Risk-Based Portfolio Sensitivity to Estimation Error

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Abstract:

Recent portfolio theory has seen an explosion in interest in risk-based portfolios which aim to lower portfolio volatility or maximize portfolio diversification. A common theme among all these is an absence of expected returns as inputs. This brings about increased robustness in the portfolio construction process as expected returns are notoriously difficult to estimate accurately. However, it does not mean that risk-based portfolios are immune to estimation risk, as the primary input to their construction is some estimate of the covariance matrix. We study six risk-based portfolios in a general framework for decomposing the covariance into separate dimensions of correlations and volatilities and illustrate risk-based portfolio sensitivity to a range of volatility and correlation estimation models. We find in a long-only fully invested equity market context, that the estimation error sensitivity varies significantly across time horizons and risk-based portfolio types, and that the simple sample historical covariance estimator is possible, but difficult to outperform in a portfolio context.

Keywords: risk-based portfolios, estimation, shrinkage, Marčenko-Pastur distribution, rotationally invariant estimator, maximum diversification portfolios, effective number of bets

1. Introduction

Since the 2008 Global Financial Crisis, there has been a significant increase in interest in portfolio construction methods that have increased diversification or risk reduction embedded into their objective functions. Such methods are collectively termed *risk-based portfolios* emphasizing that their focus is on optimally extracting information out of the risk structure of the market, as opposed to utilizing forecasts of expected returns in their construction, which as we know from Merton (1980)[i] are notoriously difficult to estimate with any accuracy.

Several additional design features may lend greater robustness to risk-based portfolios. These include, the implicit shrinkage of the sample estimates, embedded within the optimizations of some of these, notably, the Equal Risk Contribution portfolio see Maillard et al. (2009), and objective functions that do not require the inversion of the covariance matrix - as is the case in traditional mean-variance optimization (Roncalli, 2017). The finding of greater robustness of risk-based portfolios may indeed hold, but it does not mean that they are immune to estimation error, as there is still significant error to be encountered in the estimation of the covariance matrix of returns as a key input to many of these portfolios.

A rich literature has developed to address the vulnerability to estimation error of the traditional mean-variance framework, such as Michaud (1998) and Tütüncü and Koenig (2004) and more recently, innovations in random matrix theory such as Bun, Bouchaud and Potters (2016). Almost without exception, most studies on estimation have centred on the mean-variance or minimum-variance class of portfolios in their empirical application. To date the only other study of which the authors are aware that considers estimation error in a more general class of risk-based portfolios, is the Monte Carlo based analysis of Ardia, Bolliger, Boudt and Fleury (2017). We attempt to make a contribution to this void by studying six different risk-based portfolios in a empirical setting, where permutations of 10 volatility models and 12 correlation models inform a recomposed covariance matrix, used in the determination of the risk-based portfolios. Sensitivities to volatility estimates and correlation estimates are systematically isolated via a decomposition of the covariance matrix..

The scope is limited to six of those risk-based portfolios that maximize a measure of diversification in their objective functions i.e. the class of maximum diversification portfolios. The minimum variance (MV) portfolio is included as an important benchmark with a long history in portfolio theory. It can be viewed as maximizing a proxy measure of diversification i.e. portfolio variance, justifying its inclusion.

Risk-based portfolios considered in this study are:

1) the Minimum Variance Portfolio (MV), of classic Markowitz (1952) and many others such as Clarke, De Silva and Thorley (2006);
2) the Equal Weighted Portfolio (EW), examined in DeMiguel, Garlappi and Uppal (2006);
3) the Equal Risk Contribution Portfolio (ERC) of Maillard, Roncalli & Teiletche (2010);
4) the Most Diversified Portfolio (MDP) of Choueifaty and Coignard (2008);
5) the Effective Number of Bets Portfolio (EffBetsPCA) of Meucci (2009); and,

6) the Effective Number of Linear Torsion Bets Portfolio (EffBetsMLT) of Meucci, Santangelo and Deguest (2013).

Risk-based portfolio definitions are only briefly reviewed here. See Du Plessis & van Rensburg (2017) for a more detailed analysis of these. Three broad classes of volatility models are considered: EWMA, GARCH and intra-day range estimators. Correlation forecasting models covered range from basic shrinkage estimators such as average correlation (Elton and Gruber, 1973, plesiochronous correlations of Choueifaty, Coignard and Reynier (2013), to the more sophisticated eigenvalues-clipping approach of Bouchaud and Potters (2011), and the very recent rotationally invariant estimator (RIE) of Bouchaud, Bun and Potters (2016). Classic Ledoit and Wolf shrinkage (2003, 2004) are also added as a benchmark, albeit that these operate on the whole covariance matrix A perfect foresight 'upper bound' benchmark, called the 'oracle estimator' is also added for both volatilities and correlations. The historical sample estimate is used as a 'baseline' benchmark. In all cases, a long only constraint and full investment budget constraint is applied to the portfolios. **Appendix A** details all the models along with a glossary of the model abbreviations used.

The remainder of the paper is organized as follows. Section 2 briefly defines the risk based portfolios. Section 3 defines the volatility and correlation models in use. Section 4 discussed the dataset and in particular, the step-wise methodological approach followed to elicit risk-based portfolio sensitivity to estimation error. Section 5 discusses the empirical results and Section 6 concludes. Prior research is discussed in context in Section 2 to 4.

2. Risk-Based Portfolios

On Notation

We consider a market of N risky assets (or positions generally) observed at the daily frequency for most except the intraday range estimators below, which defines a vector of raw returns $r_t = (r_{1,t}, r_{2,t}, \dots, r_{N,t})$ for each day $t = 1, \dots, T$. The portfolio weight vector is denoted by $x = (x_1, \dots, x_N)$, optimal weights by x^*, Σ is the estimated $N \times N$ covariance matrix of returns, and given the subscript Σ_{hist} when necessary to refer to the sample historical covariance matrix. E is the historical sample correlation matrix, Ω the estimated correlation matrix $\mathbb{I}_N$ the identity matrix and e the unit vector. The in-sample conditioning window period is denoted m, with last in-sample observation increment T (also referring to the length of the available dataset, as will be clear from context), current time, t, and out-of-sample holding period, $T + \tau$. At each portfolio formation (or rebalance) date both the conditioning window and holding period are moved forward in time by τ. Operator $dg(\cdot)$ refers to diagonalization, such the result is a diagonal matrix with its argument on the principle diagonal and zeros everywhere else.

Minimum Variance (MV)

The minimum variance portfolio is well familiar as the left most point on the classic Markowitz (1952) frontier (ignoring the risk free asset). Its sensitivity to estimation error has been well documented, along with many attempts to improve upon its estimation such as Michaud (1998) and Tütüncü and Koenig (2004). This portfolio can be solved for via the optimization setup:

$$x^*_{MV} = argmin\ f(x)$$
$$s.t.\ 1'x = 1; \quad 0 \le x \le 1$$
$$where: f(x) = x'\Sigma x$$

Equal Weighting (EW)

The Equally Weighted portfolio is simply defined via

$$x^*_{EW} \equiv \frac{1}{N} \quad for\ N\ \in \mathbb{R}^N$$

requiring no optimization in its solution. There may be certain situations where the estimates of volatilities and correlations are so unreliable as to make it preferable to abandon optimization in favour of an $1/N$ allocation. It is included as an important benchmark, especially following DeMiguel, Garlappi and Uppal (2006), who demonstrate that this portfolio often dominates many of the others in terms of risk-adjusted return and turnover.

Equal Risk Contribution (ERC)

The Equal Risk Contribution portfolio, as defined and studied extensively in Maillard et al. (2009) and Roncalli (2014), maximizes the uniformity in the Euler risk contributions[ii]. While, for N>2 and non-uniform correlations there exists no analytical solution, the ERC portfolio can be solved via non-linear convex minimization, as in

$$x^*_{ERC} = argmin\, f(x)$$
$$s.t.\, 1'x = 1; \quad 0 \leq x \leq 1$$
$$where: f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i(\Sigma x)_i - x_j(\Sigma x)_j\right)^2$$

where $x_i(\Sigma x)_i$ represents the scaled Euler risk contributions. The ERC portfolio exhibits many appealing empirical properties, including multi-period stability exhibited by low turnover, high capacity (it always takes a non-zero position in every asset in the universe) and robustness to estimation error (see specifically, Demey, Maillard and Roncalli, 2010).

Diversification Ratio and *Most Diversified Portfolio* (MDP)

Taking a slightly different approach, Choueifaty and Coignard (2008) define the *Diversification Ratio, DR,* as the weighted average volatility divided by the portfolio volatility:

$$DR \equiv \frac{x'\sigma}{\sqrt{x'\Sigma\, \mathrm{x}}}$$

The DR measure has a range of $DR = 1$ for full concentration and $DR = \sqrt{N}$ for maximum diversification. It is interesting to note that DR is precisely the same concept as the Tasche (2008) Diversification Index, albeit that the latter is inverted with the weighted average volatility in the denominator. The portfolio that maximizes this ratio is then called the *Most Diversified Portfolio*, solvable via:

$$x^*_{MDP} = argmax\, f(x)$$
$$s.t.\, 1'x = 1; \quad 0 \leq x \leq 1$$
$$where: f(x) = \frac{x'\sigma}{\sqrt{x'\Sigma\, \mathrm{x}}}$$

The *Most Diversified Portfolio* has several interesting theoretical properties, as elaborated upon by Choueifaty, Froidure and Reynier (2013), such as that every security selected by the objective function is less correlated to the final portfolio than every stock excluded by it and that all stocks within the final portfolio have the same correlation to this portfolio.

Effective Number of Bets via PCA (EffBetsPCA)

The Effective Number of Bets via PCA portfolio of Meucci (2009) maximizes the uniformity of allocation to orthogonal risk factors (called *principal portfolios*, after Partovi and Caputo, 2004), via an eigen-decomposition of the covariance. It is solved via

$$x_{PCA}^{*} = argmax\ f(x)$$
$$s.t\ .1'x = 1;\ \ 0 \leq x \leq 1$$
$$wher\ : f(x) = exp\{-p_{PCA}{'}logp_{PCA}\}$$

Where p_{PCA} is the diversification distribution of principal portfolio contributions to portfolio risk, given by

$$p_{PCA} \equiv \frac{(E{'}^{-1}x)\circ(E{'}\Sigma x)}{x{'}\Sigma x}$$

Where E is the matrix of eigenvectors decreasingly responsible for portfolio risk, and ∘ the Hadamard (element-wise) product. This is a theoretically appealing diversification method which incorporates key aspects of the diversification problem (weights, volatilities, correlations, dimension reduction, long/short invariance) in its solution. It has been criticized in Meucci, Santangelo and DeGuest (2014), for a lack of uniqueness and instability, as will be illustrated in Section 5 below extreme sensitivity to estimation error.

Effective Bets via Minimum Linear Torsion (EffBetsMLT)

As a solution to the shortcomings of the maximum Effective Number of Bets via PCA portfolio noted above, Meucci, Santangelo and DeGuest (2014), introduce the maximum Effective Number of Bets via Minimum Linear Torsion (MLT) portfolio. Instead of focusing on orthogonal sources of risk via *principal portfolios* in its decomposition, the EffBetsMLT portfolio is solved via a de-correlating transformation (MLT) that finds the least disrupts the original factors used to inform the portfolio selection. Similarly to EffBetsPCA its objective function is given by

$$x_{MLT}^{*} = argmax\ f(x)$$
$$s.t\ .1'x = 1;\ \ 0 \leq x \leq 1$$
$$where: f(x) = \exp\{-p_{MLT}{'}logp_{MLT}\}$$

Where p_{MLT} is the diversification distribution of de-correlated minimum linear torsion portfolio contributions to portfolio risk, given by

$$p_{MLT} \equiv \frac{({t_{MT}}'^{-1}x)\circ({t_{MT}}'\Sigma x)}{x'\Sigma x}$$

Where t_{MT} is the torsion matrix. In a departure from the ‘uniform prior’ perspective taken in all the other risk-based portfolios discussed here, the EffBetsMLT portfolio is well adapted to a factor-based management paradigm where the risk of the portfolio is denominated in pre-specified factors, whereupon diversification is imposed. In keeping with the analysis of maximum diversification portfolios, it is applied in this study to the original asset returns without any prior dimension reduction into a pre-defined factor portfolio. The EffBetsMLT portfolio was originally designed to impose diversification upon a pre-defined factor portfolio differing slightly from its use as a maximum diversification portfolio derived from underlying asset returns, as is the case here (see Meucci, Santangelo and DeGuest, 2014).

3. Covariance Estimation

By decomposing the covariance matrix via $\Sigma = diag(\sigma)\,\Omega\,\mathrm{diag}(\sigma)$ it becomes possible to separately control the estimation of the correlations and the volatilities and ideally choose the best models for each independently of the other. The objective is to test models of volatility that theoretically promise an improvement to the historical standard deviation estimate, specifically in the context of risk-based portfolio construction under the null hypothesis of no improvement.

Volatility Models

Historical Sample Volatility (volHist)

The sample historical volatility estimator needs no introduction. Given its wide acceptance, ease of calculation and statistical property of being an unbiased if inefficient estimator (see Meucci, 2009, for a thorough discussion). It will form a baseline benchmark for the empirical evaluation of all other volatility estimators.

Oracle Volatility (oracleVol)

As its name implies, the oracle estimator is the perfect look-ahead estimator, of exactly the same form as the historical sample volatility estimator, (volHist), except using the returns $r_{T+\tau}$ from the out-of-sample holding period $T+\tau$, i.e. one rebalance period ahead, in the estimation of the volatilities at time T. Out-of-sample returns $r_{T+\tau}$ are of course inaccessible in practice, but having partitioned the dataset into conditioning windows and holding periods allows the use of the oracle estimator as a benchmark for the upper limit of volatility estimation improvements.

EWMA Volatility [volEWMA98; volEWMA96; volEWMA94]

The first of the conditional volatility models is the Exponentially Weighted Moving Average volatility estimator, defined as:

$$\hat{\sigma}_i^2 = (1-\lambda){r^2}_{t-i} + \lambda\hat{\sigma}_{t-1}^2$$

Where λ is the decay coefficient. The potential usefulness of EWMA model beyond the historical sample estimator lies in the introduction of both a reaction term $(1-\lambda){r^2}_{t-i}$ and a persistence term $\lambda\hat{\sigma}_{t-1}^2$. More weight is placed on recent observations and the smaller λ the more reactive is the estimate to the most recent observation. Three different values λ = [0.98, 0.96, 0.92] are considered with corresponding time periods that contribute half of the weight given by $\frac{\log(2)}{(1-\lambda)} \approx [35, 17, 9]$ days. These are respectively denoted volEWMA98, volEWMA96 and volEWMA92.

GARCH [volGARCH(1,1)N, volGARCH(1,1)t, volGARCH(GJR)N, volGARCH(GJR)t]

Of the 140+ GARCH models defined in Bollerslev's (2008) "*Glossary to ARCH (GARCH)"* which reflects the rich literature that has evolved since the introduction of *Auto-Regressive Conditional Heteroskedasticity* models by Engle (1982)- two are applied here:

1) First, the symmetric "plain vanilla" GARCH(1,1) model of Bollerslev (1986), as defined as:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

   Where ε_{t-1}^2 is the innovation term, α determines the reaction sensitivity of the model to volatility shocks, β measures the persistence in conditional volatility and the constant parameter ω, together with the sum $\alpha + \beta$ determines the long term volatility or unconditional variance ($\bar{\sigma}^2$) - via $\bar{\sigma}^2 = \frac{\omega}{1-(\alpha+\beta)}$.

   Two versions of the model are in used in this study: one where the innovation term ε_{t-1}^2 is assumed to come from a normal distribution, called **volGARCH(1,1)N**, and one where it is assumed to arise from a Student t-distribution with degrees of freedom empirically calibrated for each of the individual returns series $r_t = (r_{1,t}, r_{2,t}, \dots, r_{N,t})$, called **volGARCH(1,1)t**.

2) Second, the asymmetric GJR-GARCH model of Glosten, Jagannathan and Runkle (1993), defined as:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \lambda\mathbb{1}\{\varepsilon_{t-1} < 0\}\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

   Where the term additional to the GARCH(1,1) model above, $\lambda\mathbb{1}\{\varepsilon_{t-1} < 0\}\varepsilon_{t-1}^2$, contains an indicator function $\mathbb{1}\{\cdot\}$ that amplifies the volatility estimate by λ in the event that the most recent return was negative. This 'leverage effect' describes the empirical finding, especially in equity markets, that volatility increases much more after a negative shock than after a positive shock of same magnitude [reference needed here??]. The long term variance is given by $\bar{\sigma}^2 = \frac{\omega}{1-\left(\alpha+\beta+\frac{1}{2}\lambda\right)}$, where the rest of the parameters have the same interpretation as for vanilla GARCH. As with the vanilla GARCH model, two varieties of GJR-GARCH model are used – one with Gaussian innovations, **volGARCH(GJR)N**, and another with t-distributed innovations, **volGARCH(GJR)t**.

The interest in adding GARCH volatility models to the study lies in their ability to finely calibrate the volatility estimate to each individual security in the portfolio, with time-varying reaction and persistence functions, as opposed to the EWMA approach where these are fixed across time and across securities. In addition, in the case of the GJR-GARCH model, the ability to capture the leverage effect may further enhance the estimate.

Intraday Range Volatility [volRangePK, volRangeGK, volRangeRS, volRangeYZ]

All of the volatility estimators up to this point take as inputs the close-to-close daily returns, ignoring any information that may be exploitable in the intra-day trading range. Four classic intraday range volatility models are added to the analysis, viz: Parkinson (1980), Garman & Klass (1980), Rogers & Satchell (1991) and Yang & Zhang (2000) and are defined as follows, with additional notation explained below:

1) Parkinson (1980) [**volRangePK]:** $\sigma_{PK}^2(t) = \frac{1}{4\,log(2)}\,(h(t) - l(t))^2$
2) Garman & Klass (1980) [**volRangeGK]:** $\sigma_{GK}^2(t) = \frac{1}{2}\,[h(t) - l(t)]^2 - 2log(2)c^2(t)$
3) Rogers & Satchell (1991) [**volRangeRS]:** $\sigma_{RS}^2(t) = h(t)[h(t) - c(t)] + l(t)[l(t) - c(t)]$
4) Yang & Zhang (2000) [**volRangeYZ]:** $\sigma_{YZ}^2(t_{m-1}, t_m) = \sigma_{OJ}^2(t_{m-1}, t_m) + k\sigma_{STDEV}^2((t_{m-1}, t_m) + (1-k)\sigma_{RS}^2(t_{m-1}, t_m)$

Where the open, high, low, close log-prices of the trading day are denoted O_t, H_t, L_t, C_t , respectively, and then:

- $o_t = O_t - C_{t-1}$ is the overnight jump
- $c_t = C_t - O_t$ is the normalized closing price
- $h_t = H_t - O_t$ is the normalized high price
- $l_t = L_t - O_t$ is the normalized low price
- $r_t = C_t - C_{t-1}$ is the daily close-to-close return, and
- In the case of Yang & Zhang (2000),
    - σ_{OJ}^2 is the variance of the overnight jump (o_t)
    - $k = \frac{0.34}{1.34 + \frac{N_D + 1}{N_{D-1}}}$ is a parameter to be calibrated with N_D the number of sample days

These estimators differ in the extent of intraday information utilized and estimator efficiency (see Baltas & Kosowski, 2015). In order of increasing information, volRangePK uses intraday high and low prices, but assumes drift term of zero; volRangeGK adds the closing price, still with zero-drift; volRangeRS adds a time-varying, non-zero drift, and volRangeYZ adds the overnight jump to all of the above. They have also been demonstrated to be theoretically multiple times more efficient than the simple historical sample volatility estimator (see the original texts cited earlier in this section). This last property is their most interesting feature and reason for inclusion in this study: if an estimator is, for example, five times more efficient than another then, simply put, should need five times less data to achieve the same performance. This might allow for high quality volatility estimates in situations where the length of the conditioning window is limited or, all else being equal, superior estimates from the same data.

Correlation Models

The correlation models used in this study are briefly described below, from basic estimators to sophisticated shrinkage approaches and then finally, breaking with the format of separately modelling volatilities and correlations, the well-known Ledoit and Wolf (2003 & 2004) approach that operates directly on the covariance matrix is included as a shrinkage benchmark.

Cross-Sectional Standardization

Two versions of all the correlation models are included. First, a version denoted by the prefix **[noNorm_]** where the inputs to the correlations are the raw sample returns $r_t = (r_{1,t}, r_{2,t}, \dots, r_{N,t})$. Second, a version **[csvNorm_]** that inputs demeaned and standardized returns, following Bouchaud and Potters (2011). In this case, let, $r_{i,t} \rightarrow \tilde{r}_{i,t} \equiv {r_{i,t}}/{\hat{\sigma}_{i,t}}$ with $\hat{\sigma}_{i,t}$ the cross-sectional volatility given by

$$\hat{\sigma}_{i,t} = \sqrt{\frac{1}{N}\sum_{j=1}^{N} r_{jt}^2}$$

whereby the final returns fed into the correlation estimator are stationary to a first-order approximation.The motivation for using the cross-sectional volatility $\hat{\sigma}_{i,t}$ in the normalization is that this performs a shrinkage function in itself, and as demonstrated in Couillet, Mammoun and Pascal (2016), serves as a robust estimator of the covariance.[iii]

Historical Sample Correlation (corrHist)

Like the sample volatility estimator, the Pearson sample correlation estimator needs no introduction. It is likewise used as a baseline benchmark for all the other correlation estimators

Oracle Correlation (oracleCorr)

As its name implies and in parallel to the oracle volatility estimator, the oracle correlation estimator applies the Pearson sample correlation with perfect foresight to the out of sample returns $r_{T+\tau}$, as a indication of the upper limit of what should be possible with improved correlation estimation.

Average Correlation (corrAve)

In the average correlation estimator the cross-sectional average correlation of the sample returns at time T is simply used as the best estimate of the out of sample correlations for all pairwise correlations. This idea first occurred in Elton and Gruber (1973). It is an extreme view on the information content of the cross-section, such that no pair of correlations can be estimated more accurately than any other and only the level of average correlation at time T in the time series is reliable information. The Ledoit and Wolf (2004) approach shrinks towards this as a target, and the average correlation estimator may perform well in certain situations where only very short conditioning windows are available.

Plesiochronous Correlation (corrPlesio)

The plesiochronous estimator (from the Greek 'plesio', which means near) takes a different approach and questions the reliability of the choice of the size of the time-step dt and attempts to address this issue by taking averages of correlations measured over different tenors of dt. This is similar to the estimator understood to be in use in Choueifaty, Coignard and Reynier (2013)[iv] and defined as follows:

Let $i = [1,2,\dots,k]$ denote returns series $r_t^{(i)} = \left(r_{1,t}^{(i)}, r_{2,t}^{(i)}, \dots, r_{N,t}^{(i)}\right)$ of different tenors, such that if $i = 1$, the returns series are denominated in one-day returns; for $i = 5$, in five-day returns and so on. Then the correlation estimate Ω is composed of weighted slices of sub-matrices of different tenors, simply as:

$$\Omega_{ples} \equiv \sum_{i=1}^{k} \frac{1}{k} \cdot \Omega_i$$

Where $k = 5$ is the maximum tenor of returns and corresponding number of slices of the final estimate, Ω_{ples}. It is important to limit the size of the highest tenor point, such that conditioning window m is large enough that correlations estimated from sub-windows of size ${}^{m}/_{k}$ would contain enough data-points not to lose significance.

Eigenvalue-Clipping Shrinkage Estimator (corrEigenClip)

For the last two correlation estimators, it is useful to introduce as in Bouchaud and Potters (2011), the eigen-decomposition of the sample correlation matrix as;

$$E = \sum_{k=1}^{N} \lambda_k u_k u_k'$$

Where λ_k refers to the set of eigenvalues sorted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, with $u_1, u_{2,} \cdots, u_N$ the corresponding eigenvectors of E.The eigenvalues-clipping estimator of Bouchaud and Potter (2011), proceeds as follows: Keep the $[N\alpha]$ top eigenvalues and shrink the others to constant γ that preserves the trace of the resulting correlation matrix, such that $Tr(\Omega_{clip}) = Tr(E) = N$, thus:

$$\Omega_{clip} \equiv \xi_k^{clip} u_k u_k', \quad \text{where: } \xi_k^{clip} = \begin{cases} \lambda_k & if\ k \leq [N\alpha] \\ \gamma & otherwise \end{cases}$$

The usual procedure for choosing the cut point α, is to assumed that all sample eigenvalues beyond the upper edge of the Marčenko-Pastur density contain some signal and can therefore be kept; the rest are suspected to contain noise and are set to constant γ. However, there are two potential shortcomings with this choice of α: First, as reported in Bouchaud, Bun and Potters (2016), this treatment ignores the fact that the largest empirical eigenvalues are typically overestimated. Second, the trace-preservation constraint placed on γ typically implies that it would be set to the average of the remaining eigenvalues to be 'clipped'. Being an average, some of these will necessarily be above γ and some below, and the smallest eigenvalues will then be augmented. This has the contra-intended effect of increasing the suspected noise.

Rotationally Invariant Shrinkage Estimator (corrRIE)

In response to the shortcomings of the previous estimator, Bouchaud, Bun and Potters (2016) introduce the Rotationally Invariant (RIE) estimator, following the arguments of Ledoit and Péché (2011) and Bun and Knowles (2016). The *rotational invariance* of the estimator implies that one does not have any knowledge of the structure of the true eigenvectors, so it is best to leave the eigenvectors u_k of E unchanged.

In summary (see Bun, Bouchaud and Potters, 2017 for details):

$$\Omega_{RIE} \equiv \sum_{k=1}^{N} \xi_k^{RIE} u_k u_k'$$

Where the adjusted eigenvalues now become:

$$\xi_k^{RIE} = \frac{\lambda_k}{|1 - q + q z_k s_k(z_k)|^2}$$

In which:

- $q = \frac{N}{T}$,
- $z_k = \lambda_k - i/\sqrt{N}$ is a complex variable $\forall\, k \in [1, N]$
- $s_k(z_k) = \frac{1}{N} \sum_{j=1; j\neq k}^{N} \frac{1}{z_k - \lambda_j}$

To which is added a small sample correction via:

$$\Gamma_k = \sigma^2 \frac{\left|1 - q + q z_k g_{mp}(z_k)\right|^2}{\lambda_k}$$

Where:

- $g_{mp}(z_k) = \frac{z_k + \sigma^2(q-1) - \sqrt{z_k - \lambda_N}\sqrt{z_k - \lambda_+}}{2 q z_k \sigma^2}$ is the Stieltjes transform of the rescaled Marčenko-Pastur distribution
- $\lambda_+ = \lambda_N \left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right)^2$,
- $\sigma^2 = \frac{\lambda_N}{(1-\sqrt{q})^2}$, and λ_N is the smallest empirical eigenvalue.

Now finally,

$$\hat{\xi}_k = \begin{cases} \Gamma_k\, \xi_k^{RIE} \; if \; \Gamma_k > 1 \\ \xi_k^{RIE} \; otherwise \end{cases}$$

The RIE estimator was designed especially to provide a solution in cases where the ratio $q = \frac{N}{T} \rightarrow O(1)$, which often occurs in large scale portfolios, where $N \rightarrow T$ as a result of a limit on data availability. Bouchaud, Bun and Potters (2016) report empirical results where the RIE estimator results in significant reduction in out of sample portfolio volatility relative to a selection of other estimators.

Ledoit & Wolf (2003, 2004) Shrinkage Estimators (LWSMM & LWCC)

The familiar Ledoit and Wolf (2003 and 2004) estimators operate directly on the covariance matrix and have the form:

$$\Omega_{LW} \equiv \alpha E + (1-\alpha)[(1-\rho)\mathbb{I}_N + \rho ee']$$

Where e is the unit vector, $\mathbb{I}_N$ the identity matrix and α a coefficient that determines the shrinkage intensity, to be established via optimization. ρ is the single market factor in the case of **LWSMM** (Ledoit and Wolf 2003) and the constant correlation in the case of **LWCC** (Ledoit and Wolf 2004)

4. Data & Methodology

The dataset that forms the basis of the empirical analysis is the equity market of the Johannesburg Stock Exchange (JSE) over the time period March 1997 to September 2015. A daily point-in-time database was reconstructed from JSE, Bloomberg ® and Thomson Reuters Datastream ® data, taking into account every ticker that is known to have existed over this time, to minimize survivorship bias. A total of 898 unique tickers exist in this dataset. From the total universe existing at each time, a reduced, eligible universe, was filtered, according to two liquidity filters. First, the universe was sorted on a 250-day simple moving average of value traded, and then cut at rank number $N = 60$. Second, a zero-trade day filter set at 15% was applied such that any security with a number of non-trading days greater than 15% of the trailing sample length was excluded. These filters are deliberately set to be quite strict on liquidity, as the objective is to produce results that are of practical relevance to large investors or risk managers.

Approach:

In order to systematically develop the sensitivity profiles to sources of estimation risk in the covariance, we proceed as follows:

1. First, estimation error of all volatility and correlation models is evaluated on both a root-mean-squared-error (RMSE) as well as a rank-order correlation criterion, over 22 different holding period and estimation window combinations. Note that we are interested in comparing the differences of sample means of RMSE across time horizons adapted from the method of Alexander (2008, ch. 8.3).
2. Thereafter, risk-based portfolio weight sensitivity to a range of separately controlled uniform correlation and volatilities is measured on a normalized Herfindahl Index, to explore the likely level of concentration in weights as a function of the level of volatilities and correlations.
3. Next, empirical performance data of the risk-based portfolios are studied, but with the exception of their being conditioned on a decomposed covariance matrix that adds the covariance information in step-wise fashion to isolate the sensitivity to correlations and volatilities. These results are reported in Figure 2. Using the

decomposition $\Sigma = dg(\sigma)\,\Omega\,dg(\sigma)$, the analysis proceeds through four levels of controlled information supplied to the covariance matrix such that:

a. $\Omega = \mathbb{I}$: First, the correlations are set to constant, via the Identity Matrix, such that all off-diagonal terms composing all $\rho_{i,j}\ i \neq j$ are zero and the correlation content of the data is effectively 'switched off'. This correlation matrix is then combined with volatilities estimated from all the models in turn, and then recomposed into the covariance. This leaves covariance matrix Σ and the risk-based portfolio optimizations in turn, with no other information other than volatilities to solve the optimal weights. Thus is possible to isolate the portfolio outcome arising from the utilization of volatility information alone.

b. $\Omega = E$: Second, the correlations are set to the sample historical correlations E, again combined with volatilities from all the volatility models to recompose the covariance matrix.. This more realistic setting allows us to assess whether any portfolio improvement can be made via superior volatility estimation in the event that correlation estimates superior to E are impossible.

c. $\sigma = k$: Turning next to the volatility component of the covariance matrix, first the volatilities are set to a constant. Unfortunately, setting this constant $k = 0$, as is the case with the correlations under a. above, will lead to zero matrices, so the next most interesting least-information case is chosen where k is set to the median of all individual asset volatilities, cross-sectionally and across time. In this data set, the value is 30.81% p.a., rounded down to $k = 30\%$. This vector of uniform volatilities is then recomposed into the covariance, along with correlation estimates from every model, allowing us to 'switch off' the volatilities, to the greatest extent possible and isolate the impact of the correlation estimates. Knowing in advance the median of all asset volatilities of course introduces look-ahead bias into the early samples drawn from the time series. This is deemed not to matter as the context is anyhow artificial and the volatility information is very nearly zeroed out be setting the volatility to a constant.

d. $\sigma = \sigma_{Hist}$: Finally the volatilities are set to the historical sample volatilities, and again recombined with correlation estimates from all the models, in an effort to assess whether any portfolio improvement is possible via superior correlation estimation if no improvement is possible upon sample correlation estimates.

By evaluating and contrasting the behaviour of the risk-based portfolios in each of the four 'controlled information content' cases above, we are able to develop rich insight into the sensitivity of risk-based portfolios to volatility and correlation estimates.

4. Finally, all volatility and correlation models are combined in search of the best empirical combination for each risk-based portfolio. These portfolios are then compared against those solved via the simple sample covariance as a benchmark, along with significance tests on the means, out-of-sample volatilities and Sharpe ratios of each. In the first two cases, the familiar t-test for difference between sample means and F-test for differences in sample variances are used. For the Sharpe ratios, the non-parametric Ledoit and Wolf (2008) test for differences in Sharpe ratios is applied.

The oracle estimator **oracleVol** and **oracleCorr** is added to each of the 'controlled information content' cases in 3a), 3b), 3c) and 3d), and in combined form to 4) in order to indicate an upper bound for possible estimation improvement.

Time Horizons

In order to increase the robustness of the results and observe model performance under different time horizons, the trials in section 3) are repeated for 22 different conditioning window (m) and out-of-sample holding period (τ) horizons. To avoid an explosion in permutations, for the combined correlation and volatility analysis, as contemplated in in section 4 above, only one $m : \tau$ horizon combination of a conditioning window of 500 days and holding period of 125 days combination is reported here.

5. Empirical Results and Discussion

This section reports the results of the quasi-analytical and empirical studies outlined in in the previous section. Read with the tables and figures appended at the end of the paper, it progresses through:

1. Estimation error of volatility models assessed at underlying stock level – Tables 1 & 2.
2. Estimation error of correlation models assessed at underlying stock level – Tables 3 & 4.
3. Risk-based portfolio weight sensitivity under uniform volatilities and correlations.
4. Select performance metrics (out-of-sample returns, volatilities and turnover) for risk-based portfolios under conditions of a 'controlled information content' covariance matrix (with correlation or volatility information switched 'on' or 'off' within the covariance matrix, as developed in part 4 of *Data & Methodology* – Figure 2, Panels 1-5).
5. Empirical performance rank order distributions of all possible volatility-correlation model combinations along six performance metrics for each portfolio – Figure 3.
6. Full scale performance metrics for each risk based portfolio showing the detail of the results summarized in Figure 3. This is shown in Tables 5 – 9.

Refer to *Covariance Estimation* or Appendix A for model codes and abbreviations. Given the size of the dataset depicted, extensive use is made of heat maps to elicit the structure in the data. To aid interpretation, green is always in the desirable direction, red in the undesirable direction, although both may represent high or low numbers, depending on context.

Estimation error of volatility models

Table 1 displays the root-mean-squared-error (RMSE) of the in-sample estimated individual stock volatilities from each model against the out-of-sample realized volatilities over the holding period. 22 different holding period τ and conditioning window (m) sizes (collectively referred to as 'horizons') are reported. Each datapoint is the time-series average of the RMSE for each $m : \tau$ pair, which in turn summarizes the predictive error across the sample of stocks existing at each rebalance period. The leftmost data column gives the sample historical estimator (volHist) as the benchmark. Significance at the 5% level is indicated by bold italic underlined fonts, against the null of no difference to the RMSE of the benchmark volHist model. Models producing a lower RMSE are more desirable.

The most important findings are:

- The choice of time horizon often matters much more than the model, as evidence by the magnitude of the last row of Table 1 (median across horizons), vs the last column (median across models).
- There are not many models that perform significantly better than the benchmark volHist on this out-of-sample predictive testand many perform significantly worse. .As can be seen in Table 1, the GARCH class of models perform poorly, especially in the short horizon end of the trials. There is only one horizon combination, 500 : 20 where the GARCH class shows significant superior performance to volHist, albeit still outperformed by the EWMA class. Interestingly, the intraday range class performs particularly poorly, even at the short horizon end, with exception of volRangePK, the older (1980) model in the very short 20 day: 20 day end. The more sophisticated volRangeYZ estimator appears uniformly poor across all horizons.
- Despite their theoretical simplicity, the EWMA estimators appear to have the lowest error in estimating out of sample volatility. Particularly volEWMA98 outperform volHist and most other models over most many of the horizon, with altogether the lowest error

In confirmation of these findings, **Table 2** reports rank order correlations of individual stock volatilities using Kendall's Tau as a measure. This test evaluates which model can achieve the best rank-order correspondence between the in-sample and out of sample ranked individual stock volatilities. Considering the rank order in addition to the RMSE of individual volatilities aims to lend robustness to the assessment[v]. Significance against the null hypothesis of no difference vs the volHist benchmark is similarly indicated. Note that high numbers are preferred in this case.

The findings agree almost perfectly with Table 1, in that principally:

- GARCH models as a class do not perform very well, and do not in a single instance outperform volHist.
- Range models perform rather poorly and worse the more advanced and recently developed the model.
- The volEWMA98 estimator is the only one that significantly outperforms the volHist benchmark but in only a few of the horizons tested.
- In addition, it appears that all models perform poorly over a long : short τ combination, e.g. 500 : 20 or 250 : 20. More closely matched m and τ sizes appear more accurate.

Estimation Error of Correlation Models

Turing now to correlations, **Table 3** reports the results of applying the RMSE test to estimated vs realized individual pairwise correlations for all holding periods and all correlations models. Note that horizons where $m < 60$ are not feasible as the matrices will no longer be positive definite, for containing fewer observations than variables. In parallel to the volatility evaluation, significance at the 5% level against the null of no difference to the historical sample benchmark estimator (corrHist), is again indicated in bold italic underlined font. Here we find that:

- In contrast the case with volatilities, many more of the models outperform the historical sample estimator (corrHist).
- In particular, the Rotationally Invariant estimator (corrRIE), with or without cross-sectional volatility (CSV) normalization, outperforms in nearly every period tested and its normalized version, csvNorm_corrRIE has the lowest error of all models.
- Differently from the case for volatilities, the median error across time horizons is now closer to the median across models, suggesting that the time horizon is no longer as dominant as the in the case of volatilities.
- Similarly to the finding in Table 2, holding periods shorter than 40 days become uniformly poor in the result across all models, even if estimated with a long window.

Table 4 repeats the rank order correlation analysis of Table 2, this time for all the individual pairwise correlations. The notable outlier is the average correlation (corrAve), but this is to be expected as an average cannot yield a ranking of the cross section. In this analysis the RIE estimator in non-normalized form (noNorm_corrRIE) performs exceptionally well.

Weight sensitivity under uniform volatilities and correlations

To briefly explore the question of whether risk based portfolios are sensitive to the average level of volatilities or correlations, **Figure 1a and 1b** report results of a controlled studied where respectively the volatilities are correlations are set to a constant, uniformly applied over the cross section, and then varied over a feasible range. Sensitivities are given by the Herfindahl index on portfolio weights, to indicate deviation from uniformity, for every rebalance date in the time series from 1997 to 2017. Contrasting Figure 1a with 1b, we see immediately that when the level of volatilities are set to uniform across the cross-section (every stock is given the same volatility), the portfolio weights range within the same bounds, irrespective of the actual level of volatility. Not so for correlations: portfolio weights are very sensitive to the level of uniform correlations, for two classes of risk-based portfolio, viz. MV and EffBetsPCA which both show extreme ranges in response to varying the level of uniform correlations. ERC, MDP and EffBetsMLT do not appear to show any sensitivity to the *level* of uniform correlations.

'Controlled information content' portfolio behaviour

Figure 2 reports the results of the 'controlled information content' study which 'switches on' various components of the covariance matrix, as described in *Data and Methodology*. To guide interpretation, each panel of Figure 2 reports three portfolio metrics (mean return, out-of-sample portfolio volatility and portfolio turnover, in that order) for, from left-most in red, the equally weighted portfolio, then case where $\Omega = \mathbb{I}$ and the volatility models are varied (in olive green), then the case where $\Omega = E$ and volatility models are varied (in bright green), to the case where $\sigma = k$ and correlation models are varied (in dark blue), to finally the case where $\sigma = \sigma_{Hist}$ in magenta and correlations models are varied and combined with historical correlations. The height of the boxplot bars represent all the horizon combinations encountered in Tables 1 to 4. The oracle estimator is added here for the first time.

The following highlights from many possible observations are noteworthy:

- As mentioned earlier, horizon effects are often greater than model effects, as witnessed by the height of the bars relative to the distances between model averages.
- We note that the tightest compression in all three metrics occurs when $\Omega = \mathbb{I}$ and the correlations are zeroed out, showing that all risk-based portfolios are more sensitive to correlations than to volatilities, but differentially so.
- The oracle estimator (second in every group) shows the best out of sample volatility, but not necessarily the best return or turnover metrics. In the case of volatility, this is reassuring, in that it suggests both that, the oracle estimator is well defined as a benchmark for estimation models, as well as that effort expended in estimation improvements will yield a benefit.
- Whereas nearly all of the risk-based portfolios improve upon the EW portfolio, it is harder to see how most of the models convincingly outperform the historical sample estimator for that same portfolio (the first in every group).

Empirical rank order distributions of combined volatility and correlation models

Figure 3 shows six different performance metrics for all the risk-based portfolios, in a high-low ranked order. The equally weighted portfolio appears as a dark dotted line. Vertical lines represent the position of the sample historical estimators (volHist and corrHist), combining into the sample covariance matrix E, in the rank order of all other possible permutations (122 in total) of volatility and correlation models combinations, with matching colours identifying the portfolio. Everything else other than the covariance model is held constant. Several interesting observations follow:

- Considering the range of the curves and their shape across risk-based portfolios, especially along dimensions of Sharpe ratio and turnover, in order of sensitivity to the covariance model combination, EffBetsPCA is the most sensitive, followed by MV, then MDP, then ERC, and finally EffBetsMLT shows the least degree of sensitivity.
- The location of the sample historical covariance estimator, is not in the top-ranked position, on any of the metrics. It appears possible to improve on this benchmark, but not equally across portfolios. Considering the flatness of the curve beyond this point in the cases of MDP and ERC for instance, it seems unlikely that such improvements would be significant. Not so for EffBetsPCA, where large performance enhancements appear possible. (Tables 5 to 9 display significance tests for these.)
- Inspecting the turnover panel as an additional guide to portfolio sensitivity to the covariance MDP shows a unique drop around position 100. The results reveal that this is the position below which all the average correlation permutations (corrAve) are found. This shows the sensitivity of MDP to information in the correlation structure, beyond that of any other portfolio.

- Perhaps a remarkable finding is in the panel representing distributional skewness. Whereas EW, ERC and EffBetsMLT all share a negative skew in the order of -0.2 , almost completely invariant to the covariance, MDP, MV and especially EffBetsPCA are able to achieve a skew in their distributions at much higher levels, reaching positive numbers and in the case of EffBetsPCA, numbers > +1.0. This is rarely seen in long-only, fully invested equity portfolios.

Full scale permutations of volatility and correlation models

Complete performance data summarized in Figure 3 are exhibited in **Tables 5 to 9** for reference purposes. Colour maps aid interpretation and range statistics are displayed. The position of the oracle estimator (called oracleCov) is also shown, along with that of the equally weighted portfolio. A single horizon of 500 : 125 days is used for all portfolios. Significance below the 5% level against of a null of no difference to the sample historical covariance estimator (composed out of volHist and corrHist), is indicated by bold text and boxes around the datapoints. The sample historical benchmarks (volHist and corrHist) are stated in the first rows and first columns. Observations are too many to itemize, but in general:

- The sample historical estimators is difficult to outperform reliably. This occurs typically when the EWMA class volatility estimators are combined with the RIE correlation shrinkage model, albeit infrequently.
- The usefulness of a rich model of the covariance will depend critically on the type of risk based portfolio within which it is used. For instance, in the case of ERC there exists not a single model combination that is able to reliably outperform the sample historical estimator and very few that are able to reliably underperform. This illustrates the strong internal shrinkage embedded in the ERC portfolio objective function, first noted by Roncalli (2013).
- In unreported results, nearly all of the risk-based portfolio combinations outperform the EW portfolio on all metrics except turnover (with strong significance observed for returns, out-of-sample volatility and Sharpe ratios), despite the fact that expected returns form no part of their construction. This finding is worthy of study in its own right.
- It is the covariance matrix of individual asset returns that is in use in all instances in this study. In unreported results, there exists a preliminary indication that when the dimension is reduced from $N = 60$ that constitutes the dimension of most portfolios here, the probably of reliably outperforming the sample covariance estimator may increase. Pre-specified factor models where the risk-based portfolio is solved upon the factor portfolios rather than the underlying assets, may be such a candidate. We leave this for future work.

6. Conclusion

This study aims to connect a rich literature and several sophisticated recent methods in covariance estimation to the domain of risk-based portfolio construction. Although increasingly popular of late, the systematic study of covariance estimation in the context of general risk based portfolio construction has mostly escaped attention, other than for the recent study by Ardia et al. (2017). Insofar as some studies of estimation touched on portfolio construction, these have almost exclusively focused on mean-variance or minimum-variance portfolios We make a contribution to this under-researched area via illustrating the impact of covariance estimation for six risk based portfolios, in a long-only equity market setting. In general the finding is that the simple sample covariance estimator is difficult to outperform and that reliable outperformance strongly depends on the precise risk-based portfolio objective function, with certain of these showing strong sensitivity to covariance estimation and others showing none whatsoever. Several interesting properties of risk-based portfolios are uncovered, notably the ability to produce a strong positive distributional skew, and in general, strong outperformance of the equally weighted portfolio on many dimensions of evaluation, despite no consideration of expected returns in their construction process

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Table 1: Estimation Error of Volatility Models

RMSE of Volatility Models: Estimated vs Out-Of-Sample Volatility of Individual Volatilities

Holding Period	Conditioning Window	**volHist (benchmark)**	volEWMA92	volEWMA96	volEWMA98	volGARCH(1,1)N	volGARCH(1,1)t	volGARCH(GJR)N	volGARCH(GJR)t	volRangePK	volRangeGK	volRangeRS	volRangeYZ	*Median (Models)*
125	500	0.122	0.123	0.109	***0.104***	0.126	0.128	0.128	1.019	0.125	0.129	***0.132***	***0.174***	0.127
	250	0.115	0.124	0.110	0.105	0.128	0.286	0.136	0.205	0.117	0.121	***0.124***	***0.169***	0.124
	125	0.113	0.125	0.111	0.106	0.127	0.257	***0.139***	***0.177***	0.113	0.118	0.122	***0.171***	0.123
60	500	0.130	0.122	***0.115***	***0.112***	0.122	0.199	0.119	0.121	0.133	***0.136***	***0.140***	***0.184***	0.126
	250	0.123	0.123	0.116	***0.114***	0.123	0.225	0.121	0.219	0.124	0.128	***0.132***	***0.183***	0.124
	125	0.118	0.123	0.116	***0.112***	0.123	***0.490***	0.126	0.302	0.119	0.124	***0.129***	***0.187***	0.124
40	500	0.133	0.125	***0.117***	***0.114***	0.128	0.180	0.128	0.129	0.136	***0.139***	***0.143***	***0.189***	0.131
	250	0.126	0.127	0.118	***0.116***	0.128	0.200	0.128	0.193	0.127	0.131	***0.135***	***0.184***	0.128
	125	0.120	0.127	0.118	***0.114***	0.128	***0.433***	***0.131***	0.379	0.122	***0.126***	***0.131***	***0.186***	0.127
20	500	0.145	***0.131***	***0.125***	***0.125***	***0.133***	0.160	***0.132***	***0.134***	0.146	***0.150***	***0.153***	***0.201***	0.140
	250	0.137	0.132	***0.126***	***0.125***	0.134	0.172	0.133	0.214	0.138	***0.141***	***0.146***	***0.197***	0.138
	125	0.130	0.132	***0.125***	***0.123***	0.134	***0.487***	***0.136***	***0.341***	0.131	***0.135***	***0.140***	***0.194***	0.135
500	500	0.127	0.129	0.115	0.107	0.143	0.141	0.140	0.143	0.132	0.136	0.138	0.160	0.137
375	375	0.124	0.141	0.130	0.121	0.144	0.144	0.143	0.146	0.130	0.134	0.136	***0.173***	0.138
250	250	0.117	0.126	0.113	0.108	0.123	0.131	0.128	0.132	0.118	0.121	0.125	***0.171***	0.124
200	200	0.124	0.126	0.116	***0.114***	0.130	0.135	0.132	0.137	0.125	0.128	0.132	***0.173***	0.129
160	160	0.115	***0.141***	0.128	0.118	***0.135***	1.605	***0.134***	0.388	0.116	0.121	0.125	***0.161***	0.131
125	125	0.113	0.125	0.111	0.106	0.127	0.257	***0.139***	***0.177***	0.113	0.118	0.122	***0.171***	0.123
80	80	0.118	***0.130***	0.120	0.117	***0.132***	***1.988***	***0.133***	***0.487***	0.118	0.123	***0.128***	***0.162***	0.129
60	60	0.121	0.124	0.118	0.121	***0.130***	***2.567***	***0.135***	***2.065***	0.120	0.125	***0.131***	***0.175***	0.128
40	40	0.120	0.123	0.118	***0.134***	***0.135***	***4.886***	***0.143***	***3.048***	0.117	0.122	***0.127***	***0.178***	0.131
20	20	0.135	0.135	***0.147***	***0.182***	***0.149***	***7.605***	***0.157***	***5.802***	***0.128***	0.132	***0.139***	***0.197***	0.148
Median (Horizons)		0.122	0.126	0.117	0.114	0.129	0.241	0.133	0.210	0.124	0.128	0.132	0.177	

Table 1 reports the estimation error of the volatility models in the form of Root-Mean-Squared-Error (RMSE) of volatility estimate. Numbers displayed are the time series averages over the entire study period (03-Mar-1998 to 31-May-2017) of the cross-sectional RMSE of in-sample to out-of-sample volatilities for every stock at each rebalance point, for 22 different conditioning window and holding period horizon combinations. The comparison benchmark is the sample historical volatility estimator, shown in the first column. Colour scales aid observation with red in the unfavourable direction and green showing better-than-benchmark estimators. Median errors across all horizons are shown in the final row and across models in the final column. Bold italic and underlined fonts indicate significance at the 5% level or below.

Table 2: In-sample to Out-of-Sample Pairwise Rank Order Correlation of Correlation Models

Rank Order Correlation of Volatility Models: Estimated vs Out-of-Sample Rank Order of Individual Volatilities - Kendall's Tau														
Holding Period	Conditioning Window	**volHist (benchmark)**	volEWMA92	volEWMA96	volEWMA98	volGARCH(1,1)N	volGARCH(1,1)t	volGARCH(GJR)N	volGARCH(GJR)t	volRangePK	volRangeGK	volRangeRS	volRangeYZ	*Median (Models)*
125	500	0.504	***0.466***	0.503	0.522	***0.462***	***0.459***	***0.461***	***0.459***	***0.445***	***0.420***	***0.398***	***0.378***	0.460
	250	0.516	***0.456***	0.493	0.515	***0.463***	***0.453***	***0.458***	***0.453***	***0.468***	***0.444***	***0.421***	***0.410***	0.457
	125	0.498	***0.457***	0.491	0.501	***0.452***	***0.437***	***0.436***	***0.427***	***0.466***	***0.442***	***0.418***	***0.413***	0.447
60	500	0.466	***0.416***	0.461	***0.493***	***0.428***	***0.423***	***0.430***	***0.427***	***0.410***	***0.390***	***0.371***	***0.359***	0.425
	250	0.477	***0.415***	0.460	***0.491***	***0.428***	***0.419***	***0.430***	***0.417***	***0.435***	***0.413***	***0.392***	***0.384***	0.423
	125	0.483	***0.414***	***0.459***	0.483	***0.419***	***0.407***	***0.408***	***0.393***	***0.445***	***0.422***	***0.398***	***0.396***	0.417
40	500	0.438	***0.406***	0.445	***0.468***	***0.414***	***0.409***	***0.415***	***0.412***	***0.390***	***0.372***	***0.351***	***0.341***	0.411
	250	0.451	***0.406***	0.445	***0.466***	***0.417***	***0.408***	***0.413***	***0.407***	***0.412***	***0.393***	***0.372***	***0.367***	0.410
	125	0.457	***0.404***	***0.442***	0.458	***0.404***	***0.391***	***0.397***	***0.390***	***0.424***	***0.405***	***0.383***	***0.380***	0.404
20	500	0.384	0.377	***0.404***	***0.419***	0.382	0.378	0.385	0.381	***0.345***	***0.330***	***0.313***	***0.304***	0.379
	250	0.398	***0.376***	0.402	***0.417***	***0.378***	***0.370***	***0.377***	***0.369***	***0.366***	***0.350***	***0.333***	***0.326***	0.373
	125	0.408	***0.374***	***0.401***	***0.413***	***0.372***	***0.360***	***0.365***	***0.357***	***0.378***	***0.362***	***0.343***	***0.339***	0.369
500	500	0.496	0.448	0.491	0.491	0.435	0.427	***0.430***	0.421	***0.419***	***0.397***	***0.369***	***0.357***	0.429
375	375	0.528	0.471	0.497	0.517	***0.450***	***0.437***	***0.448***	***0.438***	***0.445***	***0.427***	***0.406***	***0.394***	0.446
250	250	0.534	***0.445***	0.499	0.529	***0.471***	***0.457***	***0.459***	***0.445***	***0.475***	***0.450***	***0.425***	***0.408***	0.458
200	200	0.506	***0.434***	0.480	0.505	***0.461***	***0.452***	***0.450***	***0.447***	***0.458***	***0.436***	***0.415***	***0.401***	0.451
160	160	0.513	***0.433***	***0.483***	0.508	***0.447***	***0.442***	***0.432***	***0.439***	***0.468***	***0.442***	***0.417***	***0.415***	0.442
125	125	0.498	***0.457***	0.491	0.501	***0.452***	***0.437***	***0.436***	***0.427***	***0.466***	***0.442***	***0.418***	***0.413***	0.447
80	80	0.477	***0.421***	***0.462***	0.475	***0.428***	***0.409***	***0.410***	***0.397***	***0.448***	***0.425***	***0.403***	***0.405***	0.423
60	60	0.467	***0.415***	***0.449***	0.462	***0.407***	***0.389***	***0.390***	***0.373***	***0.435***	***0.413***	***0.389***	***0.388***	0.410
40	40	0.421	***0.399***	0.416	0.418	***0.373***	***0.341***	***0.360***	***0.337***	***0.404***	***0.384***	***0.363***	***0.363***	0.379
20	20	0.357	***0.346***	***0.350***	***0.351***	***0.313***	***0.279***	***0.295***	***0.273***	0.350	***0.333***	***0.311***	***0.313***	0.323
Median (Horizons)		0.477	0.416	0.461	0.487	0.428	0.414	0.422	0.414	0.435	0.413	0.391	0.382	

Table 2 reports the rank order correlation between In-Sample (IS) and Out-of-Sample (OOS) volatility estimates for all models under study. Numbers displayed are the time series averages of the rank order correlation using Kendall's Tau as a measure, for each cross-section of stocks in the IS vs OOS periods, for each of 22 different holding period and conditioning window combinations. Colour scales aid interpretation with green blocks representing a better estimate (higher rank order correlation) and red blocks an inferior estimate. Median rank order correlations across all horizons are shown in the final row and across models in the final column. Bold italic and underlined fonts indicate significance at the 5% level or below.

Table 3: Estimation Error of Correlation Models

RMSE of Correlation Models: Estimate vs Out-Of-Sample Error of Individual Pairwise Correlations

Holding Period	Conditioning Window	**noNorm_corrHist (Benchmark)**	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	noNorm_LWSMM	noNorm_LWCC	*Median (Models)*
125	500	0.153	***0.147***	***0.183***	***0.179***	***0.159***	***0.150***	***0.157***	0.152	***0.152***	***0.147***	0.153	***0.151***	0.152
	250	0.160	***0.151***	***0.185***	***0.180***	***0.169***	0.158	***0.158***	***0.150***	***0.156***	***0.149***	***0.157***	***0.155***	0.158
	125	0.176	***0.164***	***0.190***	0.183	***0.189***	0.177	***0.171***	***0.158***	***0.167***	***0.157***	***0.169***	***0.165***	0.170
60	500	0.182	***0.177***	***0.207***	***0.204***	***0.187***	***0.180***	***0.185***	0.181	***0.180***	***0.176***	***0.181***	***0.180***	0.181
	250	0.186	***0.179***	***0.209***	***0.205***	***0.195***	0.186	***0.185***	***0.178***	***0.183***	***0.177***	***0.184***	***0.182***	0.185
	125	0.197	***0.189***	***0.210***	***0.207***	***0.209***	***0.200***	***0.192***	***0.184***	***0.188***	***0.183***	***0.190***	***0.187***	0.191
40	500	0.204	***0.200***	***0.227***	***0.224***	***0.210***	0.203	***0.207***	0.203	***0.203***	***0.199***	***0.204***	***0.202***	0.203
	250	0.207	***0.201***	***0.227***	***0.224***	***0.215***	0.207	***0.206***	***0.200***	***0.204***	***0.199***	***0.205***	***0.203***	0.206
	125	0.215	***0.208***	***0.229***	***0.225***	***0.227***	***0.219***	***0.211***	***0.204***	***0.208***	***0.203***	***0.210***	***0.207***	0.210
20	500	0.259	***0.255***	***0.277***	***0.274***	***0.264***	0.258	***0.261***	0.258	***0.258***	***0.255***	***0.259***	***0.258***	0.258
	250	0.261	***0.256***	***0.277***	***0.274***	***0.267***	0.261	***0.260***	***0.255***	***0.258***	***0.254***	***0.259***	***0.258***	0.260
	125	0.266	***0.261***	***0.277***	***0.275***	***0.276***	***0.270***	***0.262***	***0.257***	***0.260***	***0.256***	***0.262***	***0.259***	0.262
500	500	0.135	0.124	***0.164***	0.156	0.142	0.127	***0.139***	0.129	***0.133***	0.123	0.134	***0.131***	0.134
375	375	0.136	0.125	***0.166***	***0.157***	***0.144***	0.131	0.139	0.128	***0.133***	0.123	0.135	***0.131***	0.134
250	250	0.143	0.134	***0.173***	***0.166***	***0.156***	0.142	***0.141***	***0.132***	***0.139***	***0.132***	***0.140***	***0.138***	0.141
200	200	0.161	0.154	***0.183***	***0.179***	***0.173***	0.163	***0.158***	***0.151***	***0.155***	***0.150***	***0.157***	***0.153***	0.157
160	160	0.165	***0.154***	***0.184***	***0.179***	***0.178***	0.166	***0.161***	***0.150***	***0.158***	***0.149***	***0.160***	***0.157***	0.161
125	125	0.176	***0.164***	***0.190***	0.183	***0.189***	0.177	***0.171***	***0.158***	***0.167***	***0.157***	***0.169***	***0.165***	0.170
80	80	0.198	***0.191***	0.201	0.199	***0.215***	***0.207***	***0.189***	***0.182***	***0.184***	***0.181***	***0.186***	***0.181***	0.190
60	60	0.218	***0.210***	***0.210***	***0.209***	***0.240***	***0.231***	***0.208***	***0.199***	***0.201***	***0.197***	***0.202***	***0.195***	0.208
Median (Horizons)		0.184	0.178	0.204	0.201	0.192	0.183	0.185	0.180	0.182	0.177	0.183	0.180	

Table 3 reports the estimation error of the correlation models in the form of Root-Mean-Squared-Error (RMSE) of pairwise correlation estimate. Numbers displayed are the time series averages over the entire study period (03-Mar-1998 to 31-May-2017) of the cross-sectional RMSE of in-sample to out-of-sample correlations for every stock at each rebalance point, for 20 different conditioning window and holding period horizon combinations. The comparison benchmark is the sample historical correlation estimator, shown in the first column. Colour scales aid observation with red in the unfavourable direction and green showing better-than-benchmark estimators. Median errors across all horizons are shown in the final row and across models in the final column. Bold italic and underlined fonts indicate significance at the 5% level or below.

Table 4: In-sample to Out-of-Sample Pairwise Rank Order Correlation of Correlation Models

Rank Order Correlation of Correlation Models: Estimated vs Out-of-Sample Rank Order of Individual Pairwise Correlations - Kendall's Tau

Holding Period	Conditioning Window	**noNorm_corrHist (Benchmark)**	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	noNorm_LWSMM	noNorm_LWCC	*Median (Models)*
125	500	**0.418**	***0.406***	***0.012***	***0.012***	0.413	0.413	***0.408***	***0.399***	***0.422***	0.413	0.417	0.418	0.413
	250	**0.408**	***0.398***	***-0.006***	***-0.014***	0.399	***0.399***	***0.420***	***0.416***	***0.418***	0.413	0.410	0.408	0.408
	125	**0.374**	***0.367***	***-0.020***	***-0.013***	***0.360***	***0.361***	***0.395***	***0.397***	***0.392***	***0.393***	***0.382***	0.374	0.374
60	500	**0.353**	***0.345***	***0.004***	***-0.002***	***0.346***	***0.347***	***0.344***	***0.337***	***0.357***	0.351	0.354	0.353	0.347
	250	**0.349**	***0.342***	***0.003***	***0.002***	***0.340***	***0.340***	***0.359***	***0.356***	***0.358***	***0.355***	0.351	0.349	0.349
	125	**0.327**	***0.320***	***0.004***	***0.002***	***0.312***	***0.311***	***0.346***	***0.345***	***0.343***	***0.342***	***0.333***	0.327	0.327
40	500	**0.317**	***0.310***	***-0.008***	***-0.002***	***0.311***	***0.311***	***0.309***	***0.303***	***0.321***	0.315	0.318	0.317	0.311
	250	**0.314**	***0.307***	***-0.001***	***0.001***	***0.305***	***0.305***	***0.322***	***0.319***	***0.322***	0.318	0.316	0.314	0.314
	125	**0.297**	***0.290***	***-0.001***	***0.001***	***0.284***	***0.283***	***0.313***	***0.311***	***0.312***	***0.308***	***0.303***	0.297	0.297
20	500	**0.249**	***0.245***	***-0.005***	***-0.003***	***0.244***	***0.245***	***0.242***	***0.239***	***0.252***	0.249	0.250	0.249	0.245
	250	**0.247**	***0.243***	***0.000***	***0.000***	***0.240***	***0.240***	***0.253***	***0.251***	***0.253***	***0.251***	***0.249***	0.247	0.247
	125	**0.236**	***0.230***	***0.003***	***-0.001***	***0.225***	***0.224***	***0.247***	***0.245***	***0.247***	***0.244***	***0.240***	0.236	0.236
500	500	**0.470**	0.455	***-0.019***	***-0.017***	0.461	0.463	0.458	0.452	0.474	0.462	0.469	0.470	0.462
375	375	**0.473**	0.452	***0.014***	***0.012***	0.464	0.458	0.466	0.456	***0.481***	0.464	0.473	0.473	0.464
250	250	**0.458**	***0.440***	***-0.011***	***-0.012***	0.448	***0.444***	***0.473***	0.461	***0.471***	0.457	0.460	0.458	0.457
200	200	**0.411**	***0.396***	***0.000***	***-0.008***	0.401	***0.399***	***0.429***	0.421	***0.424***	0.417	0.414	0.411	0.411
160	160	**0.406**	***0.394***	***0.015***	***0.011***	***0.393***	***0.390***	***0.427***	***0.422***	***0.424***	***0.417***	***0.411***	0.406	0.406
125	125	**0.374**	***0.367***	***-0.020***	***-0.013***	***0.360***	***0.361***	***0.395***	***0.397***	***0.392***	***0.393***	***0.382***	0.374	0.374
80	80	**0.320**	***0.309***	***0.009***	***-0.001***	***0.300***	***0.296***	***0.344***	***0.339***	***0.342***	***0.334***	***0.329***	0.320	0.320
60	60	**0.278**	***0.271***	***0.002***	***0.001***	***0.257***	***0.257***	***0.299***	***0.296***	***0.294***	***0.289***	***0.285***	0.278	0.278
Median (Horizons)		**0.351**	0.344	0.000	-0.001	0.343	0.343	0.353	0.350	0.358	0.353	0.352	0.351	

Table 4 reports the rank order correlation between In-Sample (IS) and Out-of-Sample (OOS) volatility estimates for all models under study. Numbers displayed are the time series averages of the rank order correlation using Kendall's Tau as a measure, for each cross-section of stocks in the IS vs OOS periods, for each of 20 different holding period and conditioning window combinations. Colour scales aid interpretation with green blocks representing a better estimate (higher rank order correlation) and red blocks an inferior estimate. Median rank order correlations across all horizons are shown in the final row and across models in the final column. Bold italic and underlined fonts indicate significance at the 5% level or below.

Figure 1a: Herfindahl Index of Portfolio Weights under Uniform Volatilities

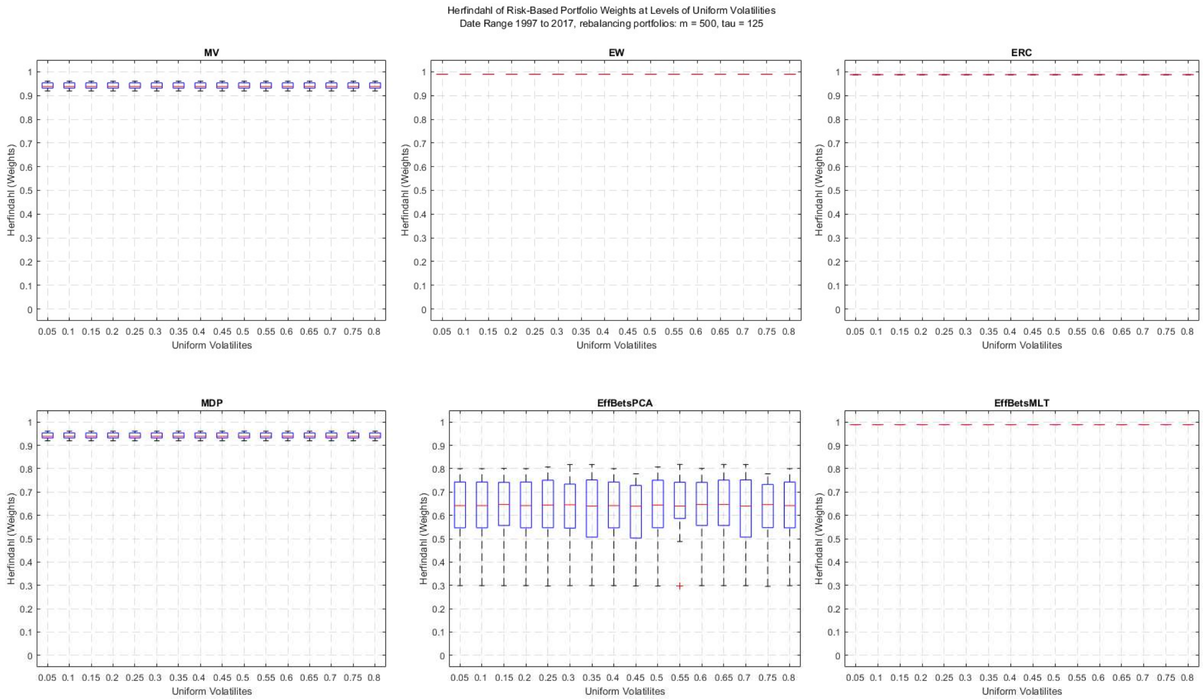


Figure 1b: Herfindahl Index of Portfolio Weights under Uniform Correlations

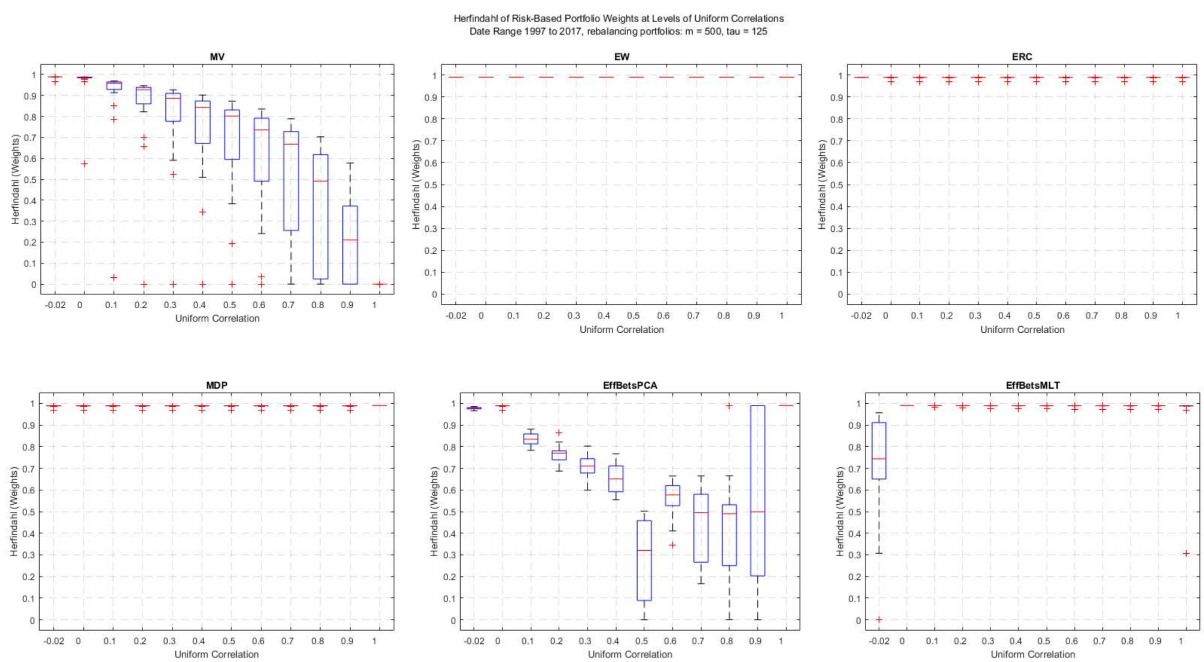


Figure 1a and 1b normalized Herfindahl indices of portfolio weights for each risk-based portfolio. The Herfindahl indices $H(x) \in [0,1]$ are scaled such that maximum concentration is reached at 0 and full diversification (i.e. equal weight) is at 1. Datapoints are collected from rebalancing portfolios with $m = 500;\ \tau = 125$ with either the volatilities (Figure 1a) or the correlations (Figure 1b) controlled to be uniform, over the ranges as depicted on the x-axis. Each set of uniform volatility and correlation points represent a full run through the 1997 to 2017 dataset.

Figure 2 - Panel 1: MV Portfolio Empirical Performance Under Controlled Covariance

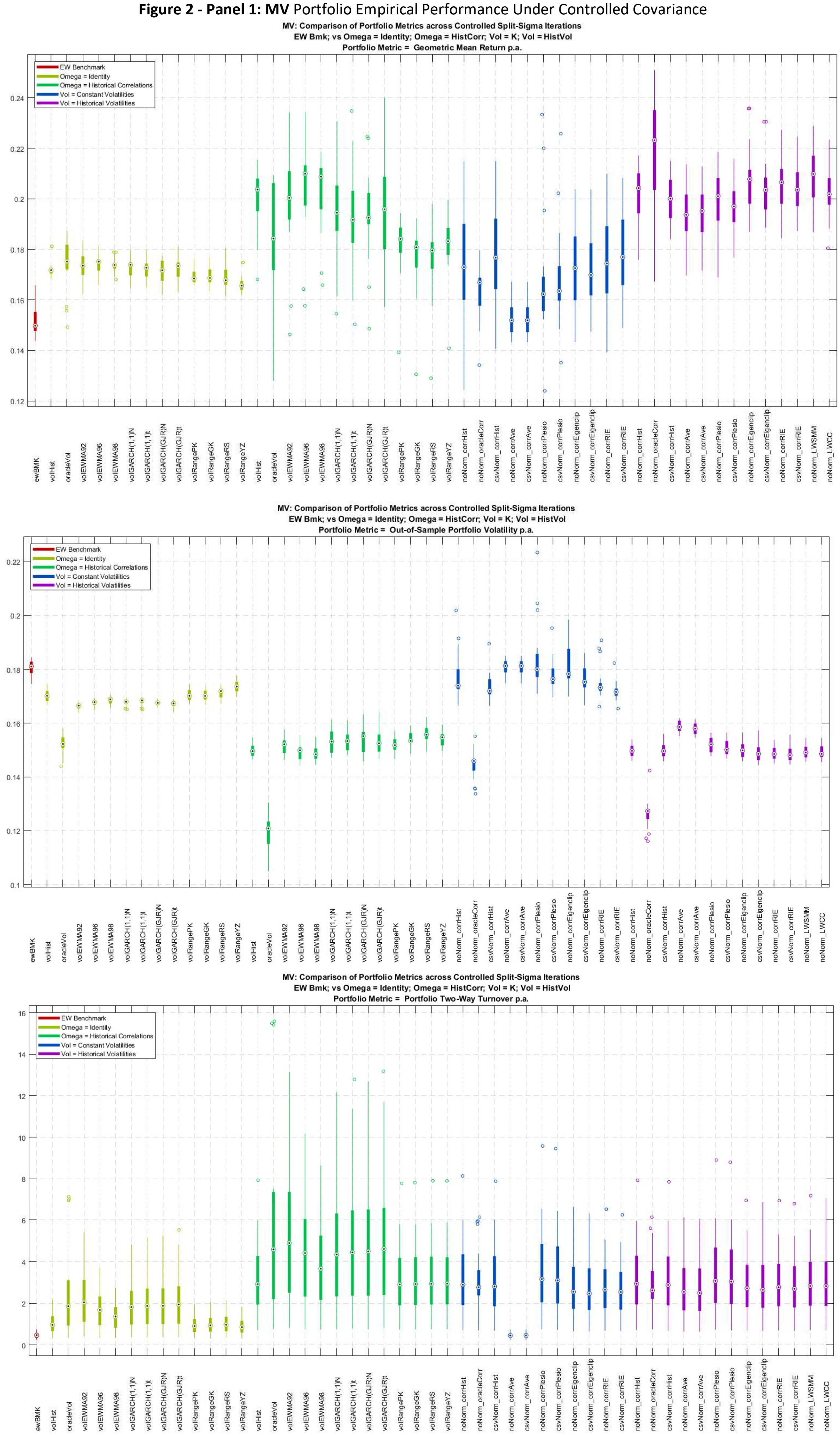
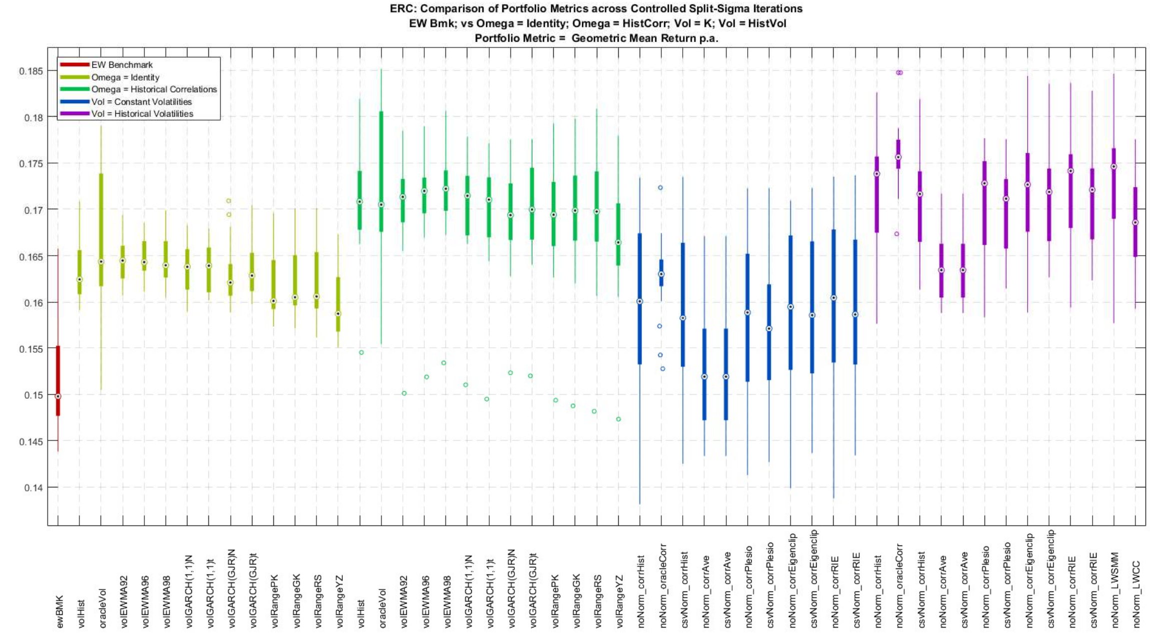
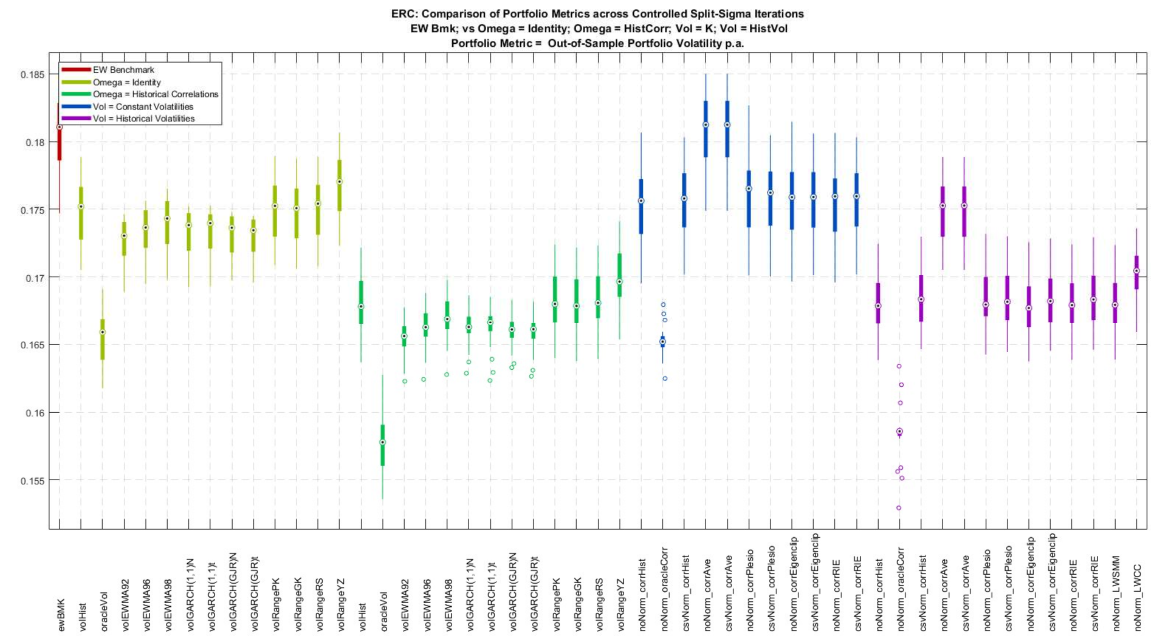


Figure 2 - Panel 2: ERC Portfolio Empirical Performance Under Controlled Covariance





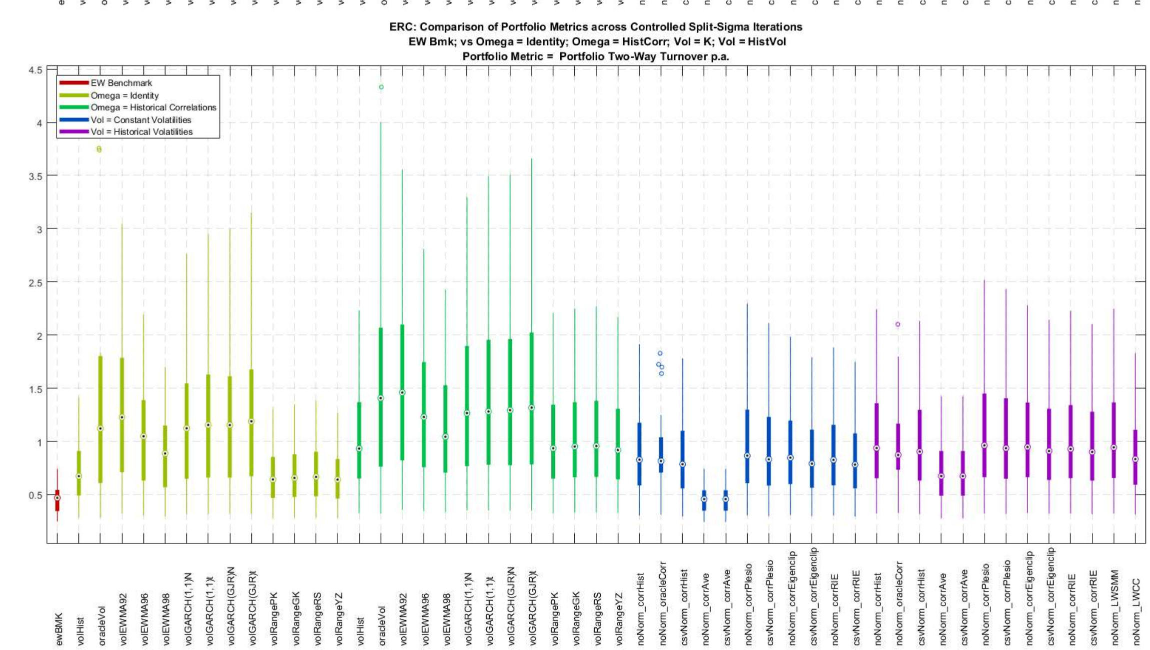
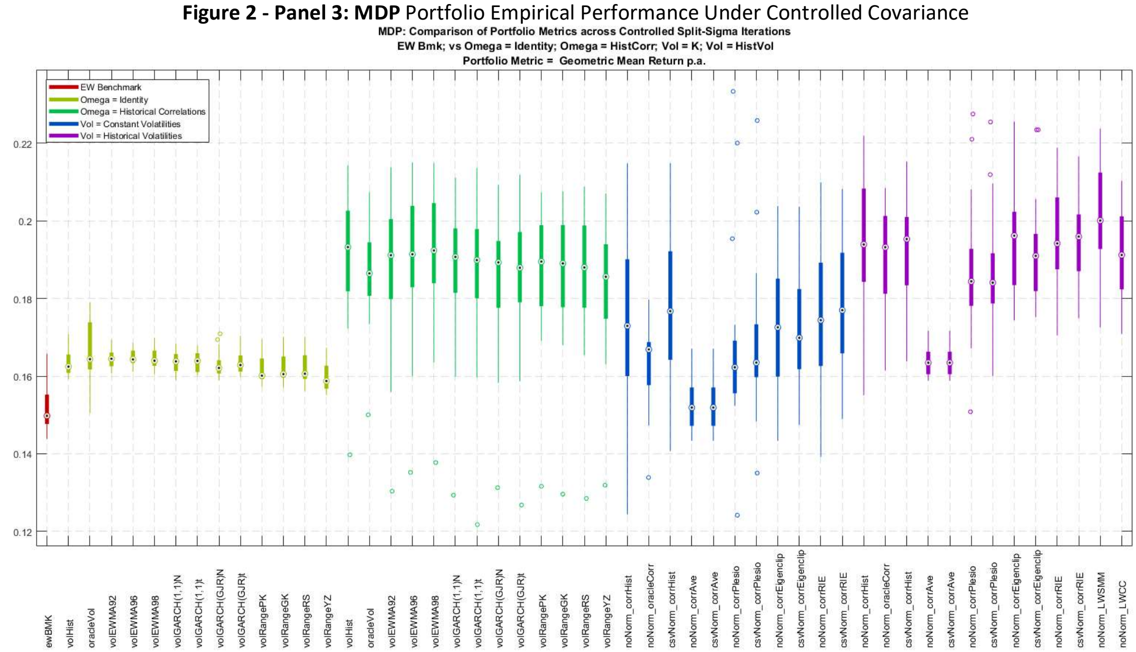
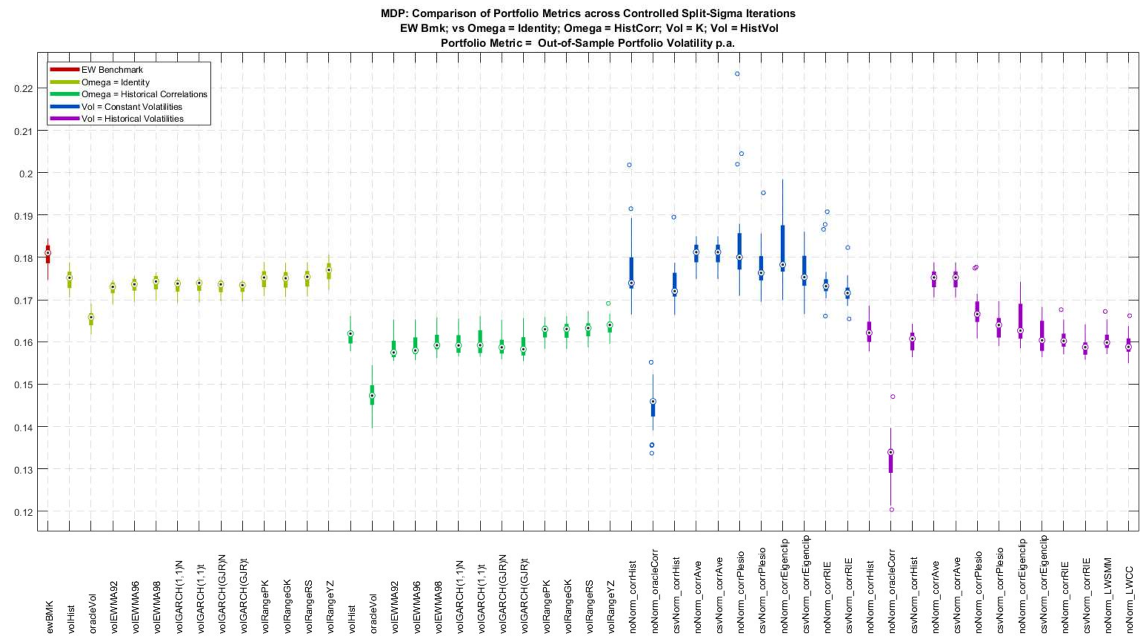


Figure 2 - Panel 3: MDP Portfolio Empirical Performance Under Controlled Covariance





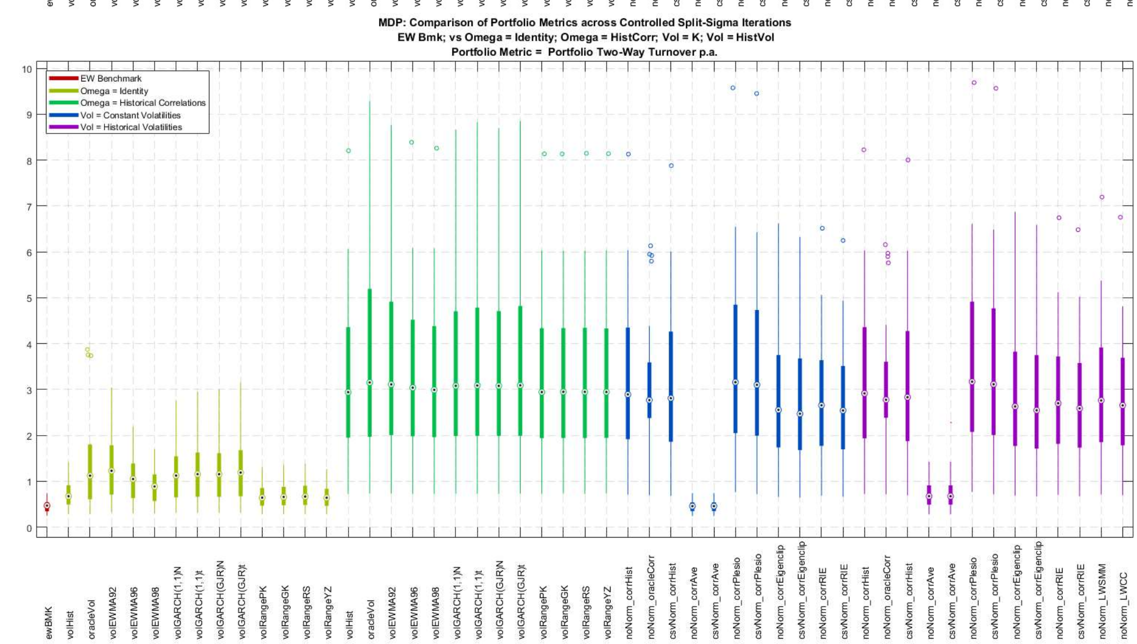


Figure 2 - Panel 4: EffBetsPCA Portfolio Empirical Performance Under Controlled Covariance

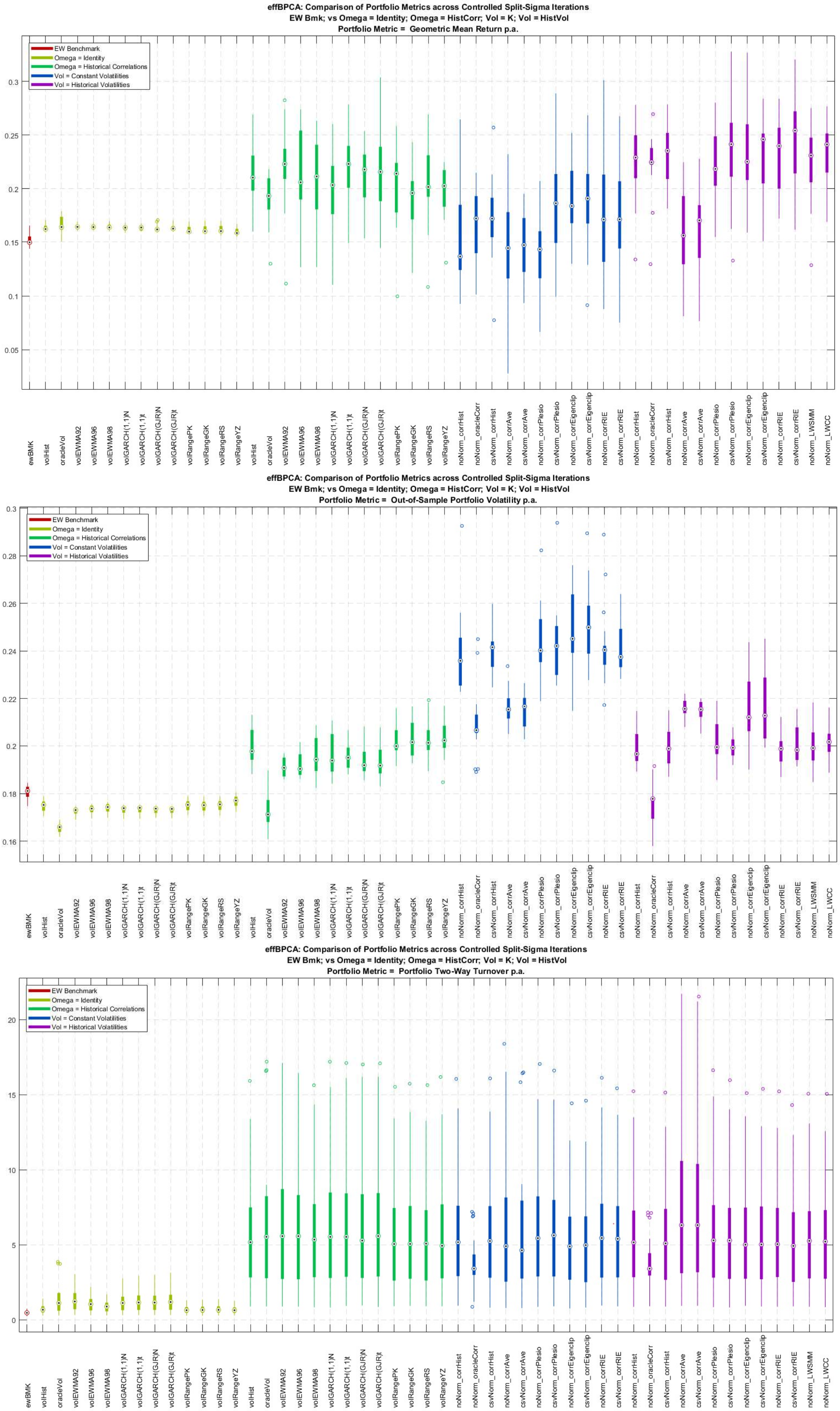
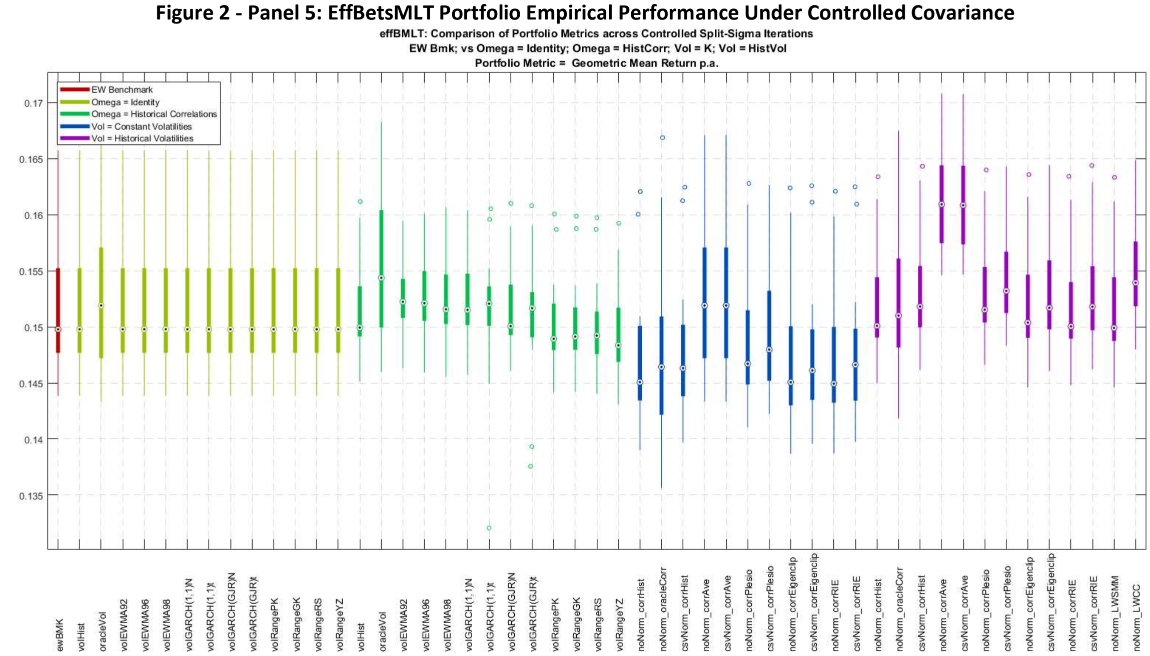
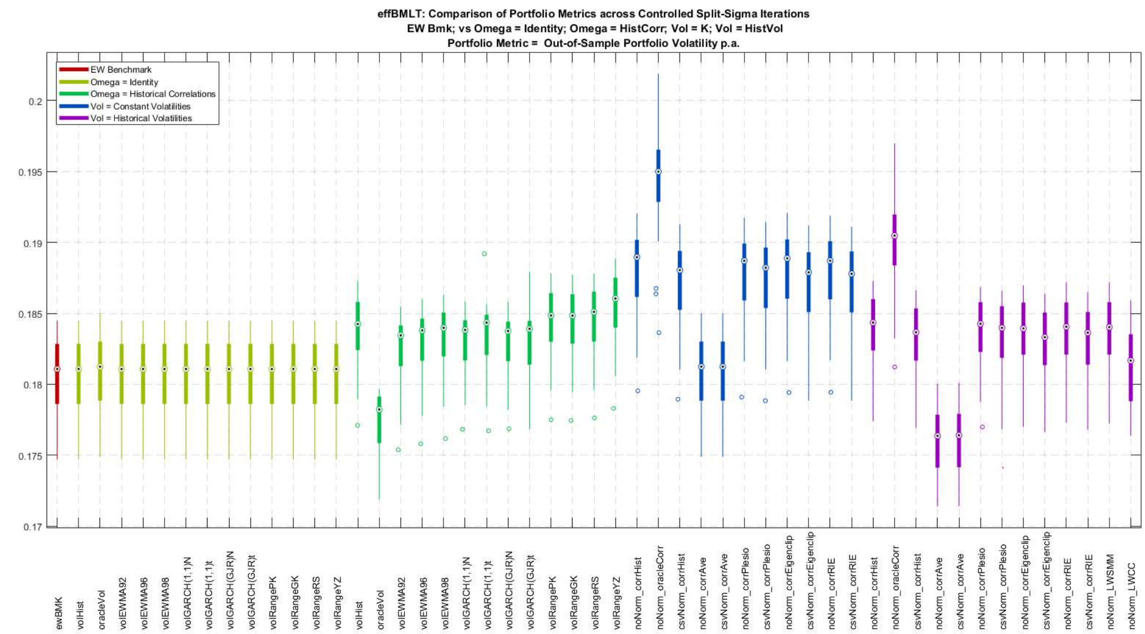


Figure 2 - Panel 5: EffBetsMLT Portfolio Empirical Performance Under Controlled Covariance





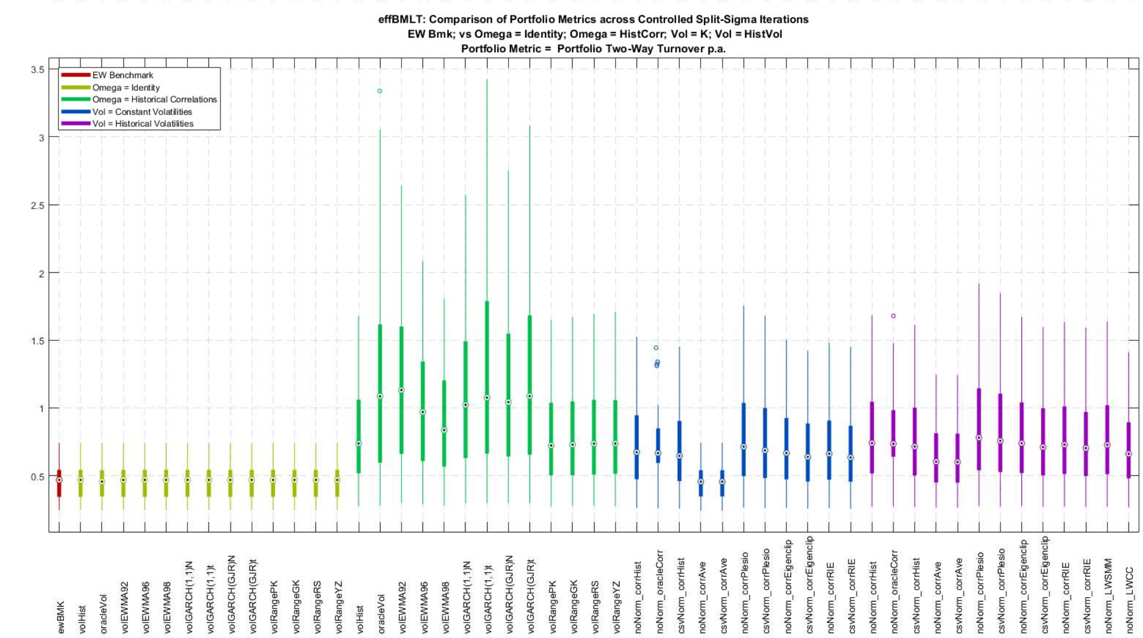


Figure 3: Rank Order Distributions of Volatility-Correlation Model Empirical Performance Metrics

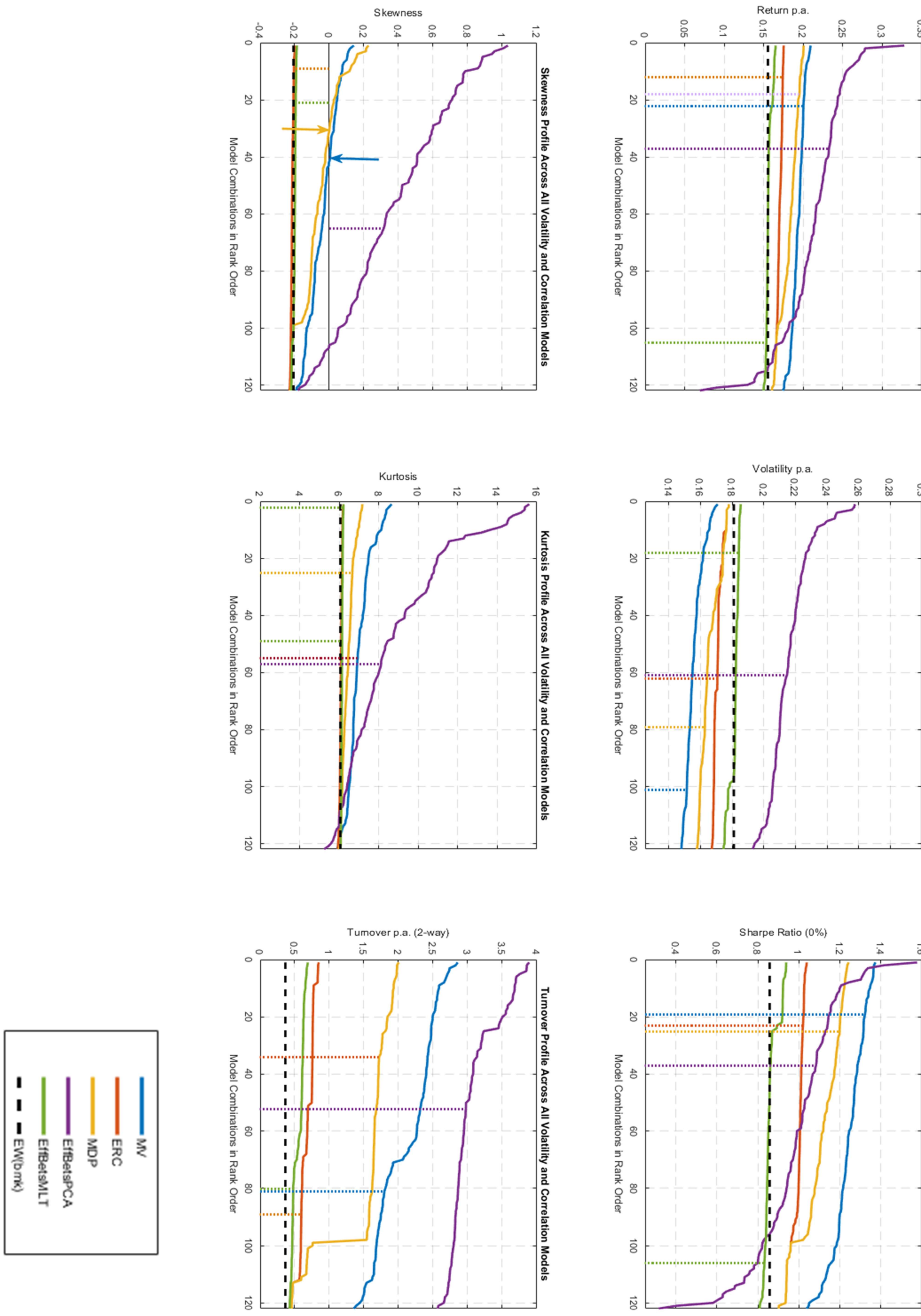


Table 5 – Panel 1: **MV** Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

MV — Geometric Mean Return p.a.

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	20.02%	20.36%	18.37%	18.39%	19.18%	19.28%	20.92%	20.88%	**20.47%**	20.83%	19.87%	**Min (Ex-Ora.)**
volEWMA92	19.14%	18.72%	17.60%	17.58%	18.21%	18.12%	18.96%	18.76%	19.20%	18.77%	18.51%	17.41%
volEWMA96	19.82%	19.51%	19.39%	19.20%	19.12%	19.18%	19.81%	19.76%	19.94%	19.66%	19.54%	
volEWMA98	19.98%	19.98%	20.23%	20.01%	19.38%	19.55%	19.84%	19.99%	20.23%	20.17%	19.94%	**Max (Ex-Ora.)**
volGARCH(1,1)N	19.55%	19.93%	18.82%	18.89%	19.24%	19.47%	19.58%	19.69%	19.76%	20.07%	19.50%	20.92%
volGARCH(1,1)t	19.95%	20.03%	18.44%	18.60%	19.61%	19.83%	20.09%	19.99%	20.14%	20.20%	19.69%	
volGARCH(GJR)N	18.87%	19.08%	17.75%	17.74%	18.19%	18.55%	18.50%	18.70%	19.00%	19.20%	18.56%	**Oracle**
volGARCH(GJR)t	18.99%	19.16%	17.41%	17.58%	18.45%	18.83%	18.99%	18.99%	19.08%	19.24%	18.67%	21.15%
volRangePK	18.97%	20.29%	19.68%	19.63%	18.65%	19.09%	19.86%	20.70%	19.51%	20.76%	19.71%	
volRangeGK	18.52%	19.80%	19.64%	19.58%	18.38%	18.78%	19.56%	20.39%	19.01%	20.25%	19.39%	**Range**
volRangeRS	18.28%	19.52%	19.62%	19.51%	18.29%	18.66%	19.33%	20.17%	18.78%	19.98%	19.21%	**(Ex-Oracle)**
volRangeYZ	19.14%	19.66%	18.47%	18.39%	18.60%	18.76%	19.79%	19.96%	19.62%	20.15%	19.25%	3.51%
Mean Across Vol.	19.27%	19.67%	18.78%	18.76%	18.78%	19.01%	19.60%	19.83%	19.56%	19.94%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
20.64%	19.93%	21.15%	15.53%

MV — Out-of-Sample Volatility p.a.

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	15.13%	15.19%	**16.19%**	**16.15%**	15.26%	15.18%	15.34%	15.32%	15.11%	15.16%	15.41%	**Min (Ex-Ora.)**
volEWMA92	15.15%	15.10%	**16.06%**	**15.99%**	15.18%	15.14%	15.30%	15.22%	15.13%	15.10%	15.34%	14.80%
volEWMA96	14.91%	14.90%	**15.95%**	**15.87%**	14.95%	14.92%	15.08%	15.01%	14.89%	14.89%	15.14%	
volEWMA98	14.83%	14.81%	**15.85%**	**15.77%**	14.88%	14.84%	14.97%	14.93%	14.80%	14.80%	15.05%	**Max (Ex-Ora.)**
volGARCH(1,1)N	**16.06%**	**15.76%**	**16.84%**	**16.75%**	**16.17%**	**15.85%**	**16.22%**	**16.08%**	**16.04%**	**15.77%**	16.15%	17.09%
volGARCH(1,1)t	15.34%	15.23%	**16.58%**	**16.43%**	15.43%	15.25%	**15.60%**	15.45%	15.33%	15.24%	15.59%	
volGARCH(GJR)N	**15.77%**	**15.60%**	**17.09%**	**16.93%**	**15.94%**	**15.67%**	**16.15%**	**16.00%**	**15.74%**	**15.61%**	16.05%	**Oracle**
volGARCH(GJR)t	15.25%	15.15%	**16.58%**	**16.43%**	15.40%	15.23%	15.46%	15.36%	15.23%	15.16%	15.53%	11.29%
volRangePK	15.42%	15.30%	**16.29%**	**16.19%**	**15.63%**	15.40%	**15.71%**	15.56%	15.40%	15.28%	15.62%	
volRangeGK	15.51%	15.41%	**16.42%**	**16.36%**	**15.71%**	15.51%	**15.77%**	**15.64%**	15.50%	15.39%	15.72%	**Range**
volRangeRS	**15.59%**	15.49%	**16.71%**	**16.63%**	**15.80%**	**15.59%**	**15.84%**	**15.70%**	**15.59%**	15.46%	15.84%	**(Ex-Oracle)**
volRangeYZ	15.46%	15.40%	**16.61%**	**16.55%**	**15.69%**	15.49%	**15.77%**	15.57%	15.43%	15.36%	15.73%	2.29%
Mean Across Vol.	15.37%	15.28%	16.43%	16.34%	15.50%	15.34%	15.60%	15.49%	15.35%	15.27%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
15.13%	15.14%	**11.29%**	18.11%

MV — Sharpe Ratio (Cash at 0%)

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	1.32	1.34	1.13	1.14	1.26	1.27	1.36	1.36	**1.35**	1.37	1.29	**Min (Ex-Ora.)**
volEWMA92	1.26	1.24	1.10	1.10	1.20	1.20	1.24	1.23	1.27	1.24	1.21	1.04
volEWMA96	1.33	1.31	1.22	1.21	1.28	1.29	1.31	1.32	1.34	1.32	1.29	
volEWMA98	1.35	1.35	1.28	1.27	1.30	1.32	1.33	1.34	1.37	1.36	1.33	**Max (Ex-Ora.)**
volGARCH(1,1)N	1.22	1.26	1.12	1.13	1.19	1.23	1.21	1.22	1.23	1.27	1.21	1.37
volGARCH(1,1)t	1.30	1.32	1.11	1.13	1.27	1.30	1.29	1.29	1.31	1.33	1.27	
volGARCH(GJR)N	1.20	1.22	1.04	1.05	1.14	1.18	1.15	1.17	1.21	1.23	1.16	**Oracle**
volGARCH(GJR)t	1.25	1.26	1.05	1.07	1.20	1.24	1.23	1.24	1.25	1.27	1.21	1.87
volRangePK	1.23	1.33	1.21	1.21	1.19	1.24	1.26	1.33	1.27	1.36	1.26	
volRangeGK	1.19	1.28	1.20	1.20	1.17	1.21	1.24	1.30	1.23	1.32	1.23	**Range**
volRangeRS	1.17	1.26	1.17	1.17	1.16	1.20	1.22	1.28	1.20	1.29	1.21	**(Ex-Oracle)**
volRangeYZ	1.24	1.28	1.11	1.11	1.19	1.21	1.26	1.28	1.27	1.31	1.23	0.33
Mean Across Vol.	1.25	1.29	1.14	1.15	1.21	1.24	1.26	1.28	1.28	1.31		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
1.36	1.32	**1.87**	0.86

Table 5 – Panel 2: MV Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

MV — Skewness

Holding Period: 125 days
Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	0.00	-0.10	-0.08	-0.10	-0.07	-0.11	0.08	-0.07	0.01	-0.10	-0.05	**Min (Ex-Ora.)**
volEWMA92	-0.08	-0.15	-0.14	-0.14	-0.12	-0.15	-0.07	-0.14	-0.08	-0.15	-0.12	-0.19
volEWMA96	-0.09	-0.15	-0.09	-0.09	-0.13	-0.16	-0.04	-0.13	-0.08	-0.15	-0.11	
volEWMA98	-0.06	-0.14	-0.04	-0.05	-0.11	-0.13	0.00	-0.12	-0.05	-0.14	-0.08	**Max (Ex-Ora.)**
volGARCH(1,1)N	0.07	0.02	-0.09	-0.09	0.06	0.03	0.04	0.01	0.07	0.01	0.01	0.14
volGARCH(1,1)t	-0.01	-0.08	-0.19	-0.19	-0.05	-0.08	-0.03	-0.10	-0.01	-0.09	-0.08	
volGARCH(GJR)N	0.11	0.05	-0.03	-0.05	0.08	0.03	0.11	0.05	0.10	0.04	0.05	**Oracle**
volGARCH(GJR)t	0.03	-0.02	-0.15	-0.14	-0.01	-0.03	0.03	-0.03	0.03	-0.02	-0.03	-0.10
volRangePK	0.06	-0.02	-0.04	-0.08	0.04	0.01	0.14	0.04	0.08	-0.02	0.02	
volRangeGK	0.05	-0.03	-0.04	-0.06	0.01	0.00	0.12	0.03	0.06	-0.02	0.01	**Range**
volRangeRS	0.04	-0.03	-0.04	-0.06	0.00	0.00	0.10	0.03	0.05	-0.02	0.01	**(Ex-Oracle)**
volRangeYZ	-0.02	-0.14	-0.16	-0.17	-0.09	-0.15	0.07	-0.08	-0.02	-0.13	-0.09	0.33
Mean Across Vol.	0.01	-0.07	-0.09	-0.10	-0.03	-0.06	0.05	-0.04	0.01	-0.07		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
0.01	-0.04	-0.10	-0.21

MV — Kurtosis

Holding Period: 125 days
Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	6.94	6.65	7.26	7.19	6.86	6.76	7.12	6.87	6.94	6.66	6.93	**Min (Ex-Ora.)**
volEWMA92	5.93	6.06	6.41	6.44	5.95	6.06	5.90	6.15	5.96	6.12	6.10	5.90
volEWMA96	6.31	6.43	6.71	6.72	6.40	6.47	6.20	6.57	6.32	6.50	6.46	
volEWMA98	6.72	6.63	6.92	6.88	6.75	6.75	6.74	6.88	6.71	6.67	6.77	**Max (Ex-Ora.)**
volGARCH(1,1)N	8.33	7.89	8.38	8.18	8.65	8.15	8.22	8.30	8.38	7.97	8.25	8.65
volGARCH(1,1)t	6.52	6.66	7.55	7.34	6.61	6.71	6.56	6.75	6.54	6.71	6.79	
volGARCH(GJR)N	7.48	7.43	8.52	8.13	7.83	7.61	7.90	7.88	7.49	7.52	7.78	**Oracle**
volGARCH(GJR)t	6.44	6.55	7.19	7.03	6.57	6.70	6.37	6.61	6.47	6.63	6.66	8.07
volRangePK	7.32	7.01	7.00	6.83	7.32	7.47	7.41	7.25	7.38	7.04	7.20	
volRangeGK	7.29	6.97	6.91	6.85	7.30	7.47	7.29	7.12	7.35	6.99	7.15	**Range**
volRangeRS	7.29	7.01	6.89	6.80	7.27	7.39	7.22	7.16	7.33	7.05	7.14	**(Ex-Oracle)**
volRangeYZ	6.74	6.41	6.94	6.89	6.73	6.71	6.90	6.49	6.76	6.45	6.70	2.75
Mean Across Vol.	6.94	6.81	7.22	7.11	7.02	7.02	6.99	7.00	6.97	6.86		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
6.86	6.71	8.07	6.06

MV — Turnover (2-way p.a.)

Holding Period: 125 days
Conditioning Window: 500 days
N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	180%	172%	154%	151%	192%	184%	170%	167%	176%	169%	171%	**Min (Ex-Ora.)**
volEWMA92	271%	269%	285%	283%	273%	274%	259%	258%	268%	266%	271%	135%
volEWMA96	241%	240%	256%	253%	247%	247%	229%	229%	238%	237%	242%	
volEWMA98	218%	215%	225%	221%	227%	226%	208%	206%	215%	212%	217%	**Max (Ex-Ora.)**
volGARCH(1,1)N	240%	236%	249%	247%	243%	242%	228%	226%	238%	232%	238%	285%
volGARCH(1,1)t	247%	243%	257%	254%	256%	250%	234%	231%	244%	239%	246%	
volGARCH(GJR)N	240%	236%	249%	247%	247%	242%	227%	227%	237%	233%	238%	**Oracle**
volGARCH(GJR)t	245%	243%	257%	254%	253%	248%	231%	230%	242%	240%	244%	261%
volRangePK	175%	168%	147%	144%	188%	181%	164%	162%	171%	164%	166%	
volRangeGK	177%	170%	151%	148%	189%	182%	167%	165%	174%	167%	169%	**Range**
volRangeRS	179%	172%	152%	150%	190%	183%	168%	165%	176%	168%	170%	**(Ex-Oracle)**
volRangeYZ	179%	169%	139%	135%	192%	185%	167%	162%	174%	165%	167%	150%
Mean Across Vol.	216%	211%	210%	207%	225%	220%	204%	202%	213%	208%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
177%	177%	261%	36%

Table 6 – Panel 1: ERC Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

ERC — Geometric Mean Return p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	17.41%	17.32%	16.57%	16.57%	17.28%	17.19%	17.17%	17.26%	17.43%	17.34%	17.15%	**Min (Ex-Ora.)**
volEWMA92	17.13%	17.08%	16.51%	16.51%	16.99%	16.94%	16.80%	16.99%	17.14%	17.08%	16.92%	16.05%
volEWMA96	17.36%	17.29%	16.68%	16.68%	17.22%	17.17%	17.05%	17.21%	17.37%	17.30%	17.13%	
volEWMA98	17.51%	17.43%	16.78%	16.78%	17.38%	17.31%	17.39%	17.36%	17.52%	17.44%	17.29%	**Max (Ex-Ora.)**
volGARCH(1,1)N	17.02%	16.99%	16.44%	16.44%	16.91%	16.85%	16.72%	16.91%	17.03%	17.00%	16.83%	17.52%
volGARCH(1,1)t	16.95%	16.90%	16.34%	16.34%	16.83%	16.77%	16.93%	16.81%	16.96%	16.91%	16.77%	
volGARCH(GJR)N	16.97%	16.93%	16.36%	16.36%	16.83%	16.79%	16.94%	16.85%	16.98%	16.94%	16.80%	**Oracle**
volGARCH(GJR)t	16.87%	16.83%	16.29%	16.29%	16.75%	16.70%	16.83%	16.74%	16.88%	16.84%	16.70%	18.31%
volRangePK	17.38%	17.29%	16.52%	16.52%	17.24%	17.15%	17.11%	17.23%	17.40%	17.31%	17.12%	
volRangeGK	17.47%	17.38%	16.59%	16.59%	17.32%	17.23%	17.20%	17.32%	17.49%	17.39%	17.20%	**Range**
volRangeRS	17.49%	17.40%	16.60%	16.60%	17.34%	17.25%	17.23%	17.34%	17.51%	17.41%	17.22%	**(Ex-Oracle)**
volRangeYZ	16.90%	16.80%	16.05%	16.05%	16.77%	16.65%	16.62%	16.73%	16.91%	16.81%	16.63%	1.47%
Mean Across Vol.	17.21%	17.14%	16.48%	16.48%	17.07%	17.00%	17.00%	17.06%	17.22%	17.15%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
17.47%	17.20%	18.31%	15.53%

ERC — Out-of-Sample Volatility p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	15.13%	17.14%	**17.66%**	**17.66%**	17.10%	17.13%	17.09%	17.13%	17.07%	17.14%	17.02%	**Min (Ex-Ora.)**
volEWMA92	16.73%	16.79%	17.32%	17.32%	16.75%	16.78%	16.76%	16.78%	16.72%	16.79%	16.87%	15.13%
volEWMA96	16.80%	16.87%	17.40%	17.40%	16.83%	16.86%	16.83%	16.86%	16.80%	16.87%	16.95%	
volEWMA98	16.88%	16.95%	17.47%	17.47%	16.91%	16.94%	16.91%	16.94%	16.88%	16.94%	17.03%	**Max (Ex-Ora.)**
volGARCH(1,1)N	16.86%	16.92%	17.45%	17.45%	16.89%	16.91%	16.89%	16.91%	16.85%	16.92%	17.00%	17.79%
volGARCH(1,1)t	16.83%	16.89%	17.41%	17.41%	16.86%	16.88%	16.85%	16.88%	16.82%	16.88%	16.97%	
volGARCH(GJR)N	16.81%	16.88%	17.41%	17.41%	16.84%	16.87%	16.83%	16.87%	16.80%	16.88%	16.96%	**Oracle**
volGARCH(GJR)t	16.80%	16.86%	17.39%	17.39%	16.83%	16.86%	16.82%	16.85%	16.80%	16.86%	16.95%	15.36%
volRangePK	17.08%	17.14%	**17.65%**	**17.65%**	17.11%	17.14%	17.12%	17.14%	17.08%	17.14%	17.23%	
volRangeGK	17.06%	17.12%	**17.63%**	**17.63%**	17.09%	17.12%	17.10%	17.12%	17.06%	17.12%	17.20%	**Range**
volRangeRS	17.07%	17.14%	**17.65%**	**17.65%**	17.10%	17.13%	17.11%	17.13%	17.07%	17.14%	17.22%	**(Ex-Oracle)**
volRangeYZ	17.20%	17.27%	**17.79%**	**17.79%**	17.23%	17.26%	17.24%	17.27%	17.20%	17.27%	17.35%	2.65%
Mean Across Vol.	16.77%	17.00%	17.52%	17.52%	16.96%	16.99%	16.96%	16.99%	16.93%	17.00%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
17.07%	17.17%	**15.36%**	18.11%

ERC — Sharpe Ratio (Cash at 0%)

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	1.02	1.01	**0.94**	**0.94**	1.01	1.00	1.00	1.01	1.02	1.01	1.00	**Min (Ex-Ora.)**
volEWMA92	1.02	1.02	0.95	0.95	1.01	1.01	1.00	1.01	1.02	1.02	1.00	0.90
volEWMA96	1.03	1.02	0.96	0.96	1.02	1.02	1.01	1.02	1.03	1.03	1.01	
volEWMA98	1.04	1.03	0.96	0.96	1.03	1.02	1.03	1.02	1.04	1.03	1.02	**Max (Ex-Ora.)**
volGARCH(1,1)N	1.01	1.00	0.94	0.94	1.00	1.00	0.99	1.00	1.01	1.01	0.99	1.04
volGARCH(1,1)t	1.01	1.00	**0.94**	**0.94**	1.00	0.99	1.01	1.00	1.01	1.00	0.99	
volGARCH(GJR)N	1.01	1.00	**0.94**	**0.94**	1.00	0.99	1.01	1.00	1.01	1.00	0.99	**Oracle**
volGARCH(GJR)t	1.00	1.00	**0.94**	**0.94**	1.00	0.99	1.00	0.99	1.00	1.00	0.99	1.19
volRangePK	1.02	1.01	**0.94**	**0.94**	1.01	1.00	1.00	1.01	1.02	1.01	0.99	
volRangeGK	1.02	1.01	**0.94**	**0.94**	1.01	1.01	1.01	1.01	1.03	1.02	1.00	**Range**
volRangeRS	1.02	1.02	**0.94**	**0.94**	1.01	1.01	1.01	1.01	1.03	1.02	1.00	**(Ex-Oracle)**
volRangeYZ	**0.98**	**0.97**	**0.90**	**0.90**	**0.97**	**0.96**	**0.96**	**0.97**	**0.98**	**0.97**	0.96	0.14
Mean Across Vol.	1.02	1.01	0.94	0.94	1.01	1.00	1.00	1.00	1.02	1.01		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
1.02	**1.00**	1.19	0.86

Table 6 – Panel 2: **ERC** Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

ERC — **Skewness**

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	-0.20	-0.21	-0.21	-0.21	-0.21	-0.21	-0.20	-0.20	-0.20	-0.21	-0.21	**Min (Ex-Ora.)**
volEWMA92	-0.22	-0.23	-0.22	-0.22	-0.23	-0.23	-0.22	-0.22	-0.22	-0.23	-0.22	-0.23
volEWMA96	-0.21	-0.22	-0.21	-0.21	-0.22	-0.22	-0.21	-0.21	-0.21	-0.22	-0.21	
volEWMA98	-0.21	-0.21	-0.21	-0.21	-0.21	-0.22	-0.20	-0.21	-0.21	-0.21	-0.21	**Max (Ex-Ora.)**
volGARCH(1,1)N	-0.22	-0.22	-0.22	-0.22	-0.23	-0.23	-0.21	-0.22	-0.22	-0.22	-0.22	-0.20
volGARCH(1,1)t	-0.22	-0.22	-0.22	-0.22	-0.22	-0.23	-0.21	-0.22	-0.22	-0.22	-0.22	
volGARCH(GJR)N	-0.22	-0.22	-0.22	-0.22	-0.22	-0.23	-0.21	-0.22	-0.22	-0.22	-0.22	**Oracle**
volGARCH(GJR)t	-0.21	-0.22	-0.22	-0.22	-0.22	-0.22	-0.21	-0.22	-0.21	-0.22	-0.22	-0.24
volRangePK	-0.21	-0.21	-0.22	-0.22	-0.21	-0.22	-0.20	-0.21	-0.20	-0.21	-0.21	
volRangeGK	-0.21	-0.22	-0.22	-0.22	-0.22	-0.22	-0.20	-0.21	-0.21	-0.22	-0.22	**Range**
volRangeRS	-0.22	-0.22	-0.22	-0.22	-0.22	-0.23	-0.21	-0.22	-0.21	-0.22	-0.22	**(Ex-Oracle)**
volRangeYZ	-0.21	-0.22	-0.22	-0.22	-0.22	-0.22	-0.20	-0.21	-0.21	-0.22	-0.21	0.03
Mean Across Vol.	-0.21	-0.22	-0.22	-0.22	-0.22	-0.22	-0.20	-0.21	-0.21	-0.22		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
-0.20	-0.21	-0.24	-0.21

ERC — **Kurtosis**

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	6.10	6.18	6.20	6.20	6.14	6.18	6.06	6.18	6.10	6.18	6.15	**Min (Ex-Ora.)**
volEWMA92	6.07	6.13	6.15	6.15	6.11	6.14	6.02	6.13	6.07	6.13	6.11	5.92
volEWMA96	6.08	6.14	6.16	6.16	6.12	6.15	6.03	6.14	6.07	6.14	6.12	
volEWMA98	6.09	6.16	6.17	6.17	6.13	6.16	6.04	6.16	6.09	6.16	6.13	**Max (Ex-Ora.)**
volGARCH(1,1)N	6.09	6.15	6.16	6.16	6.13	6.16	6.04	6.14	6.09	6.15	6.13	6.20
volGARCH(1,1)t	6.04	6.10	6.12	6.12	6.08	6.12	5.99	6.09	6.04	6.10	6.08	
volGARCH(GJR)N	6.04	6.10	6.12	6.12	6.08	6.11	5.99	6.09	6.04	6.10	6.08	**Oracle**
volGARCH(GJR)t	6.03	6.09	6.11	6.11	6.06	6.10	5.98	6.08	6.03	6.08	6.07	6.16
volRangePK	6.01	6.08	6.11	6.11	6.04	6.09	5.95	6.07	6.01	6.08	6.05	
volRangeGK	6.00	6.07	6.09	6.09	6.03	6.07	5.94	6.06	6.00	6.07	6.04	**Range**
volRangeRS	5.99	6.06	6.07	6.07	6.03	6.07	5.94	6.05	5.99	6.06	6.03	**(Ex-Oracle)**
volRangeYZ	5.98	6.05	6.09	6.09	6.01	6.05	5.92	6.04	5.97	6.05	6.02	0.28
Mean Across Vol.	6.04	6.11	6.13	6.13	6.08	6.12	5.99	6.10	6.04	6.11		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
6.10	6.15	6.16	6.06

ERC — **Turnover (2-way p.a.)**

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	60%	58%	47%	47%	61%	59%	60%	58%	60%	58%	57%	**Min (Ex-Ora.)**
volEWMA92	84%	83%	76%	76%	84%	83%	85%	83%	84%	83%	82%	45%
volEWMA96	75%	75%	67%	67%	75%	75%	76%	75%	75%	75%	74%	
volEWMA98	69%	68%	59%	59%	69%	68%	69%	68%	69%	68%	67%	**Max (Ex-Ora.)**
volGARCH(1,1)N	76%	75%	67%	67%	76%	75%	77%	75%	76%	75%	74%	85%
volGARCH(1,1)t	76%	75%	68%	68%	76%	75%	76%	75%	76%	75%	74%	
volGARCH(GJR)N	76%	76%	68%	68%	77%	76%	77%	76%	77%	76%	75%	**Oracle**
volGARCH(GJR)t	77%	76%	69%	69%	77%	77%	77%	76%	77%	76%	75%	92%
volRangePK	60%	58%	46%	46%	61%	59%	60%	58%	60%	58%	57%	
volRangeGK	61%	59%	46%	46%	62%	60%	62%	59%	61%	59%	58%	**Range**
volRangeRS	61%	59%	47%	47%	62%	60%	62%	59%	61%	59%	58%	**(Ex-Oracle)**
volRangeYZ	59%	57%	45%	45%	61%	59%	60%	58%	59%	57%	56%	39%
Mean Across Vol.	69%	68%	59%	59%	70%	69%	70%	69%	69%	68%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
60%	56%	92%	36%

Table 7 – Panel 1: MDP Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

MDP — Geometric Mean Return p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	19.51%	19.80%	16.57%	16.57%	18.46%	18.38%	18.73%	19.09%	19.73%	20.03%	18.69%	**Min (Ex-Ora.)**
volEWMA92	18.75%	18.98%	16.51%	16.51%	17.59%	17.67%	18.07%	18.32%	19.01%	19.23%	18.06%	16.05%
volEWMA96	19.06%	19.28%	16.68%	16.68%	17.89%	17.98%	18.41%	18.63%	19.32%	19.54%	18.35%	
volEWMA98	19.32%	19.54%	16.78%	16.78%	18.16%	18.20%	18.69%	18.92%	19.57%	19.79%	18.57%	**Max (Ex-Ora.)**
volGARCH(1,1)N	18.58%	18.95%	16.44%	16.44%	17.37%	17.47%	17.85%	18.33%	18.79%	19.19%	17.94%	20.04%
volGARCH(1,1)t	18.46%	18.83%	16.34%	16.34%	17.31%	17.39%	17.74%	18.19%	18.68%	19.07%	17.83%	
volGARCH(GJR)N	18.46%	18.82%	16.36%	16.36%	17.24%	17.34%	17.75%	18.21%	18.69%	19.09%	17.83%	**Oracle**
volGARCH(GJR)t	18.14%	18.63%	16.29%	16.29%	17.00%	17.15%	17.44%	17.98%	18.37%	18.87%	17.62%	19.13%
volRangePK	19.34%	19.70%	16.52%	16.52%	18.24%	18.18%	18.55%	19.07%	19.56%	19.95%	18.56%	
volRangeGK	19.36%	19.75%	16.59%	16.59%	18.23%	18.21%	18.62%	19.16%	19.58%	20.01%	18.61%	**Range**
volRangeRS	19.31%	19.73%	16.60%	16.60%	18.16%	18.16%	18.63%	19.18%	19.55%	20.01%	18.59%	**(Ex-Oracle)**
volRangeYZ	18.75%	19.05%	16.05%	16.05%	17.67%	17.54%	17.97%	18.33%	18.99%	19.30%	17.97%	3.99%
Mean Across Vol.	18.92%	19.25%	16.48%	16.48%	17.78%	17.81%	18.20%	18.62%	19.15%	19.51%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM **20.04%** noNorm_LWCC 19.46% oracleCov 19.13% Equal Weight Portfolio 15.53%

MDP — Out-of-Sample Volatility p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	16.28%	16.22%	**17.66%**	**17.66%**	16.61%	16.39%	**17.21%**	**16.79%**	16.29%	16.20%	16.73%	**Min (Ex-Ora.)**
volEWMA92	15.84%	**15.80%**	**17.32%**	**17.32%**	16.19%	16.00%	**16.79%**	16.38%	15.84%	**15.78%**	16.32%	15.78%
volEWMA96	15.90%	15.85%	**17.40%**	**17.40%**	16.25%	16.05%	**16.81%**	16.42%	15.90%	15.84%	16.38%	
volEWMA98	16.01%	15.95%	**17.47%**	**17.47%**	16.35%	16.14%	**16.90%**	16.51%	16.01%	15.93%	16.47%	**Max (Ex-Ora.)**
volGARCH(1,1)N	16.01%	15.97%	**17.45%**	**17.45%**	16.34%	16.17%	**16.98%**	16.52%	16.02%	15.95%	16.49%	17.79%
volGARCH(1,1)t	15.98%	15.93%	**17.41%**	**17.41%**	16.33%	16.15%	**16.96%**	16.50%	15.99%	15.91%	16.46%	
volGARCH(GJR)N	15.94%	15.92%	**17.41%**	**17.41%**	16.27%	16.11%	**16.93%**	16.48%	15.95%	15.90%	16.43%	**Oracle**
volGARCH(GJR)t	15.95%	15.89%	**17.39%**	**17.39%**	16.29%	16.10%	**16.93%**	16.47%	15.96%	15.87%	16.43%	12.85%
volRangePK	16.41%	16.29%	**17.65%**	**17.65%**	16.74%	16.48%	**17.57%**	**17.03%**	16.43%	16.28%	16.85%	
volRangeGK	16.41%	16.28%	**17.63%**	**17.63%**	16.73%	16.47%	**17.54%**	**17.01%**	16.42%	16.27%	16.84%	**Range**
volRangeRS	16.43%	16.30%	**17.65%**	**17.65%**	**16.76%**	16.49%	**17.56%**	**17.04%**	16.44%	16.29%	16.86%	**(Ex-Oracle)**
volRangeYZ	16.47%	16.38%	**17.79%**	**17.79%**	**16.81%**	16.56%	**17.61%**	**17.11%**	16.48%	16.36%	16.94%	2.01%
Mean Across Vol.	16.13%	16.07%	17.52%	17.52%	16.47%	16.26%	17.15%	16.69%	16.14%	16.05%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM 16.26% noNorm_LWCC 16.19% oracleCov **12.85%** Equal Weight Portfolio 18.11%

MDP — Sharpe Ratio (Cash at 0%)

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	1.20	1.22	0.94	0.94	**1.11**	1.12	1.09	1.14	1.21	1.24	1.12	**Min (Ex-Ora.)**
volEWMA92	1.18	1.20	0.95	0.95	1.09	1.10	1.08	1.12	1.20	1.22	1.11	0.90
volEWMA96	1.20	1.22	0.96	0.96	1.10	1.12	1.10	1.13	1.21	1.23	1.12	
volEWMA98	1.21	1.23	0.96	0.96	1.11	1.13	1.11	1.15	1.22	1.24	1.13	**Max (Ex-Ora.)**
volGARCH(1,1)N	1.16	1.19	0.94	0.94	**1.06**	**1.08**	1.05	1.11	1.17	1.20	1.09	1.24
volGARCH(1,1)t	1.15	1.18	0.94	0.94	**1.06**	**1.08**	1.05	1.10	1.17	1.20	1.09	
volGARCH(GJR)N	1.16	1.18	0.94	0.94	**1.06**	**1.08**	1.05	1.10	1.17	1.20	1.09	**Oracle**
volGARCH(GJR)t	1.14	1.17	0.94	0.94	**1.04**	**1.06**	**1.03**	1.09	1.15	1.19	1.08	1.49
volRangePK	1.18	1.21	**0.94**	**0.94**	**1.09**	1.10	1.06	1.12	1.19	1.23	1.10	
volRangeGK	1.18	1.21	**0.94**	**0.94**	**1.09**	1.11	1.06	1.13	1.19	1.23	1.11	**Range**
volRangeRS	1.18	1.21	**0.94**	**0.94**	**1.08**	1.10	1.06	1.13	1.19	1.23	1.11	**(Ex-Oracle)**
volRangeYZ	**1.14**	1.16	**0.90**	**0.90**	**1.05**	**1.06**	**1.02**	1.07	1.15	1.18	1.06	0.34
Mean Across Vol.	1.17	1.20	0.94	0.94	1.08	1.10	1.06	1.12	1.19	1.22		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM **1.23** noNorm_LWCC 1.20 oracleCov **1.49** Equal Weight Portfolio 0.86

Table 7 – Panel 2: **MDP** Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

MDP — Skewness

Holding Period: 125 days; Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	0.00	-0.10	-0.21	-0.21	-0.04	-0.10	0.16	0.01	0.01	-0.09	-0.06	**Min (Ex-Ora.)**
volEWMA92	-0.08	-0.15	-0.22	-0.22	-0.10	-0.16	0.06	-0.05	-0.07	-0.15	-0.11	-0.22
volEWMA96	-0.05	-0.14	-0.21	-0.21	-0.08	-0.14	0.09	-0.04	-0.04	-0.13	-0.09	
volEWMA98	-0.02	-0.12	-0.21	-0.21	-0.06	-0.13	0.12	-0.02	-0.01	-0.11	-0.08	**Max (Ex-Ora.)**
volGARCH(1,1)N	-0.04	-0.11	-0.22	-0.22	-0.07	-0.12	0.12	0.00	-0.02	-0.10	-0.08	0.23
volGARCH(1,1)t	-0.02	-0.11	-0.22	-0.22	-0.06	-0.11	0.14	0.01	-0.01	-0.10	-0.07	
volGARCH(GJR)N	-0.04	-0.10	-0.22	-0.22	-0.07	-0.11	0.13	0.00	-0.02	-0.10	-0.07	**Oracle**
volGARCH(GJR)t	-0.02	-0.10	-0.22	-0.22	-0.05	-0.11	0.15	0.01	0.00	-0.09	-0.06	-0.11
volRangePK	0.04	-0.08	-0.22	-0.22	-0.01	-0.08	0.23	0.05	0.06	-0.07	-0.03	
volRangeGK	0.03	-0.10	-0.22	-0.22	-0.03	-0.10	0.21	0.04	0.04	-0.08	-0.04	**Range**
volRangeRS	0.02	-0.10	-0.22	-0.22	-0.04	-0.11	0.21	0.04	0.03	-0.09	-0.05	**(Ex-Oracle)**
volRangeYZ	-0.01	-0.11	-0.22	-0.22	-0.05	-0.12	0.16	0.02	0.00	-0.10	-0.06	0.45
Mean Across Vol.	-0.01	-0.11	-0.22	-0.22	-0.05	-0.12	0.15	0.01	0.00	-0.10		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
0.03	-0.04	-0.11	-0.21

MDP — Kurtosis

Holding Period: 125 days; Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	6.66	6.36	6.20	6.20	6.71	6.69	6.85	6.26	6.74	6.38	6.51	**Min (Ex-Ora.)**
volEWMA92	6.19	6.16	6.15	6.15	6.38	6.45	6.22	5.98	6.23	6.18	6.21	5.98
volEWMA96	6.35	6.24	6.16	6.16	6.48	6.53	6.36	6.07	6.39	6.25	6.30	
volEWMA98	6.52	6.33	6.17	6.17	6.60	6.63	6.56	6.19	6.57	6.34	6.41	**Max (Ex-Ora.)**
volGARCH(1,1)N	6.51	6.37	6.16	6.16	6.65	6.75	6.60	6.16	6.58	6.38	6.43	7.17
volGARCH(1,1)t	6.48	6.26	6.12	6.12	6.60	6.64	6.64	6.11	6.55	6.29	6.38	
volGARCH(GJR)N	6.45	6.28	6.12	6.12	6.59	6.66	6.58	6.09	6.53	6.31	6.37	**Oracle**
volGARCH(GJR)t	6.48	6.24	6.11	6.11	6.60	6.62	6.67	6.10	6.55	6.27	6.38	6.98
volRangePK	7.00	6.43	6.11	6.11	6.92	6.83	7.16	6.31	7.07	6.46	6.64	
volRangeGK	7.00	6.43	6.09	6.09	6.92	6.84	7.14	6.31	7.06	6.46	6.63	**Range**
volRangeRS	7.03	6.44	6.07	6.07	6.94	6.86	7.17	6.32	7.08	6.47	6.65	**(Ex-Oracle)**
volRangeYZ	6.61	6.24	6.09	6.09	6.60	6.55	6.69	6.02	6.68	6.26	6.38	1.19
Mean Across Vol.	6.61	6.32	6.13	6.13	6.67	6.67	6.72	6.16	6.67	6.34		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
6.79	6.46	6.98	6.06

MDP — Turnover (2-way p.a.)

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	172%	164%	47%	47%	193%	184%	159%	156%	166%	158%	145%	**Min (Ex-Ora.)**
volEWMA92	181%	174%	76%	76%	200%	194%	170%	168%	176%	169%	158%	45%
volEWMA96	177%	169%	67%	67%	197%	189%	165%	163%	171%	164%	153%	
volEWMA98	174%	167%	59%	59%	195%	187%	162%	159%	168%	161%	149%	**Max (Ex-Ora.)**
volGARCH(1,1)N	177%	171%	67%	67%	197%	190%	166%	163%	171%	165%	153%	200%
volGARCH(1,1)t	176%	171%	68%	68%	198%	191%	164%	162%	171%	165%	153%	
volGARCH(GJR)N	176%	170%	68%	68%	197%	190%	165%	162%	171%	165%	153%	**Oracle**
volGARCH(GJR)t	176%	170%	69%	69%	198%	191%	164%	162%	170%	164%	153%	273%
volRangePK	171%	163%	46%	46%	192%	183%	157%	154%	165%	157%	143%	
volRangeGK	172%	164%	46%	46%	192%	183%	158%	155%	166%	158%	144%	**Range**
volRangeRS	171%	164%	47%	47%	192%	183%	158%	154%	165%	158%	144%	**(Ex-Oracle)**
volRangeYZ	172%	164%	45%	45%	193%	184%	157%	154%	166%	158%	144%	155%
Mean Across Vol.	175%	168%	59%	59%	195%	187%	162%	159%	169%	162%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
168%	164%	273%	36%

Table 8 – Panel 1: EffBetsPCA Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

EffBetsPCA — Geometric Mean Return p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	23.25%	23.56%	20.37%	20.88%	21.51%	**32.77%**	21.55%	27.04%	26.30%	20.09%	23.73%	**Min (Ex-Ora.)**
volEWMA92	22.08%	20.22%	9.03%	6.91%	20.21%	16.44%	20.70%	22.72%	27.34%	20.16%	18.58%	6.91%
volEWMA96	25.14%	18.28%	16.27%	17.50%	19.06%	16.48%	20.43%	23.86%	25.82%	21.94%	20.48%	
volEWMA98	27.72%	21.29%	20.83%	19.86%	22.80%	18.75%	19.05%	22.18%	23.40%	17.65%	21.35%	**Max (Ex-Ora.)**
volGARCH(1,1)N	24.36%	22.70%	17.33%	16.18%	22.24%	24.38%	22.50%	24.78%	21.51%	22.13%	21.81%	32.77%
volGARCH(1,1)t	23.59%	27.15%	14.23%	13.83%	22.75%	24.18%	19.55%	20.95%	19.57%	24.88%	21.07%	
volGARCH(GJR)N	20.06%	24.30%	15.52%	17.73%	16.12%	21.09%	23.33%	24.66%	16.21%	23.45%	20.25%	**Oracle**
volGARCH(GJR)t	22.33%	24.04%	19.13%	16.09%	21.35%	20.00%	27.81%	23.23%	20.86%	23.20%	21.80%	19.25%
volRangePK	22.50%	24.32%	12.98%	13.83%	19.38%	23.52%	21.32%	20.09%	21.07%	23.51%	20.25%	
volRangeGK	24.62%	25.47%	15.64%	18.20%	19.96%	19.87%	20.61%	25.30%	26.62%	24.62%	22.09%	**Range**
volRangeRS	20.59%	23.51%	14.02%	15.46%	21.33%	21.64%	21.55%	22.87%	22.36%	23.14%	20.65%	**(Ex-Oracle)**
volRangeYZ	19.42%	23.68%	18.01%	18.14%	22.58%	19.43%	23.57%	22.43%	21.63%	21.59%	21.05%	25.86%
Mean Across Vol.	22.97%	23.21%	16.11%	16.22%	20.77%	21.55%	21.83%	23.34%	22.72%	22.20%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
24.03%	25.18%	19.25%	15.53%

EffBetsPCA — Out-of-Sample Volatility p.a.

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	21.46%	**20.39%**	21.22%	21.70%	21.27%	**20.79%**	**22.85%**	**23.36%**	21.17%	21.55%	21.58%	**Min (Ex-Ora.)**
volEWMA92	**20.82%**	**19.37%**	**22.37%**	21.87%	**20.53%**	**19.83%**	**22.50%**	**22.48%**	**20.78%**	**19.85%**	21.04%	19.30%
volEWMA96	**19.30%**	**20.29%**	21.49%	21.56%	**20.20%**	**20.54%**	21.86%	**22.34%**	**19.75%**	**20.19%**	20.75%	
volEWMA98	**19.60%**	**20.24%**	21.13%	21.14%	**20.77%**	**20.31%**	**22.64%**	**22.54%**	**20.57%**	**20.59%**	20.95%	**Max (Ex-Ora.)**
volGARCH(1,1)N	**20.62%**	**20.38%**	**22.19%**	22.01%	21.04%	21.10%	**23.03%**	**22.95%**	**20.50%**	**20.79%**	21.46%	25.77%
volGARCH(1,1)t	20.95%	**20.31%**	22.00%	21.99%	**20.58%**	21.65%	**23.82%**	**23.07%**	21.08%	**20.75%**	21.62%	
volGARCH(GJR)N	**19.73%**	**20.62%**	**22.63%**	**22.13%**	21.19%	**20.54%**	**22.84%**	**22.68%**	**20.75%**	**19.43%**	21.26%	**Oracle**
volGARCH(GJR)t	**20.51%**	21.03%	21.76%	21.93%	21.00%	21.09%	**23.24%**	**22.57%**	21.05%	21.51%	21.57%	16.04%
volRangePK	**20.72%**	21.33%	**22.33%**	22.02%	**20.65%**	**22.13%**	**24.62%**	**24.49%**	21.70%	21.61%	22.16%	
volRangeGK	**20.75%**	**22.17%**	21.36%	**20.84%**	21.01%	21.02%	**25.77%**	**25.57%**	**20.11%**	21.56%	22.02%	**Range**
volRangeRS	21.40%	21.02%	**23.24%**	**22.43%**	21.55%	**22.23%**	**24.54%**	**25.72%**	21.70%	21.98%	22.58%	**(Ex-Oracle)**
volRangeYZ	21.69%	20.96%	**22.35%**	22.06%	**22.20%**	21.71%	**23.98%**	**23.39%**	21.06%	22.07%	22.15%	6.46%
Mean Across Vol.	20.63%	20.68%	22.00%	21.81%	21.00%	21.08%	23.48%	23.43%	20.85%	20.99%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
21.01%	21.61%	16.04%	18.11%

EffBetsPCA — Sharpe Ratio (Cash at 0%)

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	1.08	1.16	0.96	0.96	1.01	**1.58**	0.94	1.16	1.24	0.93	1.10	**Min (Ex-Ora.)**
volEWMA92	1.06	1.04	**0.40**	**0.32**	0.98	0.83	0.92	1.01	1.32	1.02	0.89	0.32
volEWMA96	1.30	0.90	0.76	0.81	0.94	0.80	0.93	1.07	1.31	1.09	0.99	
volEWMA98	1.41	1.05	0.99	0.94	1.10	0.92	0.84	0.98	1.14	0.86	1.02	**Max (Ex-Ora.)**
volGARCH(1,1)N	1.18	1.11	0.78	0.74	1.06	1.16	0.98	1.08	1.05	1.06	1.02	1.58
volGARCH(1,1)t	1.13	1.34	0.65	0.63	1.11	1.12	0.82	0.91	0.93	1.20	0.98	
volGARCH(GJR)N	1.02	1.18	0.69	0.80	0.76	1.03	1.02	1.09	0.78	1.21	0.96	**Oracle**
volGARCH(GJR)t	1.09	1.14	0.88	0.73	1.02	0.95	1.20	1.03	0.99	1.08	1.01	1.20
volRangePK	1.09	1.14	0.58	0.63	0.94	1.06	0.87	0.82	0.97	1.09	0.92	
volRangeGK	1.19	1.15	0.73	0.87	0.95	0.95	0.80	0.99	1.32	1.14	1.01	**Range**
volRangeRS	0.96	1.12	0.60	0.69	0.99	0.97	0.88	0.89	1.03	1.05	0.92	**(Ex-Oracle)**
volRangeYZ	0.90	1.13	0.81	0.82	1.02	0.90	0.98	0.96	1.03	0.98	0.95	1.26
Mean Across Vol.	1.12	1.12	0.74	0.75	0.99	1.02	0.93	1.00	1.09	1.06		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
1.14	1.17	1.20	0.86

Table 8 – Panel 2: **EffBetsPCA** Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

EffBetsPCA — Skewness

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	0.31	0.17	-0.04	0.00	0.50	0.47	0.65	0.79	0.46	0.23	0.35	**Min (Ex-Ora.)**
volEWMA92	0.24	0.10	-0.03	-0.03	0.12	0.18	0.22	0.33	0.03	0.16	0.13	-0.18
volEWMA96	0.17	0.05	0.00	0.16	0.20	-0.06	0.22	0.25	0.22	0.26	0.15	
volEWMA98	0.28	0.18	0.08	0.13	0.14	0.27	0.16	0.21	0.41	0.24	0.21	**Max (Ex-Ora.)**
volGARCH(1,1)N	0.32	0.41	0.05	0.05	0.89	0.74	0.51	0.51	0.32	0.42	0.42	1.03
volGARCH(1,1)t	0.42	0.29	0.04	-0.02	0.46	0.64	0.60	0.40	0.60	0.37	0.38	
volGARCH(GJR)N	0.31	0.22	-0.09	-0.10	0.37	0.34	0.48	0.65	0.05	0.33	0.26	**Oracle**
volGARCH(GJR)t	0.13	0.10	-0.14	-0.05	0.30	0.54	0.42	0.35	0.70	0.59	0.29	0.17
volRangePK	1.01	0.88	-0.09	-0.12	0.25	0.58	0.73	0.55	0.86	0.96	0.56	
volRangeGK	0.32	0.87	-0.12	-0.13	0.71	0.69	0.71	0.77	0.53	0.48	0.48	**Range**
volRangeRS	0.77	0.59	-0.17	-0.18	0.64	0.77	0.77	0.73	1.03	0.94	0.59	**(Ex-Oracle)**
volRangeYZ	0.58	0.64	0.10	0.19	0.51	0.12	0.57	0.88	0.48	0.51	0.46	1.21
Mean Across Vol.	0.41	0.38	-0.03	-0.01	0.42	0.44	0.50	0.54	0.47	0.46		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
0.73	0.69	0.17	-0.21

EffBetsPCA — Kurtosis

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	8.12	7.33	5.45	5.73	9.57	8.77	9.86	11.34	8.75	8.16	8.31	**Min (Ex-Ora.)**
volEWMA92	6.91	6.40	5.82	5.24	5.99	8.32	6.27	6.55	5.68	6.33	6.35	5.24
volEWMA96	6.71	6.46	5.99	6.53	7.75	7.76	6.02	6.07	6.02	6.71	6.60	
volEWMA98	6.90	7.70	6.17	6.24	7.08	7.11	6.00	6.22	7.54	7.40	6.84	**Max (Ex-Ora.)**
volGARCH(1,1)N	7.45	8.87	5.88	6.00	14.58	10.79	7.88	8.45	6.66	9.33	8.59	15.62
volGARCH(1,1)t	8.05	7.97	6.47	5.79	7.39	10.51	9.39	8.00	9.31	8.12	8.10	
volGARCH(GJR)N	7.63	7.28	6.23	5.59	8.21	9.31	8.38	10.38	8.26	7.35	7.86	**Oracle**
volGARCH(GJR)t	7.27	8.09	6.65	6.84	7.60	10.52	7.75	7.14	10.96	10.18	8.30	7.41
volRangePK	15.40	14.48	6.51	6.61	8.75	10.94	11.43	8.86	15.19	15.37	11.35	
volRangeGK	7.49	14.80	6.38	6.42	13.64	10.79	10.91	11.02	8.32	10.44	10.02	**Range**
volRangeRS	14.16	10.02	6.97	6.60	11.18	13.17	11.50	10.56	15.62	14.52	11.43	**(Ex-Oracle)**
volRangeYZ	10.82	12.38	7.83	7.66	9.80	6.88	9.04	11.55	8.81	10.67	9.54	10.38
Mean Across Vol.	8.91	9.31	6.36	6.27	9.30	9.57	8.70	8.84	9.26	9.55		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
12.30	10.69	7.41	6.06

EffBetsPCA — Turnover (2-way p.a.)

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	297%	277%	350%	355%	294%	293%	281%	267%	303%	270%	299%	**Min (Ex-Ora.)**
volEWMA92	304%	292%	389%	385%	319%	323%	313%	304%	314%	290%	323%	257%
volEWMA96	287%	288%	370%	365%	305%	298%	295%	291%	284%	291%	307%	
volEWMA98	294%	288%	368%	366%	320%	308%	293%	287%	297%	283%	310%	**Max (Ex-Ora.)**
volGARCH(1,1)N	322%	279%	365%	369%	293%	308%	282%	314%	277%	288%	310%	389%
volGARCH(1,1)t	295%	285%	376%	370%	308%	319%	286%	297%	306%	297%	314%	
volGARCH(GJR)N	295%	304%	385%	385%	310%	302%	303%	308%	287%	282%	316%	**Oracle**
volGARCH(GJR)t	321%	277%	368%	368%	308%	306%	281%	286%	298%	278%	309%	328%
volRangePK	275%	257%	348%	345%	287%	278%	272%	284%	282%	281%	291%	
volRangeGK	294%	274%	355%	358%	281%	274%	282%	282%	271%	274%	294%	**Range**
volRangeRS	295%	265%	362%	358%	280%	266%	271%	283%	272%	258%	291%	**(Ex-Oracle)**
volRangeYZ	288%	274%	346%	345%	321%	288%	280%	276%	291%	284%	299%	132%
Mean Across Vol.	297%	280%	365%	364%	302%	297%	287%	290%	290%	281%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
286%	286%	328%	36%

Table 9 – Panel 1: EffBetsMLT Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

EffBetsMLT — **Geometric Mean Return p.a.**

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	15.33%	15.51%	**16.34%**	**16.33%**	15.44%	**15.60%**	15.43%	15.57%	15.32%	15.50%	15.64%	**Min (Ex-Ora.)**
volEWMA92	15.46%	15.63%	16.30%	16.29%	15.58%	15.74%	15.58%	15.70%	15.44%	15.62%	15.73%	14.98%
volEWMA96	15.53%	15.71%	16.43%	16.42%	15.65%	15.81%	15.63%	15.77%	15.51%	15.70%	15.82%	
volEWMA98	15.54%	**15.73%**	**16.50%**	**16.49%**	15.66%	**15.82%**	15.64%	15.79%	15.52%	15.71%	15.84%	**Max (Ex-Ora.)**
volGARCH(1,1)N	15.45%	15.61%	16.23%	16.23%	15.54%	15.69%	15.53%	15.63%	15.44%	15.59%	15.69%	16.50%
volGARCH(1,1)t	15.37%	15.54%	16.16%	16.15%	15.46%	15.61%	15.45%	15.56%	15.36%	15.52%	15.62%	
volGARCH(GJR)N	15.38%	15.54%	16.17%	16.17%	15.48%	15.63%	15.46%	15.57%	15.36%	15.52%	15.63%	**Oracle**
volGARCH(GJR)t	15.34%	15.50%	16.11%	16.11%	15.43%	15.58%	15.41%	15.52%	15.32%	15.48%	15.58%	16.10%
volRangePK	15.25%	15.44%	16.29%	16.28%	15.37%	15.54%	15.34%	15.48%	15.24%	15.43%	15.57%	
volRangeGK	15.27%	15.47%	16.35%	16.33%	15.40%	15.57%	15.36%	15.51%	15.26%	15.45%	15.60%	**Range**
volRangeRS	15.27%	15.47%	16.36%	16.34%	15.40%	15.57%	15.36%	15.51%	15.25%	15.46%	15.60%	**(Ex-Oracle)**
volRangeYZ	15.01%	15.19%	15.94%	15.93%	15.13%	15.30%	15.03%	15.17%	14.98%	15.17%	15.28%	1.52%
Mean Across Vol.	15.35%	15.53%	16.27%	16.26%	15.46%	15.62%	15.44%	15.56%	15.33%	15.51%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
15.30%	15.51%	16.10%	15.53%

EffBetsMLT — **Out-of-Sample Volatility p.a.**

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	18.43%	18.36%	**17.74%**	**17.75%**	18.41%	18.37%	18.39%	18.33%	18.42%	18.34%	18.25%	**Min (Ex-Ora.)**
volEWMA92	18.19%	18.11%	**17.46%**	**17.47%**	18.17%	18.13%	18.13%	18.07%	18.17%	18.10%	18.00%	17.46%
volEWMA96	18.24%	18.17%	**17.53%**	**17.53%**	18.23%	18.19%	18.19%	18.13%	18.23%	18.15%	18.06%	
volEWMA98	18.29%	18.22%	**17.59%**	**17.60%**	18.28%	18.24%	18.24%	18.19%	18.28%	18.21%	18.11%	**Max (Ex-Ora.)**
volGARCH(1,1)N	18.30%	18.22%	**17.56%**	**17.57%**	18.28%	18.23%	18.25%	18.19%	18.29%	18.20%	18.11%	18.55%
volGARCH(1,1)t	18.27%	18.19%	**17.53%**	**17.54%**	18.25%	18.20%	18.22%	18.17%	18.26%	18.18%	18.08%	
volGARCH(GJR)N	18.27%	18.19%	**17.53%**	**17.54%**	18.25%	18.20%	18.23%	18.17%	18.26%	18.17%	18.08%	**Oracle**
volGARCH(GJR)t	18.26%	18.18%	**17.52%**	**17.53%**	18.24%	18.19%	18.22%	18.15%	18.25%	18.16%	18.07%	18.77%
volRangePK	18.46%	18.38%	**17.75%**	**17.75%**	18.44%	18.39%	18.43%	18.36%	18.45%	18.37%	18.28%	
volRangeGK	18.45%	18.37%	**17.73%**	**17.73%**	18.42%	18.38%	18.42%	18.35%	18.44%	18.35%	18.26%	**Range**
volRangeRS	18.46%	18.37%	**17.74%**	**17.74%**	18.43%	18.39%	18.43%	18.36%	18.45%	18.36%	18.27%	**(Ex-Oracle)**
volRangeYZ	18.55%	18.46%	**17.85%**	**17.85%**	18.52%	18.48%	18.52%	18.45%	18.54%	18.45%	18.37%	1.09%
Mean Across Vol.	18.35%	18.27%	17.63%	17.63%	18.33%	18.28%	18.31%	18.24%	18.33%	18.25%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
18.41%	18.29%	18.77%	18.11%

EffBetsMLT — **Sharpe Ratio (Cash at 0%)**

Holding Period: 125 days; Conditioning Window: 500 days; N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	0.83	**0.85**	**0.92**	**0.92**	0.84	**0.85**	0.84	**0.85**	0.83	**0.84**	0.86	**Min (Ex-Ora.)**
volEWMA92	0.85	**0.86**	**0.93**	**0.93**	0.86	**0.87**	0.86	**0.87**	0.85	**0.86**	0.87	0.81
volEWMA96	0.85	**0.86**	**0.94**	**0.94**	**0.86**	**0.87**	**0.86**	**0.87**	0.85	**0.86**	0.88	
volEWMA98	**0.85**	**0.86**	**0.94**	**0.94**	**0.86**	**0.87**	**0.86**	**0.87**	**0.85**	**0.86**	0.87	**Max (Ex-Ora.)**
volGARCH(1,1)N	0.84	0.86	**0.92**	**0.92**	0.85	0.86	0.85	0.86	0.84	0.86	0.87	0.94
volGARCH(1,1)t	0.84	0.85	**0.92**	**0.92**	0.85	0.86	0.85	0.86	0.84	0.85	0.86	
volGARCH(GJR)N	0.84	0.85	**0.92**	**0.92**	0.85	0.86	0.85	0.86	0.84	0.85	0.86	**Oracle**
volGARCH(GJR)t	0.84	0.85	**0.92**	**0.92**	0.85	0.86	0.85	0.85	0.84	0.85	0.86	0.86
volRangePK	0.83	0.84	**0.92**	**0.92**	0.83	0.84	0.83	0.84	0.83	0.84	0.85	
volRangeGK	0.83	0.84	**0.92**	**0.92**	0.84	0.85	0.83	0.85	0.83	0.84	0.85	**Range**
volRangeRS	0.83	0.84	**0.92**	**0.92**	0.84	0.85	0.83	0.84	0.83	0.84	0.85	**(Ex-Oracle)**
volRangeYZ	**0.81**	0.82	0.89	0.89	0.82	0.83	**0.81**	0.82	**0.81**	0.82	0.83	0.13
Mean Across Vol.	0.84	0.85	0.92	0.92	0.84	0.85	0.84	0.85	0.84	0.85		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
0.83	0.85	0.86	0.86

Table 9 – Panel 2: EffBetsMLT Portfolio Empirical Performance Metrics: $m = 500$; $\tau = 125$; All Model Combinations

EffBetsMLT — Skewness

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	-0.19	-0.19	-0.21	-0.21	-0.19	-0.19	-0.20	-0.20	-0.19	-0.19	-0.20	**Min (Ex-Ora.)**
volEWMA92	-0.20	-0.20	-0.22	-0.22	-0.19	-0.19	-0.20	-0.20	-0.20	-0.20	-0.20	-0.22
volEWMA96	-0.19	-0.19	-0.21	-0.21	-0.19	-0.19	-0.19	-0.19	-0.19	-0.19	-0.20	
volEWMA98	-0.19	-0.19	-0.21	-0.21	-0.19	-0.19	-0.19	-0.19	-0.19	-0.19	-0.19	**Max (Ex-Ora.)**
volGARCH(1,1)N	-0.20	-0.20	-0.22	-0.22	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.19
volGARCH(1,1)t	-0.20	-0.20	-0.22	-0.22	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	
volGARCH(GJR)N	-0.20	-0.20	-0.22	-0.22	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	**Oracle**
volGARCH(GJR)t	-0.20	-0.20	-0.22	-0.22	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.15
volRangePK	-0.20	-0.20	-0.21	-0.21	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	
volRangeGK	-0.20	-0.20	-0.22	-0.22	-0.20	-0.20	-0.21	-0.21	-0.20	-0.20	-0.21	**Range**
volRangeRS	-0.21	-0.21	-0.22	-0.22	-0.20	-0.20	-0.21	-0.21	-0.21	-0.21	-0.21	**(Ex-Oracle)**
volRangeYZ	-0.20	-0.20	-0.21	-0.21	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	0.03
Mean Across Vol.	-0.20	-0.20	-0.22	-0.21	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
-0.19	-0.20	-0.15	-0.21

EffBetsMLT — Kurtosis

Holding Period: 125 days

Conditioning Window: 500 days

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	6.22	6.18	6.17	6.17	6.20	6.18	6.22	6.17	6.21	6.17	6.19	**Min (Ex-Ora.)**
volEWMA92	6.18	6.15	6.14	6.14	6.17	6.15	6.19	6.15	6.18	6.14	6.16	6.07
volEWMA96	6.19	6.15	6.14	6.14	6.17	6.15	6.19	6.15	6.18	6.14	6.16	
volEWMA98	6.19	6.15	6.15	6.15	6.17	6.15	6.19	6.15	6.19	6.15	6.17	**Max (Ex-Ora.)**
volGARCH(1,1)N	6.17	6.14	6.14	6.14	6.15	6.13	6.19	6.15	6.17	6.13	6.15	6.22
volGARCH(1,1)t	6.15	6.12	6.11	6.11	6.13	6.11	6.17	6.13	6.15	6.11	6.13	
volGARCH(GJR)N	6.15	6.11	6.10	6.10	6.13	6.11	6.17	6.12	6.15	6.10	6.12	**Oracle**
volGARCH(GJR)t	6.15	6.11	6.10	6.10	6.13	6.11	6.17	6.12	6.14	6.11	6.12	6.23
volRangePK	6.15	6.11	6.10	6.10	6.14	6.11	6.15	6.11	6.15	6.11	6.12	
volRangeGK	6.14	6.10	6.08	6.08	6.12	6.10	6.14	6.09	6.13	6.09	6.11	**Range**
volRangeRS	6.13	6.09	6.07	6.07	6.11	6.08	6.12	6.08	6.12	6.08	6.10	**(Ex-Oracle)**
volRangeYZ	6.14	6.10	6.08	6.08	6.12	6.10	6.14	6.10	6.13	6.10	6.11	0.15
Mean Across Vol.	6.16	6.12	6.11	6.11	6.14	6.12	6.17	6.13	6.16	6.12		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
6.21	6.17	6.23	6.06

EffBetsMLT — Turnover (2-way p.a.)

Holding Period: 125 days

Conditioning Window: 500 days

N = 60

Volatility Models	noNorm_corrHist	csvNorm_corrHist	noNorm_corrAve	csvNorm_corrAve	noNorm_corrPlesio	csvNorm_corrPlesio	noNorm_corrEigenclip	csvNorm_corrEigenclip	noNorm_corrRIE	csvNorm_corrRIE	*Mean Across Correlation Models*	*Summary Statistics*
volHist	48%	47%	44%	44%	50%	49%	48%	47%	48%	47%	47%	**Min (Ex-Ora.)**
volEWMA92	67%	65%	66%	66%	69%	67%	68%	67%	67%	65%	67%	43%
volEWMA96	61%	59%	59%	59%	62%	61%	62%	60%	61%	59%	60%	
volEWMA98	56%	54%	53%	53%	57%	56%	57%	55%	55%	54%	55%	**Max (Ex-Ora.)**
volGARCH(1,1)N	62%	60%	60%	59%	63%	62%	63%	61%	62%	60%	61%	69%
volGARCH(1,1)t	62%	60%	60%	59%	64%	62%	63%	61%	62%	60%	61%	
volGARCH(GJR)N	62%	61%	60%	60%	64%	62%	63%	62%	62%	61%	62%	**Oracle**
volGARCH(GJR)t	63%	62%	61%	60%	65%	63%	64%	62%	63%	62%	62%	70%
volRangePK	47%	45%	43%	43%	49%	47%	47%	46%	47%	45%	46%	
volRangeGK	47%	45%	44%	43%	49%	48%	47%	46%	46%	45%	46%	**Range**
volRangeRS	47%	46%	44%	44%	49%	48%	47%	46%	47%	45%	46%	**(Ex-Oracle)**
volRangeYZ	48%	46%	43%	43%	50%	49%	48%	46%	48%	46%	47%	26%
Mean Across Vol.	56%	54%	53%	53%	58%	56%	56%	55%	56%	54%		

Integrated Covariance Models and Benchmarks

noNorm_LWSMM	noNorm_LWCC	oracleCov	Equal Weight Portfolio
48%	46%	70%	36%

Appendix A: Glossary of Portfolio Codes and Model Abbreviations

Portfolio Codes

MV	Minimum Variance Portfolio with weights x_{MV}
EW	Equal Weighted Portfolio with weights x_{EW}
ERC	Equal Risk Contribution Portfolio with weights x_{ERC}
MDP	Most Diversified Portfolio with weights x_{MDP}
EffBetsPCA	Effective Number of Bets via PCA Torsion Portfolio with weights x_{PCA}
EffBetsMLT	Effective Number of Bets via Minimum Linear Torsion Portfolio with weights x_{MLT}

Volatility Model Abbreviations

volHist	Historical trailing standard deviation with specified window
oracleVol	'Oracle Volatilities': perfect-look ahead out-of-sample volatilities used for the in-sample portfolio estimation
volEWMA	Exponentially Weighted Moving Average Volatility with specified decay coefficient λ (e.g. volEMWA96 denotes EMWA volatility with $\lambda = 0.96$
volGARCH	Generalized Autoregressive Conditional Heteroskedasticity (GARCH) volatility models with specified form and innovation distribution (e.g. volGARCH(1,1)N denotes vanilla GARCH with a Guassian innovation distribution; volGARCH(GJR)t denotes GJR-GARCH with a t innovation distribution
volRangePK	Intra-day range volatility estimator of Parkinson (1980)
volRangeGK	Intra-day range volatility estimator of Garman & Klass (1980)
volRangeRS	Intra-day range volatility estimator of Rogers & Satchell (1991)
volRangeYZ	Intra-day range volatility estimator of Yang & Zhang (2000)

Correlation Model Abbreviations

corrHist	Historical sample correlations yielding sample correlation matrix E
oracleCorr	'Oracle Correlations': perfect-look ahead out-of-sample correlations used for the in-sample portfolio estimation
corrAve	Average correlations – a valid correlation matrix where each entry $\rho_{i,j}$ is set to a uniform sample average value

corrPlesio	The plesiochronous correlation matrix estimator of Choueifaty et al. (2013)
corrEigenClip	The treated correlation matrix estimator of Bouchaud and Potters (2011)
corrRIE	The rotationally invariant correlation matrix estimator of Bouchaud, Bun and Potters (2016)
LW03	The covariance matrix shrinkage estimator of Ledoit and Wolf (2003) with single market model shrinkage target
LW04CC	The covariance matrix shrinkage estimator of Ledoit and Wolf (2004) with constant correlation shrinkage target

End Notes

[i] None of the portfolios in this study include expected returns inputs. It is possible though, that there are some 'risk-based' portfolios that do include expected returns. The Efficient Maximum Sharpe Ratio Portfolio of Amenc, Goltz and Stoyanov (2011) which includes expected returns albeit as an increasing direct function of downside risk, is such an example. It may be a stretch to extend the definition of risk to include a drift (return) term, but this is exactly the argument by Roncalli (2013), defining the portfolio risk $R(x)$ as $R(x) \equiv -x'\mu + c \cdot \sqrt{x'\Sigma x}$, with c an arbitrary scalar, leading to an immediate comparison to the functional form of VaR as $VaR_{\alpha}(x) = -x'\mu + \Phi^{-1}(\alpha) \cdot \sqrt{x'\Sigma x}$, with Φ^{-1} the inverse normal cdf and α the significance level.

[ii] The concept of Euler risk contributions is based on the Euler theorem for the decomposition of multivariate functions into their weighted first partial derivatives. The risk of any one portfolio position is then defined as the product between the weight of that position and the portfolio's marginal sensitivity to a change in that weight. The Euler risk contribution is general in the risk measure and analytical expression exists for contributions under VaR and CVaR. See Du Plessis and van Rensburg (2017) for details.

[iii] The surprising empirical finding in this study is that this exceedingly simple regularization method contributes a great deal of the value of more sophisticated models such as corrEigenClip and corrRIE that typically embed it. See Section 3.

[iv] Choueifaty, Coignard and Reynier (2013) do not provide details about their approach other than that is was inspired by the Hayashi and Yoshida (2005) estimator for overcoming asynchronous arrival times in high frequency estimation. The development of the plesiochronous estimator [corrPlesio] in our study is informed by discussions the corresponding author had with its original developers (Messrs. Choueifaty and Froidure) and who accepts all responsibility for errors and omissions in its design. It was nonetheless deemed interesting enough for inclusion here.

[v] The ability of a volatility model to correct predict the rank order of out-of-sample volatilities should also be useful in the capture of the 'low-volatility effect', which is exactly a phenomenon that exists in the rank order of cross-sectional volatilities. Low volatility portfolios are not directly studied here, but many of the risk-based portfolios in the text take exposure to low volatility factors. See for example Roncalli (2013).