfolios of a given size N. The boxplot of the bagged out-of-sample pretest estimator denoted by *opB* shows how stable is this strategy and how few negative outliers does it have in comparison to other ones. For instance, for $N \geq 40$ the out-of-sample CE's of the pretest estimators despite of having very similar medians are very different from each other in terms of the number of negative outliers. Comparing boxplots *in* with *inB* and *op* with *opB* shows that bagging always helps to stabilize the out-of-sample performance of the pretest estimators and works exceptionally well in the combination with the time-adaptive out-of-sample significance level choice. In particular, the out-of-sample CE of the proposed estimator is almost as stable as

Figure 2: Boxplots of the out-of-sample CE for $T = 120$.



Each boxplot corresponds to the distribution of the out-of-sample CE computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an evaluation horizon of $H = 500$ observations, risk aversion parameter $\gamma = 1$, in-sample estimation window length $T = 120$. X-axes denote different ways of computing portfolio weights: G from (2), TN from (3), 1/N from (4), 5% denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, *in* allows for a flexible significance level choice according to (12), *inB* is the bagged version of *in*. *op* and *opB* denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). SP corresponds to (19).

the equally weighted portfolio and at the same time it outperforms the $1/N$ benchmark in mean and has a very few outliers in comparison with the sequential performance weighting.

The source of such a good performance of the bagged pretest estimator is the transaction cost reduction. Table 2 reports the average turnover[4] of the considered strategies. As expected, the equally weighted portfolio produces the smallest turnover among the considered strategies, the turnover of the GMVP and the tangency portfolio increases with the increase in N. Bagging the pretest estimators reduces turnover and the average turnover of the proposed bagged out-of-sample pretest estimator (α^s_{op} B) is as small as the one of the equally weighted portfolio. The average turnover of sequential performance weighting is way larger and steeply increases with the portfolio size N. Moreover, our novel approach outperforms the sequential relative performance weighting also in terms of the out-of-sample Sharpe Ratio (SR) based on the net

[4]Turnover of a strategy s is computed as $TO(s) = \frac{1}{H}\sum_{t=1}^{H}\left(\sum_{j=1}^{N}|\hat{\omega}_{j,t+1}(s) - \hat{\omega}_{j,t^+}(s)|\right)$

Table 2: Turnover for $T = 120$.

	N = 10	N = 20	N = 30	N = 40	N = 50	N = 60	N = 70	N = 80	N = 90
GMVP	0.0749	0.1413	0.2036	0.2680	0.3365	0.4160	0.5127	0.6415	0.8293
Tangency	0.9084	1.4565	1.1571	0.7946	1.7806	1.5813	1.1478	6.4067	1.0036
1/N	0.0217	0.0222	0.0225	0.0225	0.0226	0.0226	0.0227	0.0227	0.0227
$\alpha = 5\%$	0.0367	0.0437	0.0789	0.0641	0.0609	0.0848	0.1673	0.2719	0.1914
α^s_{in}	0.0485	0.0559	0.1685	0.1184	0.4787	0.5698	0.2326	0.6230	0.2454
α^s_{in} B	0.0301	0.0362	0.0401	0.0689	0.0499	0.0546	0.0642	0.0712	0.1585
α^s_{op}	0.1241	0.3858	0.2446	0.2049	0.2674	0.5261	0.2833	0.6939	0.2216
α^s_{op} B	0.0264	0.0305	0.0319	0.0331	0.0336	0.0332	0.0341	0.0353	0.0382
Seq. Perf.	0.0364	0.0502	0.0669	0.1007	0.0982	0.1157	0.1352	0.1575	0.1872

Figures in the table correspond to the average turnover over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an evaluation horizon of $H = 500$ observations, in-sample estimation window length $T = 120$. $\alpha = 5\%$ denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, α^s_{in} allows for a flexible significance level choice according to (12) and α^s_{in} B is the bagged version of α^s_{in}. α^s_{op} and α^s_{op} B denote the out-of-sample significance level choice and the one combined with bagging as in (17). Seq. Perf. corresponds to (19).

returns. Figure 3 in Appendix depicts the boxplots of the Sharpe Ratios across the 500 random portfolios of size N. Similarly to the out-of-sample CE, bagging stabilizes the SR and reduces the number of outliers. Furthermore, the results are robust to the in-sample estimation window length. Tables 3 and 4 report the average out-of-sample CE and turnover of the considered strategies for $T = 180$. In this case, the tangency portfolio and the GMVP perform well and the proposed bagged pretest estimator is performing as good as the best underlying strategy. Figures 5 and 4 report the boxplots of the out-of-sample CE and SR for different portfolio sizes N and $T = 180$, where again, the most stable strategy with less outliers is the bagged out-of-sample pretest estimator, performing very well for all randomly drawn portfolios.

4 Conclusions

This paper introduces a novel pretest estimator with a data-driven and time adaptive significance level choice. Our pretest estimator is designed to choose an optimal portfolio allocation strategy among M alternatives and the equally weighted portfolio as a benchmark strategy. Equally weighted portfolio is known as one of the toughest benchmarks in the empirical finance, i.e. it is very difficult for a more sophisticated theory based strategy to outperform the naive $1/N$ in both in-sample and out-of-sample comparisons. However, for certain datasets and parameter constellations it is possible for other strategies to be better than the equally weighted portfolio according to a certain portfolio evaluation criteria. Our proposed pretest estimator uses the weight estimates of the competing strategies and chooses the one, maximizing the predetermined criterion. We have constructed an estimator which results in the highest out-of-sample CE based on the net portfolio returns, the algorithm however can be easily adjusted to other performance measures.

Our estimator takes into account transaction costs and with the help of machine learning techniques computes the optimal portfolio weights. It first evaluates the pseudo-out-of-sample CE and chooses an optimal significance level in a time adaptive way. With bagging it smooths the indicator functions of the pretest estimator, which stabilizes the performance and reduces the turnover costs. We show that our estimator is robust to different portfolio sizes and in-sample estimation window lengths. We also provide evidence that the proposed estimator outperforms the underlying strategies with respect to the out-of-sample Certainty Equivalent and Sharpe Ratio.

The analysis in this paper is based on the monthly return data, in future work it needs to be shown whether our findings can be generalized to portfolio strategies estimated at higher frequencies where autocorrelation plays a major role.

References

DeMiguel, V., L. Garlappi, and R. Uppal (2009): "Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?" *Review of Financial Studies*, 22, 1915–1953.

DeMiguel, V., F. J. Nogales, and R. Uppal (2014): "Stock Return Serial Dependence and Out-of-Sample Portfolio Performance," *Review of Financial Studies*, 27, 1031–1073.

Efron, B. (1992): *Bootstrap methods: another look at the jackknife*, Springer.

Frahm, G. and C. Memmel (2010): "Dominating estimators for minimum-variance portfolios," *Journal of Econometrics*, 159, 289–302.

Jagannathan, R. and T. Ma (2003): "Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps," *The Journal of Finance*, 58, 1651–1684.

Kazak, E. and W. Pohlmeier (2017): "Testing Out-of-Sample Portfolio Performance," Tech. rep., GSDS Working Paper Series.

Kourtis, A., G. Dotsis, and R. N. Markellos (2012): "Parameter uncertainty in portfolio selection: Shrinking the inverse covariance matrix," *Journal of Banking & Finance*, 36, 2522–2531.

Ledoit, O. and M. Wolf (2003): "Improved estimation of the covariance matrix of stock returns with an application to portfolio selection," *Journal of Empirical Finance*, 10, 603–621.

——— (2004): "Honey, I Shrunk the Sample Covariance Matrix," *The Journal of Portfolio Management*, 30, 110–119.

——— (2008): "Robust performance hypothesis testing with the Sharpe ratio," *Journal of Empirical Finance*, 15, 850–859.

——— (2014): "Nonlinear Shrinkage of the Covariance Matrix for Portfolio Selection: Markowitz Meets Goldilocks," Working paper, Department of Economics, University of Zurich.

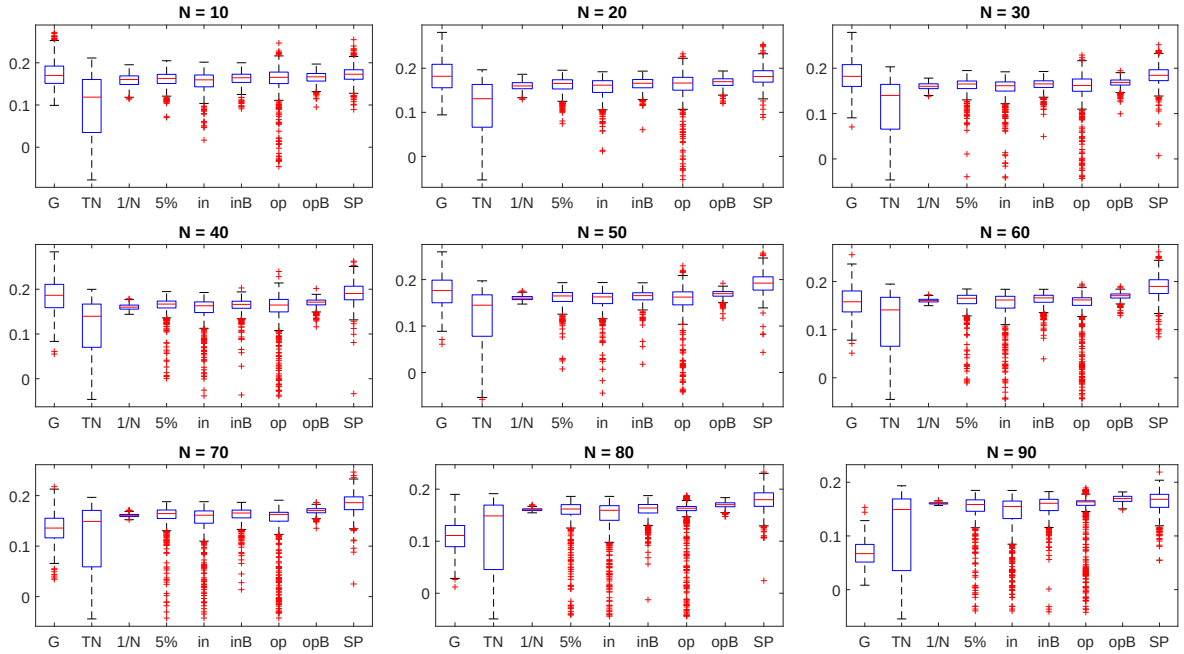
Merton, R. C. and P. A. Samuelson (1992): *Continuous-time finance*, Blackwell Boston.

Politis, D. N. and J. P. Romano (1992): "A circular block-resampling procedure for stationary data," *Exploring the limits of bootstrap*, 263–270.

Shan, K. and Y. Yang (2009): "Combining regression quantile estimators," *Statistica Sinica*, 1171–1191.

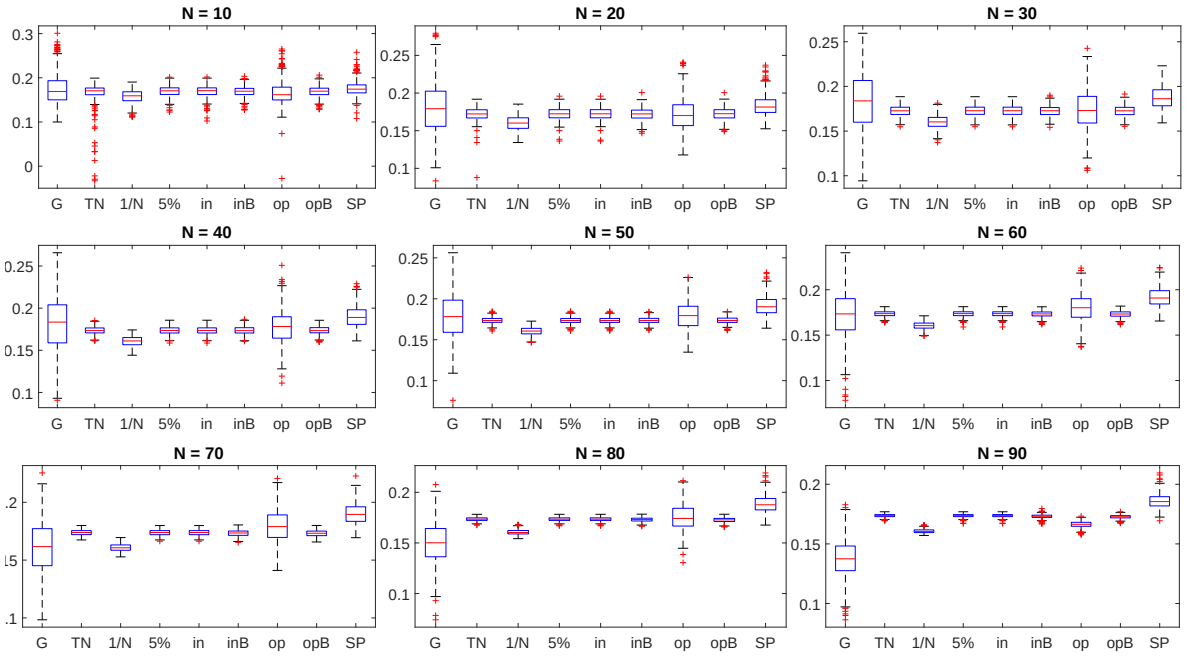
A Appendix

Figure 3: Boxplots of the out-of-sample SR for $T = 120$.



Each boxplot corresponds to the distribution of the out-of-sample SR computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an out-of-sample evaluation horizon of $H = 500$ observations, risk aversion parameter is $\gamma = 1$, in-sample estimation window length $T = 120$. X-axes denote different ways of computing portfolio weights: G from (2), TN from (3), 1/N from (4), 5% denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, *in* allows for a flexible significance level choice according to (12), *inB* is the bagged version of *in*. *op* and *opB* denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). SP corresponds to (19).

Figure 4: Boxplots of the out-of-sample SR for $T = 180$.



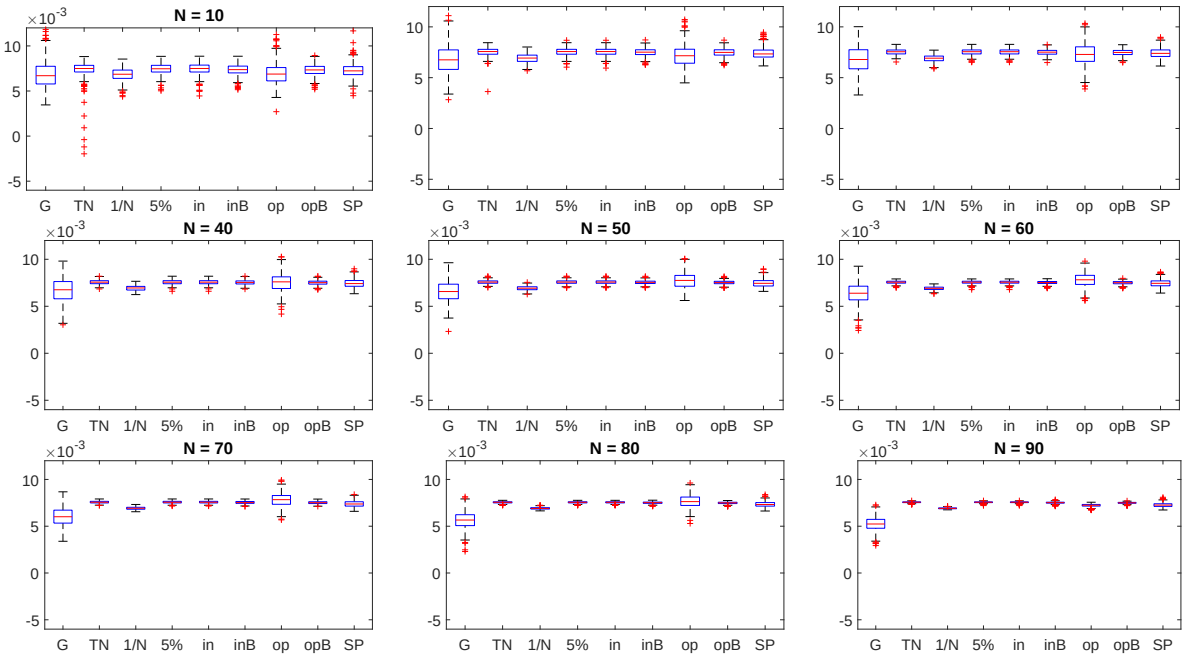
Each boxplot corresponds to the distribution of the out-of-sample SR computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an out-of-sample evaluation horizon of $H = 500$ observations, risk aversion parameter is $\gamma = 1$, in-sample estimation window length $T = 180$. X-axes denote different ways of computing portfolio weights: G from (2), TN from (3), 1/N from (4), 5% denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, *in* allows for a flexible significance level choice according to (12), *inB* is the bagged version of *in*. *op* and *opB* denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). SP corresponds to (19).

Table 3: Out-of-sample CE for $T = 180$.

	N = 10	N = 20	N = 30	N = 40	N = 50	N = 60	N = 70	N = 80	N = 90
GMVP	0.0069	0.0069	0.0068	0.0067	0.0066	0.0064	0.0060	0.0056	0.0052
Tangency	0.0060	0.0075	0.0075	0.0076	0.0076	0.0076	0.0076	0.0075	0.0076
1/N	0.0068	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069
$\alpha = 5\%$	0.0074	0.0075	0.0075	0.0076	0.0076	0.0076	0.0076	0.0075	0.0076
α^s_{in}	0.0074	0.0075	0.0075	0.0076	0.0076	0.0076	0.0076	0.0075	0.0076
α^s_{in} B	0.0074	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
α^s_{op}	0.0067	0.0072	0.0073	0.0075	0.0077	0.0078	0.0078	0.0077	0.0072
α^s_{op}	0.0073	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
Seq. Perf.	0.0073	0.0074	0.0074	0.0075	0.0075	0.0075	0.0074	0.0073	0.0073

Figures in the table correspond to the average out-of-sample CE computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an out-of-sample evaluation horizon of $H = 500$ observations, risk aversion parameter is $\gamma = 1$, in-sample estimation window length $T = 180$. $\alpha = 5\%$ denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, α^s_{in} allows for a flexible significance level choice according to (12) and α^s_{in} B is the bagged version of α^s_{in}. α^s_{op} and α^s_{op} B denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). Seq. Perf. corresponds to (19).

Figure 5: Boxplots of the out-of-sample CE for $T = 180$.



Each boxplot corresponds to the distribution of the out-of-sample CE computed on net portfolio returns over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the out-of-sample CE is computed over an out-of-sample evaluation horizon of $H = 500$ observations, risk aversion parameter is $\gamma = 1$, in-sample estimation window length $T = 180$. X-axes denote different ways of computing portfolio weights: G from (2), TN from (3), 1/N from (4), 5% denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, *in* allows for a flexible significance level choice according to (12), *inB* is the bagged version of *in*. *op* and *opB* denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). SP corresponds to (19).

Table 4: Turnover for $T = 180$.

	N = 10	N = 20	N = 30	N = 40	N = 50	N = 60	N = 70	N = 80	N = 90
GMVP	0.0673	0.1238	0.1733	0.2227	0.2718	0.3213	0.3757	0.4334	0.4995
Tangency	0.0529	0.0232	0.0225	0.0224	0.0225	0.0226	0.0226	0.0226	0.0226
1/N	0.0218	0.0223	0.0225	0.0225	0.0226	0.0226	0.0226	0.0227	0.0227
$\alpha = 5\%$	0.0226	0.0226	0.0227	0.0226	0.0226	0.0227	0.0226	0.0227	0.0227
α^s_{in}	0.0235	0.0226	0.0227	0.0226	0.0226	0.0227	0.0226	0.0227	0.0227
α^s_{in} B	0.0220	0.0225	0.0228	0.0228	0.0230	0.0231	0.0231	0.0232	0.0235
α^s_{op}	0.0524	0.0626	0.0750	0.0847	0.0877	0.0869	0.0786	0.0574	0.0227
α^s_{op}	0.0220	0.0226	0.0228	0.0228	0.0229	0.0230	0.0230	0.0230	0.0231
Seq. Perf.	0.0244	0.0316	0.0396	0.0476	0.0557	0.0627	0.0675	0.0733	0.0758

Figures in the table correspond to the average turnover over a 500 randomly drawn portfolios of size N. For each randomly drawn portfolio the average turnover is computed over an out-of-sample evaluation horizon of $H = 500$ observations, in-sample estimation window length $T = 180$. $\alpha = 5\%$ denotes the pretest estimator based on the in-sample test statistic as in (9) with a fixed significance level of 5%, α^s_{in} allows for a flexible significance level choice according to (12) and α^s_{in} B is the bagged version of α^s_{in}. α^s_{op} and α^s_{op} B denote the out-of-sample significance level choice and the same pretest estimator combined with bagging as in (17). Seq. Perf. corresponds to (19).