Testing Ex-post Implications of Asset Pricing Models using Individual Stocks*

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Abstract

This paper develops an over-identified IV approach that uses past beta estimates and firm characteristics as instruments for estimating ex-post risk premia while addressing the error-in-variables problem in the two-pass cross-sectional regression method. The approach is developed in the context of large cross sections of individual stocks and short time series. We establish the N-consistency of the resulting IV ex-post risk premia estimator and obtain its asymptotic distribution along with an estimator of its asymptotic variance-covariance matrix. These results are then employed to develop new tests for various asset pricing model implications that we empirically use to evaluate a number of popular asset pricing models.

Keywords: Error-in-variables problem, instrumental variables, individual stocks, N-consistent ex-post risk premia estimator, asset pricing tests.

JEL Classification Codes: C36, C55, C58, G12.

1 Introduction

Asset pricing models suggest that an asset's average return should be related to its exposure to systematic risk. Models differ in the factors they identify as sources of relevant systematic risk. A typical model identifies a small number of pervasive risk factors and postulates that the average return on an asset is a linear function of the factor betas. The quest for the identification of relevant risk factors at the theoretical level can be traced back to the works of Sharpe (1964), Lintner (1965) and Mossin (1966) on the CAPM and Ross (1976) on the APT. On the empirical front, a long line of research on the evaluation of such models has been developed, starting with Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973).

One important aspect of empirical evaluation of an asset pricing model involves determining the cross section of test assets. On standard approach in the literature, introduced by Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973), is to perform the asset pricing tests on a small number of portfolios. Indeed, following Fama and French (1992), it has become standard practice to sort stocks according to some firm characteristic in order to form sets of portfolios, typically deciles, that are subsequently used as test assets. However, as Lewellen, Nagel, and Shanken (2010) and Daniel and Titman (2012) demonstrate, inference regarding the performance of an asset pricing model crucially depends on the choice of test assets.The method used to form the test portfolios could indeed affect the inference results in undesirable ways. As Roll (1977) points out, in the process of forming portfolios, important mispricing in individual stocks can be averaged out within portfolios, making it harder to reject the wrong model. Lo and MacKinlay (1990) are concerned about the exact opposite error: if stocks are grouped into portfolios with respect to attributes already observed to be related to average returns, the correct model may be rejected too often. In a recent contribution to the literature, Kogan and Tian (2015) question the standard practice used in the literature to form portfolio deciles by sorting firms on various characteristics, construct factors as long-short portfolio spreads, and finally using the portfolio deciles as test assets. They point out that, by searching through the firm characteristics known to be associated with substantial spreads in stock returns, it is easy to construct seemingly successful empirical factor pricing models and argue that factor model mining can be a serious concern.

Motivated by this findings, we develop a framework for estimating and evaluating asset pricing factor models using large cross sections of individual stock return data, instead of employing portfolios as test assets, as originally suggested by Litzenberger and Ramaswamy (1979). We only consider short time horizons so that we can evaluate the implications of asset pricing models locally in time. The implications we test involve ex-post risk premia, when the factors are traded portfolio returns or spreads (see Shanken (1992)), as well as measures of potential mispricing at the individual stock level.

The existing methodological literature on the estimation and evaluation of asset pricing models mainly focuses the case in which the time-series sample size, T, is large while the size of the cross section of test assets, N, is small. This scenario is suitable when portfolios, as opposed to individual stocks, are used as test assets.[1] The analysis of linear asset pricing factor models when the number

[1] The long list of related papers includes, among others, Gibbons (1982), Shanken (1985), Connor and Korajczyk (1988), Lehmann and Modest (1988), Gibbons, Ross, and Shanken (1989), Harvey (1989), Lo and MacKinlay (1990),

of test assets N is large has been the subject of a few recent papers. Gagliardini, Ossola, and Scaillet (2012) extend the two-pass cross-sectional methodology to the case of a conditional factor model incorporating firm characteristics. Their asymptotic theory, based on N and T jointly increasing to infinity at suitable rates, facilitates studying time varying risk premia. Chordia, Goyal, and Shanken (2015), building on Shanken (1992), use bias-corrected risk premia estimates in a context with individual stocks and time variation in the betas through macroeconomic variables and firm characteristics. Their focus is the relative contribution of betas and characteristics in explaining cross-sectional differences in conditional expected returns. More closely related to our paper is the recent paper by Jegadeesh, Noh, Pukthuanthong, Roll, and Wang (2015) which employs an instrumental variable (IV) approach to deal with the EIV problem in the risk premia estimation using individual stocks, where the instruments are betas estimated over separate time periods. Our paper differs from their work in the following important aspects. First, we develop a consistent estimator of the variance-covariance matrix of the ex-post risk premia estimator. In contrast, they resort to the original Fama-MacBeth approach for computing standard errors and test statistics. We provide simulation evidence that using Fama-MacBeth standard errors in our small T-large N context can lead to wrong inferences. Second, in addition to beta estimates from past periods, our overidentified IV approach also uses firm characteristics as instruments. Third, while they focus only on risk premia, we address other important implications of asset pricing models examining potential mispricing at the individual stock level.

We contribute to the extant literature by developing an instrumental variable-generalized method of moments (IV-GMM) approach for estimating ex-post risk premia when the number of assets, N, tends to infinity while the time-series length T is fixed. In addition, we propose statistics for testing ex-post asset pricing implications, in terms of risk premia and potential mispricing at the individual stock level, and develop the associated asymptotic theory. In particular, we develop the analogues, suitable for our small T–large N context, of the tests developed in Brennan, Chordia, and Subrahmanyam (1998), relating mispricing to firm characteristics, and Gibbons, Ross, and Shanken (1989), examining the magnitude of the average squared pricing error. In the standard two-pass procedure used for estimating risk premia, the second step is a regression of average returns on estimated betas. As explained in Section 6 in Shanken (1992), when T is fixed and N tends to infinity, the orthogonality condition required for consistency in the second pass is *not* satisfied rendering the two-pass CSR estimator inconsistent. This is a manifestation of the well-known EIV problem which emerges from using beta estimates instead of the true betas. Our approach uses past beta estimates and firm characteristics as instrumental variables in order to deal with the EIV problem. We establish that the overidentified IV-GMM ex-post risk estimator is N-consistent and show that it asymptotically follows a normal distribution. Finally, incorporating a cluster structure for idiosyncratic shock correlations, we obtain an N-consistent estimator of the asymptotic variance-covariance matrix which we use to develop statistics for testing ex-post asset pricing model implications.

Zhou (1991), Shanken (1992), Connor and Korajczyk (1993), Zhou (1993), Zhou (1994), Berk (1995), Hansen and Jagannathan (1997), Ghysels (1998), Jagannathan and Wang (1998), Kan and Zhou (1999), Jagannathan and Wang (2002), Chen and Kan (2004), Lewellen and Nagel (2006), Shanken and Zhou (2007), Kan and Robotti (2009), Hou and Kimmel (2010), Lewellen, Nagel, and Shanken (2010), Nagel and Singleton (2011), Ang and Kristensen (2012), Kan, Gospodinov, and Robotti (2013) and Kan, Robotti, and Shanken (2013).

We examine the performance of the IV-GMM ex-post risk premia estimator, in terms of bias reduction, and the associated test statistics, in terms of size and power properties, for empirically relevant sample sizes, in a number of Monte Carlo simulation experiments. In our empirical investigation, we use the IV-GMM estimator to test the implications of four popular asset pricing model: the CAPM, the Fama and French (1993) three-factor model (FF3), the Hou, Xue, and Zhang (2015) four-factor model (HZX4) and the Fama and French (2015) five-factor model (FF5). To make them relevant for our empirical exercise, we calibrate our simulations to the CAPM, the FF3 model and the HXZ4 model. The simulation results clearly show the significant bias reduction in the cross-sectional regression intercept and ex-post risk premia estimates achieved by the IV-GMM approach and the good performance of our asset pricing tests for relevant sample sizes. Empirically, we find that the CAPM is strongly rejected in all testing periods, while the FF3 and FF5 models are rejected in seven out of eight testing periods, with the exception of the 2005-2009 period, and overall perform similarly. Finally, according to our various tests, the HZX4 model is supported in the first half of the overall sample period, covering the years 1975-1994, while it is rejected in the second half, covering the years 1995-2004.

The rest of the paper is organized as follows. In Section 2, we describe the general econometric framework and develop the IV-GMM ex-post risk premia estimator using past beta estimates and firm characteristics as instruments. We further establish the N-consistency of the IV-GMM estimator, obtain its asymptotic distribution, provide an estimator of its asymptotic variance-covariance matrix and develop novel asset pricing tests. In Section 3, we provide Monte Carlo evidence on the finite sample behavior of the IV-GMM estimator and the associated tests. Section 4 presents empirical evidence on four popular asset pricing models. Finally, Section 5 concludes. Proofs are collected in the Appendix and additional results are delegated to the Online Appendix.

2 Econometric Framework

2.1 Model specification

Consider an economy with N traded assets and K factors. Let $\mathbf{r}_t = [\ r_{1,t} \ \cdots \ r_{N,t}\ ]'$ be the vector of returns of the N traded assets in excess of the risk-free return and $\mathbf{f}_t = [\ f_{1,t} \ \cdots \ f_{K,t}\ ]'$ be the vector of factor realizations at time t. We assume that data are available over times 1 through T, where T is finite and fixed, and formally consider the case in which the number of assets, N, tends to infinity.

We refer to the periods covering times 1 through τ_1 and $\tau_1 + 1$ through $T = \tau_1 + \tau_2$ as the pretesting and testing periods, respectively. That is, τ_1 and τ_2, that are fixed throughout our analysis, are the pretesting and testing time-series sample sizes. We are interested in testing the implications of an asset pricing model over the period from time $\tau_1 + 1$ through $T = \tau_1 + \tau_2$.

The expectations of the excess return $\mathbf{r}_t$ and the factor $\mathbf{f}_t$ are denoted by $\boldsymbol{\mu}_r = \mathrm{E}[\mathbf{r}_t]$ and $\boldsymbol{\mu}_f = \mathrm{E}[\mathbf{f}_t]$, respectively. Furthermore, the $K \times K$ factor variance-covariance matrix is denoted by $\boldsymbol{\Sigma}_f =$

$\mathrm{E}[(\mathbf{f}_t - \boldsymbol{\mu}_f)(\mathbf{f}_t - \boldsymbol{\mu}_f)']$, while the $N \times K$ excess return-factor covariance matrix is denoted by $\boldsymbol{\Sigma}_{rf} = \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\mu}_r)(\mathbf{f}_t - \boldsymbol{\mu}_f)']$. The $N \times K$ beta matrix is then defined by

$$\mathbf{B} = [\ \boldsymbol{\beta}_1 \quad \cdots \quad \boldsymbol{\beta}_N\ ]' = \boldsymbol{\Sigma}_{rf}\boldsymbol{\Sigma}_f^{-1}, \tag{1}$$

where $\boldsymbol{\beta}_i$ denotes the beta vector for the i-th asset, $i = 1, \ldots, N$. Letting $\boldsymbol{\delta}_f$ denote the vector of ex-ante risk premia, the corresponding linear beta model implies the pricing equation $\boldsymbol{\mu}_r = \mathbf{B}\boldsymbol{\delta}_f$. If the factors comprising $\mathbf{f}_t$ belong to the return space, then we have $\boldsymbol{\delta}_f = \boldsymbol{\mu}_f$.

Defining the residual $\mathbf{u}_t = \mathbf{r}_t - \mathbf{B}\mathbf{f}_t$, we can then write $\mathbf{u}_t = (\mathbf{r}_t - \boldsymbol{\mu}_r) - \mathbf{B}(\mathbf{f}_t - \boldsymbol{\mu}_f)$, which implies $\mathrm{E}[\mathbf{u}_t] = \mathbf{0}_N$ and $\mathrm{E}[\mathbf{u}_t\mathbf{f}_t'] = \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\mu}_r)\mathbf{f}_t' - \mathbf{B}(\mathbf{f}_t - \boldsymbol{\mu}_f)\mathbf{f}_t'] = \boldsymbol{\Sigma}_{rf} - \mathbf{B}\boldsymbol{\Sigma}_f = \mathbf{0}_{N\times K}$, where $\mathbf{0}_N$ and $\mathbf{0}_{N\times K}$ denote the $N \times 1$ vector and $N \times K$ matrix of zeros, respectively. Hence, by imposing the pricing restriction $\boldsymbol{\mu}_r = \mathbf{B}\boldsymbol{\delta}_f$, we obtain the following time-series regression representation:

$$\mathbf{r}_t = \mathbf{B}(\mathbf{f}_t + (\boldsymbol{\delta}_f - \boldsymbol{\mu}_f)) + \mathbf{u}_t, \text{ with } \quad \mathrm{E}[\mathbf{u}_t] = \mathbf{0}_N, \quad \mathrm{E}[\mathbf{u}_t\mathbf{f}_t'] = \mathbf{0}_{N\times K}. \tag{2}$$

Over the testing period, covering times $t = \tau_1 + 1, \ldots, \tau_1 + \tau_2$, the data generating process in (2) implies that

$$\bar{\mathbf{r}}_2 = \mathbf{1}_N\lambda_0 + \mathbf{B}\boldsymbol{\lambda}_f + \bar{\mathbf{u}}_2 \tag{3}$$

with

$$\lambda_0 = 0, \quad \boldsymbol{\lambda}_f = \bar{\mathbf{f}}_2 + (\boldsymbol{\delta}_f - \boldsymbol{\mu}_f),$$

where we use the generic notation $\bar{\mathbf{y}}_2 = \frac{1}{\tau_2}\sum_{t=\tau_1+1}^{\tau_1+\tau_2}\mathbf{y}_t$ for a vector time series $\{\mathbf{y}_{\tau_1+1}, \ldots, \mathbf{y}_{\tau_1+\tau_2}\}$.

Recall that, since T is finite and fixed, our analysis is conducted conditionally on the factor realizations. Hence, following Shanken (1992), among others, we refer to $\boldsymbol{\lambda}_f$ as the ex-post risk premia. When the linear factor model holds and the factors belong to the return space, the vector of ex-post risk premia $\boldsymbol{\lambda}_f$ equals the average factor realization over the testing period, namely $\bar{\mathbf{f}}_2$. The object of our inference is the $(K+1) \times 1$ vector

$$\boldsymbol{\lambda} = [\ \lambda_0 \quad \boldsymbol{\lambda}_f'\ ]'. \tag{4}$$

Defining the $N \times (K+1)$ matrix $\mathbf{X}$ by

$$\mathbf{X} = [\ \mathbf{1}_N \quad \mathbf{B}\ ], \tag{5}$$

we can rewrite equation (3) as

$$\bar{\mathbf{r}}_2 = \mathbf{1}_N \lambda_0 + \mathbf{B}\boldsymbol{\lambda}_f + \bar{\mathbf{u}}_2 = \mathbf{X}\boldsymbol{\lambda} + \bar{\mathbf{u}}_2. \tag{6}$$

2.2 Estimating ex-post risk premia

If the true beta matrix $\mathbf{B}$ were known, an N-consistent estimator of $\boldsymbol{\lambda}$ could be obtained by regressing the average excess return vector $\bar{\mathbf{r}}_2$ on a vector of ones and the beta matrix $\mathbf{B}$, under the reasonable assumption of zero limiting cross-sectional correlation between the betas and the shocks. However, the beta matrix $\mathbf{B}$ is not known and has to be estimated using the available data. Natural proxies for $\mathbf{B}$ are the time-series OLS estimators of $\mathbf{B}$ obtained using data from the pretesting period or the testing period. When τ_1 and τ_2 are fixed, as in our framework, the two-pass CSR approach with either proxy yields an inconsistent estimator. This is a manifestation of the well-known EIV problem as pointed out in Shanken (1992), Jagannathan, Skoulakis, and Wang (2010), and Kim and Skoulakis (2017).

Various approaches have been advanced in the statistics and econometrics literature for dealing with the EIV problem. One such approach, particularly suitable for the case in which multiple proxies of unobserved quantities are available, is the instrumental variable (IV) approach. Starting with the early works of Wald (1940), Reiersøl (1941), and Geary (1943), a long related literature was subsequently developed.[2] Chapter 6 of Carroll, Ruppert, Stefanski, and Crainiceanu (2006) offers a comprehensive account of the IV method, where they state that "One possible source of an instrumental variable is a second, possibly biased, measurement of the (true unobserved) regressor obtained by an independent measuring method."[3] The recent paper by Jegadeesh, Noh, Pukthuanthong, Roll, and Wang (2015) also uses an IV approach in the context of asset pricing tests. Our paper differ from the aforementioned paper in a number of important aspects. First, we use beta estimates obtained in the pretesting period and therefore, given that our pretesting and testing periods are non-overlapping and consecutive, our approach is less prone to potential serial correlations in the real data. Second, in addition to past beta estimates, we also use firm characteristics as instruments. As a result, our estimator is an overidentified IV-GMM estimator and not a standard two-stage IV estimator. Third, focusing on the case of fixed T and large N, we develop a statistic for jointly testing ex-post asset pricing implications in terms of risk-premia and potential mispricing at the individual stock level. Finally, we develop a fully operational asymptotic theory for our tests. That is, we show consistency and asymptotic normality of the joint test statistic, and N tends to infinity, and, furthermore, construct an estimator of its asymptotic variance-covariance matrix that we then use to develop novel asset pricing tests.

To develop the IV risk premia estimator, we need to introduce some notation. Define the $N \times \tau_1$ excess return matrix $\mathbf{R}_1$ and the $K \times \tau_1$ factor realization matrix $\mathbf{F}_1$, over the pretesting period, the $N \times \tau_2$ excess return matrix $\mathbf{R}_2$ and the $K \times \tau_2$ factor realization matrix $\mathbf{F}_2$, over the testing period,

[2] Durbin (1954) provides a review of the early EIV literature. Aldrich (1993) offers a historical account of the development of the IV approach to the EIV problem in the 1940s.

[3] In the Online Appendix, we illustrate how the IV method can be used in the context of a linear regression model with regressors subject to the EIV problem.

by

$$\mathbf{R}_1 = [\ \mathbf{r}_1 \quad \cdots \quad \mathbf{r}_{\tau_1}\ ], \quad \mathbf{F}_1 = [\ \mathbf{f}_1 \quad \cdots \quad \mathbf{f}_{\tau_1}\ ] \tag{7}$$

and

$$\mathbf{R}_2 = [\ \mathbf{r}_{\tau_1+1} \quad \cdots \quad \mathbf{r}_{\tau_1+\tau_2}\ ], \quad \mathbf{F}_2 = [\ \mathbf{f}_{\tau_1+1} \quad \cdots \quad \mathbf{f}_{\tau_1+\tau_2}\ ]. \tag{8}$$

Then, using the quantities defined in (7) and (8), we express the time-series OLS estimators of the beta matrix $\mathbf{B}$ over the pretesting and testing periods, denoted by $\widehat{\mathbf{B}}_1$ and $\widehat{\mathbf{B}}_2$, respectively, as follows:

$$\widehat{\mathbf{B}}_1 = \left(\mathbf{R}_1\mathbf{J}_{\tau_1}\mathbf{F}_1'\right)\left(\mathbf{F}_1\mathbf{J}_{\tau_1}\mathbf{F}_1'\right)^{-1}, \quad \widehat{\mathbf{B}}_2 = \left(\mathbf{R}_2\mathbf{J}_{\tau_2}\mathbf{F}_2'\right)\left(\mathbf{F}_2\mathbf{J}_{\tau_2}\mathbf{F}_2'\right)^{-1}, \tag{9}$$

where $\mathbf{J}_m = \mathbf{I}_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m'$, with $\mathbf{I}_m$ and $\mathbf{1}_m$ denoting the $m \times m$ identity matrix and the $m \times 1$ vector of ones, respectively, for any positive integer m.[4]

We introduce the $N \times \tau_1$ idiosyncratic shock matrix $\mathbf{U}_1$, over the pretesting period, and the $N \times \tau_2$ idiosyncratic shock matrix $\mathbf{U}_2$, over the testing period, defined by

$$\mathbf{U}_1 = [\ \mathbf{u}_1 \quad \cdots \quad \mathbf{u}_{\tau_1}\ ], \quad \mathbf{U}_2 = [\ \mathbf{u}_{\tau_1+1} \quad \cdots \quad \mathbf{u}_{\tau_1+\tau_2}\ ]. \tag{10}$$

Alternatively, letting $\mathbf{u}_{1,[i]}'$ and $\mathbf{u}_{2,[i]}'$ denote the i-th row of $\mathbf{U}_1$ and $\mathbf{U}_2$, respectively, for $i = 1, \ldots, N$, we can write

$$\mathbf{U}_1 = [\ \mathbf{u}_{1,[1]} \quad \cdots \quad \mathbf{u}_{1,[N]}\ ]', \quad \mathbf{U}_2 \ = \ [\ \mathbf{u}_{2,[1]} \quad \cdots \quad \mathbf{u}_{2,[N]}\ ]'. \tag{11}$$

Observe that $\overline{\mathbf{u}}_2$, the disturbance term in equation (6), and $\mathbf{U}_2$ are related by

$$\overline{\mathbf{u}}_2 = \frac{1}{\tau_2}\mathbf{U}_2\mathbf{1}_{\tau_2}. \tag{12}$$

Noting that $\mathbf{R}_1 = \mathbf{B}\mathbf{F}_1 + \mathbf{U}_1$ and $\mathbf{R}_2 = \mathbf{B}\mathbf{F}_2 + \mathbf{U}_2$, we can decompose the beta estimators $\widehat{\mathbf{B}}_1$ and $\widehat{\mathbf{B}}_2$, defined in (9), respectively, into the true beta matrix $\mathbf{B}$ and the corresponding estimation error terms as follows:

$$\widehat{\mathbf{B}}_1 = \mathbf{B} + \mathbf{U}_1\mathbf{G}_1, \quad \widehat{\mathbf{B}}_2 = \mathbf{B} + \mathbf{U}_2\mathbf{G}_2, \tag{13}$$

where the $\tau_1 \times K$ matrix $\mathbf{G}_1$ and the $\tau_2 \times K$ matrix $\mathbf{G}_2$ are defined by

$$\mathbf{G}_1 = \mathbf{J}_{\tau_1}\mathbf{F}_1'\left(\mathbf{F}_1\mathbf{J}_{\tau_1}\mathbf{F}_1'\right)^{-1}, \quad \mathbf{G}_2 = \mathbf{J}_{\tau_2}\mathbf{F}_2'\left(\mathbf{F}_2\mathbf{J}_{\tau_2}\mathbf{F}_2'\right)^{-1}. \tag{14}$$

[4]Standard matrix algebra shows that $\mathbf{J}_m$ is a symmetric and idempotent matrix, and that $\mathrm{tr}\left(\mathbf{J}_m\right) = m - 1$.

To illustrate the effect of the beta estimation error in the case that we use $\widehat{\mathbf{B}}_2$ as a beta matrix proxy in the second-pass CSR, we observe that equation (6) is reexpressed as $\bar{\mathbf{r}}_2 = \widehat{\mathbf{X}}_2\boldsymbol{\lambda} + (\mathbf{X} - \widehat{\mathbf{X}}_2)\boldsymbol{\lambda} + \bar{\mathbf{u}}_2$ or

$$\bar{\mathbf{r}}_2 = \widehat{\mathbf{X}}_2\boldsymbol{\lambda} + \boldsymbol{\omega}_2, \tag{15}$$

where

$$\widehat{\mathbf{X}}_2 = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_2\ ], \tag{16}$$

$$\boldsymbol{\omega}_2 = (\mathbf{X} - \widehat{\mathbf{X}}_2)\boldsymbol{\lambda} + \bar{\mathbf{u}}_2. \tag{17}$$

Using equations (12) and (13), the disturbance term $\boldsymbol{\omega}_2$ defined in (17) can be expressed as

$$\boldsymbol{\omega}_2 = -(\mathbf{U}_2\mathbf{G}_2)\boldsymbol{\lambda}_f + \bar{\mathbf{u}}_2 = \mathbf{U}_2\left(\frac{1}{\tau_2}\mathbf{1}_{\tau_2} - \mathbf{G}_2\boldsymbol{\lambda}_f\right) = \mathbf{U}_2\mathbf{g}_2, \tag{18}$$

where the $\tau_2 \times 1$ vector of $\mathbf{g}_2$ is given by

$$\mathbf{g}_2 = \frac{1}{\tau_2}\mathbf{1}_{\tau_2} - \mathbf{G}_2\boldsymbol{\lambda}_f = \frac{1}{\tau_2}\mathbf{1}_{\tau_2} - \mathbf{J}_{\tau_2}\mathbf{F}_2'\left(\mathbf{F}_2\mathbf{J}_{\tau_2}\mathbf{F}_2'\right)^{-1}\boldsymbol{\lambda}_f. \tag{19}$$

It follows from equations (16) and (17) that the orthogonality condition, necessary for consistency, is violated in the cross-sectional regression (15). In the next subsection, we start our analysis by developing a consistent ex-post risk premia estimator using an instrumental variable approach.

It follows from the expressions (13) and (18) that the regressor and disturbance terms in the cross-sectional regression (15) are correlated through the beta estimation error contained in $\widehat{\mathbf{B}}_2$. Hence, ignoring the error-in-variables problem, one would obtain an inconsistent ex-post risk premia estimator. We develop an instrumental variable approach to deal with the error-in-variables problem using past beta estimates and firm characteristics as instruments. Next, we explain that, under mild assumptions, $\widehat{\mathbf{B}}_1$ can serve as an instrumental variable for constructing an N-consistent estimator of $\boldsymbol{\lambda}$ using the cross-sectional regression (15).

Assumption 1 **(i)** *As* $N \to \infty$, $\frac{1}{N}\mathbf{U}'\mathbf{1}_N \xrightarrow{p} \mathbf{0}_T$ *and* $\frac{1}{N}\mathbf{U}'\mathbf{B} \xrightarrow{p} \mathbf{0}_{T\times K}$, *where* $\mathbf{U} = [\ \mathbf{U}_1 \quad \mathbf{U}_2\ ]$. **(ii)** *As* $N \to \infty$, $\frac{1}{N}\mathbf{U}_1'\mathbf{U}_2 \xrightarrow{p} \mathbf{0}_{\tau_1\times\tau_2}$. **(iii)** *As* $N \to \infty$, $\frac{1}{N}\mathbf{B}'\mathbf{1}_N = \frac{1}{N}\sum_{i=1}^N \boldsymbol{\beta}_i \to \boldsymbol{\mu}_\beta$. **(iv)** *As* $N \to \infty$, $\frac{1}{N}\sum_{i=1}^N(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)(\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta)' \to \mathbf{V}_\beta$, *where* $\mathbf{V}_\beta$ *is a symmetric and positive definite matrix.*

Assumption 1(i) states that, at each time t, the cross-sectional average of the shocks $u_{i,t}$ converges to zero, and the limiting cross-sectional correlation between the shocks $u_{i,t}$ and the betas $\boldsymbol{\beta}_i$ is also zero, as the number of assets N tends to ∞. For $1 \le t \le \tau_1$ and $\tau_1 + 1 \le t' \le \tau_1 + \tau_2$, Assumption 1(ii) states that the limiting cross-sectional correlation between $u_{i,t}$ and $u_{i,t'}$ vanishes. That is, the pretesting-period and the testing-period shocks are assumed to be cross-sectionally uncorrelated in the limit $N \to \infty$.[5] Assumption 1(iii) states that the limiting cross-sectional average of the betas $\boldsymbol{\beta}_i$ exists while Assumption 1(iv) states that the limiting cross-sectional variance of the betas $\boldsymbol{\beta}_i$ exists and is a symmetric and positive definite matrix.

In light of equation (13), it follows from Assumption 1 that $\widehat{\mathbf{B}}_1$ is correlated with the explanatory variable $\widehat{\mathbf{B}}_2$ in the cross-sectional regression (15), in the sense that $\widehat{\mathbf{B}}_1'\widehat{\mathbf{B}}_2/N$ converges in probability to $\mathbf{M}_\beta = \mathbf{V}_\beta + \boldsymbol{\mu}_\beta\boldsymbol{\mu}_\beta'$, which is symmetric and positive definite, and hence invertible matrix, as $N \to \infty$. Furthermore, based on equations (13) and (18), Assumptions 1(i) and 1(ii) imply that the proposed instrumental variable $\widehat{\mathbf{B}}_1$ is uncorrelated with the disturbance term $\boldsymbol{\omega}_2$ in the cross-sectional regression (15), in the sense that $\widehat{\mathbf{B}}_1'\boldsymbol{\omega}_2/N \xrightarrow{p} \mathbf{0}_K$, as $N \to \infty$. These properties are formally established in Theorem 1 below, where we establish the N-consistency of the proposed IV-GMM estimator.

In our empirical applications, the factors are returns on spread portfolios constructed after sorting stocks with respect to a certain firm characteristic, such as size and book-to-market ratio. In this context, it is expected that characteristics and betas with respect to the corresponding spread are highly correlated. We indeed provide evidence that this is the case in Section 4, where we empirically evaluate a number of popular asset pricing models. Hence, naturally, we also employ the characteristics that are related to the spread factors as instrumental variables. Moreover, we utilize additional firm characteristics, not directly related to the factors, to examine whether characteristics can explain potential mispricing at the individual stock level. For the purposes of the theoretical development, we make the following assumption on firm characteristics, for which we need to introduce some additional notation. Let $\mathbf{C} = [\ \mathbf{C}_f \quad \mathbf{C}_a\ ]$ denote the $N \times J$ matrix of characteristics observed in the pretesting period. The $N \times L$ matrix $\mathbf{C}_f$ contains the characteristics to be used as instruments, while the $N \times (J - L)$ matrix $\mathbf{C}_a$ contains the additional characteristics to be used to detect mispricing. For $i = 1, \dots, N$, let $\mathbf{c}_i'$, $\mathbf{c}_{f,i}'$, and $\mathbf{c}_{a,i}'$ be the i-th row of $\mathbf{C}$, $\mathbf{C}_f$, and $\mathbf{C}_a$, respectively.

Assumption 2 **(i)** *As* $N \to \infty$, $\mathbf{C}'\mathbf{U}_1/N \xrightarrow{p} \mathbf{V}_{cu}$, *where* $\mathbf{V}_{cu}$ *is an* $J \times \tau_1$ *matrix.* **(ii)** *As* $N \to \infty$, $\mathbf{C}'\mathbf{U}_2/N \xrightarrow{p} \mathbf{0}_{J\times\tau_2}$. **(iii)** *For each* N, $\mathbf{C}'\mathbf{1}_N/N = \frac{1}{N}\sum_{i=1}^N \mathbf{c}_i = \mathbf{0}_J$. **(iv)** *As* $N \to \infty$, $\mathbf{C}'\mathbf{C}/N \to \mathbf{V}_c$, *where* $\mathbf{V}_c$ *is a* $J \times J$ *symmetric and positive definite matrix.* **(iv)** *As* $N \to \infty$, $\mathbf{C}'\mathbf{B}/N \xrightarrow{p} \mathbf{V}_{c\beta} \equiv [\ (\mathbf{V}_{c\beta}^f)' \quad (\mathbf{V}_{c\beta}^a)'\ ]'$, *where* $\mathbf{V}_{c\beta}$, $\mathbf{V}_{c\beta}^f$, *and* $\mathbf{V}_{c\beta}^a$ *are* $J\times K$, $L\times K$, *and* $(J-L)\times K$ *matrices, respectively.*

Assumptions 2(i) and 2(ii) state that firm characteristics observed in the pretesting period are potentially correlated with idiosyncratic shocks in the pretesting period but not with those in the testing period. In light of equation (18), Assumption 2(ii) states that $\mathbf{C}$ is uncorrelated with the disturbance term $\boldsymbol{\omega}_2$. Assumption 2(iii), without loss of generality, postulates that the first cross-sectional moments of firm characteristics are zero, while Assumption 2(iv) states that the second cross-sectional moments of the firm characteristics are well-defined. Finally, Assumption 2(v) states that the firm characteristics observed in the pretesting period are correlated with the true betas.

[5] As long as the pretesting and testing periods do not overlap and the shocks over the two periods are cross-sectionally uncorrelated when $N \to \infty$, the IV approach would provide valid inference. In our analysis, we consider the two periods to be consecutive so as to mitigate the effect of potential serial correlation in the real data.

Under the aforementioned assumptions, and in particular Assumptions 1 (ii) and 2 (ii), the past beta estimates $\widehat{\mathbf{B}}_1$ and the characteristics $\mathbf{C}_f$ can be used as instruments in the estimation of ex-post risk premia, giving rise to the following overidentified IV-GMM estimator:

$$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} = \left[(\widehat{\mathbf{X}}_2'\widehat{\mathbf{Z}}_1)\widehat{\mathbf{W}}(\widehat{\mathbf{Z}}_1'\widehat{\mathbf{X}}_2)\right]^{-1}(\widehat{\mathbf{X}}_2'\widehat{\mathbf{Z}}_1)\widehat{\mathbf{W}}(\widehat{\mathbf{Z}}_1'\bar{\mathbf{r}}_2), \tag{20}$$

where the $N \times (K+L+1)$ instrument matrix $\widehat{\mathbf{Z}}_1$ is defined by

$$\widehat{\mathbf{Z}}_1 = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_1 \quad \mathbf{C}_f\ ], \tag{21}$$

and $\widehat{\mathbf{W}}$ is a $(K+L+1)\times(K+L+1)$ symmetric weighting matrix of full rank which can be computed using the available data. The weighting matrix $\widehat{\mathbf{W}}$ is assumed to converge to a symmetric and positive definite matrix $\mathbf{W}$, as $N \to \infty$. Note that, if we only use the past beta estimates as instruments, i.e., if $\widehat{\mathbf{Z}}_1 = \widehat{\mathbf{X}}_1 = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_1\ ]$, then the weighting matrix is irrelevant and the estimator assumes the usual exactly identified IV form: $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}} = (\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_2)^{-1}(\widehat{\mathbf{X}}_1'\bar{\mathbf{r}}_2)$. We will establish the N-consistency and asymptotic normality of the estimator $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$ for a generic weighting matrix and then show how to obtain the efficient IV-GMM estimator by suitably selecting $\mathbf{W}$ and $\widehat{\mathbf{W}}$. Note that equation (15) implies

$$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} = \boldsymbol{\lambda} + \left(\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\widehat{\boldsymbol{\Omega}}\right)^{-1}\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\left(\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2/N\right), \tag{22}$$

where

$$\widehat{\boldsymbol{\Omega}} = \frac{1}{N}\widehat{\mathbf{Z}}_1'\widehat{\mathbf{X}}_2 \xrightarrow{p} \boldsymbol{\Omega}, \tag{23}$$

and $\boldsymbol{\Omega}$ is a full-rank $(K+L+1)\times(K+1)$ matrix (see equation (95) in the Appendix). Moreover, $\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2/N$ converges to a vector of zeros, and hence N-consistency of $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$ is shown. The proof of the following theorem contains the details.

Theorem 1 *Under Assumptions 1 and 2, the IV-GMM ex-post risk premia estimator $\widehat{\boldsymbol{\lambda}}_{IV}^{GMM}$, defined in (20), is an N-consistent estimator of $\boldsymbol{\lambda}$.*

Having established the N-consistency of the proposed IV-GMM estimator $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$, we proceed to define a measure of aggregate mispricing, that takes into account the time-series alphas of all individual stocks, and describe its limit as the number of stocks increases to infinity.

2.3 Two metrics of ex-post mispricing

It is common practice to evaluate asset pricing models by examining the corresponding pricing errors. Given the ex-post risk premia estimator $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$, obtained in the previous subsection, we define the vector of ex-post pricing error estimates, traditionally referred to as alphas, as follows

$$\widehat{\boldsymbol{\alpha}} = \bar{\mathbf{r}}_2 - \widehat{\mathbf{X}}_2 \widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}. \tag{24}$$

Invoking equation (6), we obtain $\widehat{\boldsymbol{\alpha}} = \bar{\mathbf{u}}_2 + \mathbf{X}\boldsymbol{\lambda} - \widehat{\mathbf{X}}_2 \widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} = \bar{\mathbf{u}}_2 - (\widehat{\mathbf{X}}_2 - \mathbf{X})\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \mathbf{X}(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda})$. Moreover, combining (13) and (16) yields $\widehat{\mathbf{X}}_2 - \mathbf{X} = [\ \mathbf{0}_N \quad \mathbf{U}_2\mathbf{G}_2\ ]$, and so, using equation (12), we obtain that

$$\widehat{\boldsymbol{\alpha}} = \mathbf{U}_2 \widehat{\mathbf{g}}_2 - \mathbf{X}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right), \tag{25}$$

where

$$\widehat{\mathbf{g}}_2 = \frac{1}{\tau_2}\mathbf{1}_{\tau_2} - \mathbf{G}_2 \widehat{\boldsymbol{\lambda}}_{\mathtt{IV},f}^{\mathtt{GMM}}, \tag{26}$$

and the $K \times 1$ vector $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV},f}^{\mathtt{GMM}}$ is determined by $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} = \left[\widehat{\lambda}_{\mathtt{IV},0}^{\mathtt{GMM}} \quad \left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV},f}^{\mathtt{GMM}}\right)'\right]'$.

In what follows, we focus on two aspects of ex-post mispricing based on which we develop associated tests. First, in the spirit of Brennan, Chordia, and Subrahmanyam (1998), we examine whether characteristics can explain any potential ex-post mispricing by estimating the so-called characteristics rewards. Second, in the spirit of Gibbons, Ross, and Shanken (1989), we investigate the magnitude of the ex-post alpha vector $\widehat{\boldsymbol{\alpha}}$ by considering the average squared pricing error.

2.3.1 A characteristics-based mispricing metric

The first mispricing metric is obtained by considering whether the ex-post alpha estimates are related to firm characteristics. Specifically, following the approach advanced by Brennan, Chordia, and Subrahmanyam (1998), we estimate the characteristics rewards by regressing $\widehat{\boldsymbol{\alpha}}$ on the matrix of characteristics $\mathbf{C} = [\ \mathbf{C}_f \quad \mathbf{C}_a\ ]$. The resulting estimator is[6]

$$\widehat{\boldsymbol{\phi}} = \left(\mathbf{C}'\mathbf{C}\right)^{-1}\left(\mathbf{C}'\widehat{\boldsymbol{\alpha}}\right). \tag{27}$$

Under the null hypothesis of correct model specification, characteristics should not be able to explain any patterns of mispricing reflected in the ex-post alphas. Namely, as we illustrate next, the vector of estimated characteristics rewards $\widehat{\boldsymbol{\phi}}$ should converge to a vector of zeros, when $N \to \infty$.

Using the definition of $\widehat{\boldsymbol{\phi}}$ in (27) and equation (25), we obtain

$$\widehat{\boldsymbol{\phi}} = \left(\mathbf{C}'\mathbf{C}/N\right)^{-1}\left[\left(\mathbf{C}'\mathbf{U}_2/N\right)\widehat{\mathbf{g}}_2 - \left(\mathbf{C}'\mathbf{X}/N\right)\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)\right]. \tag{28}$$

[6]Note that we do not include an intercept in this regression. This choice does not pose any problem given our assumption that the characteristics have zero cross-sectional means and given that we do not use the residuals of this regression in the subsequent analysis.

It follows from definitions (19) and (26) and Theorem 1 that, as $N \to \infty$,

$$\widehat{\mathbf{g}}_2 \xrightarrow{p} \mathbf{g}_2. \tag{29}$$

According to Assumption 2, as $N \to \infty$, we have $\frac{\mathbf{C}'\mathbf{C}}{N} \to \mathbf{V}_c$, $\frac{\mathbf{C}'\mathbf{U}_2}{N} \xrightarrow{p} \mathbf{0}_{J\times\tau_2}$, and $\frac{\mathbf{C}'\mathbf{X}}{N} \to [\ \mathbf{0}_J \quad \mathbf{V}_{c\beta}\ ]$. Hence, invoking Theorem 1 again, we obtain from equation (28) that, as $N \to \infty$, $\widehat{\boldsymbol{\phi}} \to \mathbf{0}_J$. The following proposition formally states the result.

Proposition 2 *Let Assumptions 1 and 2 be in effect. Then, as $N \to \infty$, the characteristics rewards estimator $\widehat{\boldsymbol{\phi}} = (\mathbf{C}'\mathbf{C})^{-1}(\mathbf{C}'\widehat{\boldsymbol{\alpha}})$, where the vector of estimated ex-post alphas $\widehat{\boldsymbol{\alpha}}$ is defined in (24), converges to $\boldsymbol{\phi} = \mathbf{0}_J$.*

In what follows, we obtain the asymptotic distribution of $\widehat{\boldsymbol{\phi}}$ based on which we then build statistics that can be used to test the null hypothesis of correct model specification.

2.3.2 Average squared pricing error

To gauge the magnitude of the ex-post alpha vector $\widehat{\boldsymbol{\alpha}}$ given by (24), we use as a metric the following average squared pricing error

$$\widehat{\mathcal{Q}} = \frac{1}{N}\widehat{\boldsymbol{\alpha}}'\widehat{\boldsymbol{\alpha}}. \tag{30}$$

Note that $\widehat{\mathcal{Q}}$ is the analogue to the well-known and widely used GRS statistic of Gibbons, Ross, and Shanken (1989) in the small T-large N context. When the asset pricing model is correctly specified, the number of test assets N is small and fixed and T tends to infinity, then all individual alphas, as well as $\widehat{\mathcal{Q}}$, vanish in the limit. In contrast, in our context $\widehat{\mathcal{Q}}$ converges to a positive quantity that we denote by $\mathcal{Q}$. To study the sampling properties of $\widehat{\mathcal{Q}}$, we make the following mild assumption on the limiting behavior of the second moment of the disturbances $u_{i,s}$, $i = 1,\ldots,N$ and $s = 1,\ldots,\tau_2$, allowing for time-series heteroscedasticity.

Assumption 3 *As $N \to \infty$, $\frac{1}{N}\mathbf{U}_2'\mathbf{U}_2 \xrightarrow{p} \mathbf{V}_2$, where $\mathbf{V}_2$ is a diagonal matrix with (s,s) element equal to a positive constant $v_{2,s}$, $s = 1,\ldots,\tau_2$.*

Next, we describe the probability limit of $\widehat{\mathcal{Q}}$. It follows from definition (30) and equation (25) that

$$\widehat{\mathcal{Q}} = \widehat{\mathbf{g}}_2'\left(\mathbf{U}_2'\mathbf{U}_2/N\right)\widehat{\mathbf{g}}_2 + (\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda})'\left(\mathbf{X}'\mathbf{X}/N\right)(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}) - 2\widehat{\mathbf{g}}_2'\left(\mathbf{U}_2'\mathbf{X}/N\right)(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}). \tag{31}$$

Note that Assumptions 1(iii) and 1(iv) together imply that the matrix $\frac{\mathbf{X}'\mathbf{X}}{N}$ converges to a finite $(K+1)\times(K+1)$ matrix and Assumption 1(i) implies that $\frac{\mathbf{U}_2'\mathbf{X}}{N} \xrightarrow{p} \mathbf{0}_{\tau_2\times(K+1)}$. Hence, using the probability limit in (29), Theorem 1 and Assumption 3, we obtain from equation (31) that $\widehat{\mathcal{Q}} \xrightarrow{p}$

$\mathbf{g}_2'\mathbf{V}_2\mathbf{g}_2$. Using fact (F2), we obtain the following proposition which characterizes the probability limit $\mathcal{Q}$ of the aggregate mispricing metric $\widehat{\mathcal{Q}}$.

Proposition 3 *Under Assumptions 1, 2, and 3, $\widehat{\mathcal{Q}} \xrightarrow{p} \mathcal{Q}$, as $N \to \infty$, where*

$$\mathcal{Q} = (\mathbf{g}_2 \otimes \mathbf{g}_2)' \mathbf{v}_2, \tag{32}$$

the vector $\mathbf{g}_2$ is defined in (19), and the vector $\mathbf{v}_2$ is given by

$$\mathbf{v}_2 = vec(\mathbf{V}_2). \tag{33}$$

To further study the sampling distribution of the aggregate mispricing metric $\widehat{\mathcal{Q}}$, we need an estimator of $\mathcal{Q} = (\mathbf{g}_2 \otimes \mathbf{g}_2)' \mathbf{v}_2$. Such an estimator can be obtained by employing estimators of $\mathbf{g}_2$ and $\mathbf{v}_2$.

First, note that $\mathbf{g}_2$ can be consistently estimated by $\widehat{\mathbf{g}}_2$ as it follows from (29). Next, we propose an estimator of the vector $\mathbf{v}_2$. To do so, we introduce the time-series regression residuals over the testing period, $\widehat{\mathbf{u}}_s = (\mathbf{r}_s - \overline{\mathbf{r}}_2) - \widehat{\mathbf{B}}(\mathbf{f}_s - \overline{\mathbf{f}}_2)$, $s = \tau_1 + 1, \ldots, \tau_1 + \tau_2$, gathered in the $N \times \tau_2$ matrix

$$\widehat{\mathbf{U}}_2 = \mathbf{R}_2 \mathbf{J}_{\tau_2} - \widehat{\mathbf{B}}_2 \mathbf{F}_2 \mathbf{J}_{\tau_2}. \tag{34}$$

Noting that $\mathbf{R}_2 = \mathbf{B}\left(\boldsymbol{\delta}_f - \boldsymbol{\mu}_f\right)\mathbf{1}_{\tau_2}' + \mathbf{B}\mathbf{F}_2 + \mathbf{U}_2$ and using equation (13) along with the property of $\mathbf{J}_{\tau_2}\mathbf{1}_{\tau_2} = \mathbf{0}_{\tau_2}$, we can express $\widehat{\mathbf{U}}_2$ as

$$\widehat{\mathbf{U}}_2 = \mathbf{U}_2 \mathbf{H}_2, \tag{35}$$

where the matrix $\mathbf{H}_2$ is defined by[7]

$$\mathbf{H}_2 = \mathbf{J}_{\tau_2} - \mathbf{J}_{\tau_2}\mathbf{F}_2'\left(\mathbf{F}_2\mathbf{J}_{\tau_2}\mathbf{F}_2'\right)^{-1}\mathbf{F}_2\mathbf{J}_{\tau_2}. \tag{36}$$

We assume throughout that the $\tau_2 \times \tau_2$ Hadamard product matrix $\mathbf{H}_2 \odot \mathbf{H}_2$ is of full rank, and hence invertible.[8] The following proposition provides an N-consistent estimator of $\mathbf{v}_2$.

[7] Standard matrix algebra shows that $\mathbf{H}_2$ is symmetric and idempotent. Moreover, it follows from the properties of the trace operator that $\operatorname{tr}(\mathbf{H}_2) = \tau_2 - K - 1$.

[8] It is straightforward to establish that the matrix $\mathbf{H}_2$ is equal to the projection matrix $\mathbf{M}_2 = \mathbf{I}_{\tau_2} - \widetilde{\mathbf{F}}_2(\widetilde{\mathbf{F}}_2'\widetilde{\mathbf{F}}_2)\widetilde{\mathbf{F}}_2'$, where $\widetilde{\mathbf{F}}_2 = [\ \mathbf{1}_{\tau_2} \quad \mathbf{F}_2'\ ]$. The Hadamard product $\mathbf{M}_2 \odot \mathbf{M}_2$, and its invertibility, has come up in early studies of linear models with heteroscedasticity. Hartley, Rao, and Kiefer (1969) and Rao (1970) provide sufficient conditions for the invertibility of $\mathbf{M}_2 \odot \mathbf{M}_2$, while Mallela (1972) provides a necessary and sufficient condition. It follows from the results in Mallela (1972) that a necessary condition for the invertibility of $\mathbf{H}_2 \odot \mathbf{H}_2$ is $\tau_2 \geq 2K + 3$. When this condition is satisfied and the factors are normally distributed, extensive simulation evidence suggests that $\mathbf{H}_2 \odot \mathbf{H}_2$ is indeed invertible. In fact, $\mathbf{H}_2 \odot \mathbf{H}_2$ is always invertible in our empirical applications, where $\tau_2 = 60$ and K takes values up to 5.

Proposition 4 *Under Assumption 3, the vector*

$$\widehat{\mathbf{v}}_2 = \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right] vec\left(\widehat{\mathbf{U}}_2'\widehat{\mathbf{U}}_2/N\right) \tag{37}$$

is an N-consistent estimator of $\mathbf{v}_2$, where $\widehat{\mathbf{U}}_2$ is defined in (34), $\mathbf{H}_2$ is defined in (36), and $\boldsymbol{\mathcal{S}}$ is the $\tau_2^2 \times \tau_2$ selection matrix such that the $(\tau_2(s-1)+s, s)$ element of $\boldsymbol{\mathcal{S}}$ is 1, for $s = 1, \ldots, \tau_2$, and all other elements are zero.

The following theorem follows from Theorem 1 and Propositions 3 and 4.

Theorem 5 *Let Assumptions 1 and 3 be in effect. Then, as $N \to \infty$,*

$$\widehat{\mathcal{Q}} - \left(\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2\right)' \widehat{\mathbf{v}}_2 \xrightarrow{p} 0. \tag{38}$$

In the previous two subsections, we describe the limits of the ex-post risk premia estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}$, the characteristics rewards estimator $\widehat{\boldsymbol{\phi}}$, and the aggregate mispricing metric $\widehat{\mathcal{Q}}$. In the following subsection, we obtain their asymptotic distributions based on which we will devise empirical test statistics.

2.4 Asymptotic distributions

The objective in this subsection is to obtain the asymptotic distributions of the following three statistics: (i) $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}} - \boldsymbol{\lambda}\right)$, (ii) $\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)$ and (iii) $\sqrt{N}\left(\widehat{\mathcal{Q}} - \left(\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2\right)' \widehat{\mathbf{v}}_2\right)$. To state the results, we need to introduce the following $\mathcal{T}$-dimensional random vector

$$\mathbf{e}_i = \left[\begin{array}{cccccc} \mathbf{e}_{1,i}' & \mathbf{e}_{2,i}' & \mathbf{e}_{3,i}' & \mathbf{e}_{4,i}' & \mathbf{e}_{5,i}' & \mathbf{e}_{6,i}' \end{array}\right]', \tag{39}$$

where

$$\mathbf{e}_{1,i} = \mathbf{u}_{2,[i]}, \quad \mathbf{e}_{2,i} = \mathbf{u}_{2,[i]} \otimes \mathbf{u}_{1,[i]}, \quad \mathbf{e}_{3,i} = \mathbf{u}_{2,[i]} \otimes \boldsymbol{\beta}_i, \tag{40}$$

$$\mathbf{e}_{4,i} = \mathbf{u}_{2,[i]} \otimes \mathbf{c}_{f,i}, \quad \mathbf{e}_{5,i} = \mathbf{u}_{2,[i]} \otimes \mathbf{c}_{a,i}, \quad \mathbf{e}_{6,i} = \text{vec}\left(\mathbf{u}_{2,[i]}\mathbf{u}_{2,[i]}' - \mathbf{V}_2\right), \tag{41}$$

and $\mathcal{T} = \tau_2(\tau_1 + \tau_2 + K + J + 1)$. Assumptions 1(i), 1(ii), 2(ii), and 3 together imply that $\frac{1}{N}\sum_{i=1}^N \mathbf{e}_i \xrightarrow{p} \mathbf{0}_{\mathcal{T}}$, as $N \to \infty$. In order to obtain the asymptotic distributions of interest, we make the following mild assumption, where $\xrightarrow{d}$ denotes convergence in distribution, postulating that $\mathbf{e}_i$ satisfies a cross-sectional central limit theorem.

Assumption 4 *As $N \to \infty$, $\frac{1}{\sqrt{N}}\sum_{i=1}^N \mathbf{e}_i \xrightarrow{d} N(\mathbf{0}_{\mathcal{T}}, \mathbf{V}_e)$, where $\mathbf{V}_e$ is a symmetric and positive definite $\mathcal{T} \times \mathcal{T}$ matrix.*

2.4.1 Asymptotic distribution of the ex-post risk premia estimator

Note that equation (22) yields

$$\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}-\boldsymbol{\lambda}\right)=\left(\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\widehat{\boldsymbol{\Omega}}\right)^{-1}\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\left(\frac{1}{\sqrt{N}}\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2\right), \tag{42}$$

where $\widehat{\boldsymbol{\Omega}}=\frac{1}{N}\widehat{\mathbf{Z}}_1'\widehat{\mathbf{X}}_2$. It is shown in the proof of Theorem 1 that $\widehat{\boldsymbol{\Omega}}\xrightarrow{p}\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the full-rank $(K+L+1)\times(K+1)$ matrix defined in equation (95) in the Appendix. Letting

$$\widehat{\boldsymbol{\Psi}}_\lambda=\left(\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\widehat{\boldsymbol{\Omega}}\right)^{-1}\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}, \tag{43}$$

we have

$$\widehat{\boldsymbol{\Psi}}_\lambda\xrightarrow{p}\boldsymbol{\Psi}_\lambda, \tag{44}$$

where

$$\boldsymbol{\Psi}_\lambda=(\boldsymbol{\Omega}'\mathbf{W}\boldsymbol{\Omega})^{-1}\boldsymbol{\Omega}'\mathbf{W}. \tag{45}$$

Hence, to determine the asymptotic distribution of $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}$, it suffices to determine the asymptotic distribution of $\frac{1}{\sqrt{N}}\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2$. It turns out that $\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2=\boldsymbol{\Pi}_\lambda\sum_{i=1}^N\mathbf{e}_i$, where $\boldsymbol{\Pi}_\lambda$ is a suitable matrix (see equations (101) and (102) in the Appendix) and $\mathbf{e}_i$ is defined in (39). It follows that $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}-\boldsymbol{\lambda}\right)=\boldsymbol{\Psi}_\lambda\boldsymbol{\Pi}_\lambda\frac{1}{\sqrt{N}}\sum_{i=1}^N\mathbf{e}_i+o_p(1)$. The proof of the following theorem, which provides the asymptotic distribution of the estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}$, contains the details.

Theorem 6 *Under Assumptions 1, 2, and 4, as* $N\to\infty$, $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{IV}^{GMM}-\boldsymbol{\lambda}\right)\xrightarrow{d}N\left(\mathbf{0}_{K+1},\mathbf{V}_\lambda\right)$, *where*

$$\mathbf{V}_\lambda=\boldsymbol{\Psi}_\lambda\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda'\boldsymbol{\Psi}_\lambda', \tag{46}$$

$\boldsymbol{\Psi}_\lambda$ *is defined in equation (45),* $\boldsymbol{\Pi}_\lambda$ *is defined in equation (102) in the Appendix, and* $\mathbf{V}_e$ *is defined in Assumption 4.*

It follows from a standard argument, typically employed in a GMM context, that the optimal (most efficient) IV-GMM estimator is obtained when the weighting matrix is $\mathbf{W}^*=(\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda')^{-1}$, in which case we obtain $\mathbf{V}_\lambda=\left(\boldsymbol{\Omega}'\left(\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda'\right)^{-1}\boldsymbol{\Omega}\right)^{-1}$.[9] In the following subsection, we obtain an N-consistent estimator of $\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda'$, based on which an N-consistent estimator of $\mathbf{V}_\lambda$ is readily constructed using

[9] Note that $\mathbf{1}_{\tau_2}'\mathbf{g}_2=1$ which implies that $\mathbf{g}_2\neq\mathbf{0}_{\tau_2}$. Hence, it follows from equation (102) in the Appendix that $\boldsymbol{\Pi}_\lambda$ has full rank equal to $K+L+1$ and so $\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda'$ is invertible, given that $\mathbf{V}_e$ is positive definite according to Assumption 4.

equation (46). As expected, the optimal IV-GMM estimator $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$ is at least as efficient as the IV estimator $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}$. We formally establish this property in the Online Appendix.

2.4.2 Asymptotic distribution of the characteristics rewards estimator

We next turn to the asymptotic distribution of the characteristics rewards estimator $\widehat{\boldsymbol{\phi}}$ defined in equation (27). Under the correct model specification, $\widehat{\boldsymbol{\phi}} \xrightarrow{p} \boldsymbol{\phi} = \mathbf{0}_J$, as stated in Proposition 2. Combining equations (28), (42), and (43) yields

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = -\left(\mathbf{C}'\mathbf{C}/N\right)^{-1}\left[\left(\mathbf{C}'\mathbf{X}/N\right)\widehat{\boldsymbol{\Psi}}_{\lambda}\left(\frac{\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2}{\sqrt{N}}\right) - \left(\frac{\mathbf{C}'\mathbf{U}_2}{\sqrt{N}}\right)\widehat{\mathbf{g}}_2\right]. \tag{47}$$

It follows from Assumption 4 and equations (29) and (18) that $\left(\frac{\mathbf{C}'\mathbf{U}_2}{\sqrt{N}}\right)\widehat{\mathbf{g}}_2 = \frac{\mathbf{C}'\boldsymbol{\omega}_2}{\sqrt{N}} + o_p(1)$. Hence, recalling that $\widehat{\mathbf{Z}}_1 = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_1 \quad \mathbf{C}_f\ ]$ and $\mathbf{C} = [\ \mathbf{C}_f \quad \mathbf{C}_a\ ]$, we obtain that

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = \boldsymbol{\Psi}_{\phi}\left(\frac{\widehat{\boldsymbol{\mathcal{Z}}}_1'\boldsymbol{\omega}_2}{\sqrt{N}}\right) + o_p(1), \tag{48}$$

where

$$\widehat{\boldsymbol{\mathcal{Z}}}_1 = [\ \widehat{\mathbf{Z}}_1 \quad \mathbf{C}_a\ ] = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_1 \quad \mathbf{C}\ ], \tag{49}$$

and $\boldsymbol{\Psi}_{\phi}$ is a suitable matrix (see equation (109) in the Appendix). Moreover, it turns out that $\widehat{\boldsymbol{\mathcal{Z}}}_1'\boldsymbol{\omega}_2 = \boldsymbol{\Pi}_{\phi}\sum_{i=1}^{N}\mathbf{e}_i$, where $\boldsymbol{\Pi}_{\phi}$ is given in (112) in the Appendix. It follows that $\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = \boldsymbol{\Psi}_{\phi}\boldsymbol{\Pi}_{\phi}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{e}_i + o_p(1)$. The proof of the following theorem, which provides the asymptotic distribution of the characteristics rewards estimator $\widehat{\boldsymbol{\phi}}$, contains the details.

Theorem 7 *Under Assumptions 1, 2, and 4, as $N \to \infty$, we have*

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) \xrightarrow{d} N\left(\mathbf{0}_J, \mathbf{V}_{\phi}\right), \tag{50}$$

where

$$\mathbf{V}_{\phi} = \boldsymbol{\Psi}_{\phi}\boldsymbol{\Pi}_{\phi}\mathbf{V}_e\boldsymbol{\Pi}_{\phi}'\boldsymbol{\Psi}_{\phi}', \tag{51}$$

$\boldsymbol{\Psi}_{\phi}$ *and* $\boldsymbol{\Pi}_{\phi}$ *are defined by* (109) *and* (112) *in the Appendix, respectively, and* $\mathbf{V}_e$ *is defined in Assumption 4.*

2.4.3 Asymptotic distribution of the average squared pricing error

Next we provide the asymptotic distribution of the centered version of the aggregate mispricing metric $\widehat{\mathcal{Q}}$. Recall equation (31) which states that

$$\widehat{\mathcal{Q}} = \widehat{\mathbf{g}}_2' \left(\mathbf{U}_2'\mathbf{U}_2/N\right) \widehat{\mathbf{g}}_2 - 2\widehat{\mathbf{g}}_2' \left(\mathbf{U}_2'\mathbf{X}/N\right) (\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}) + (\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda})'(\mathbf{X}'\mathbf{X}/N)(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}).$$

Assumption 1 implies that $\mathbf{U}_2'\mathbf{X}/N \xrightarrow{p} \mathbf{0}_{r_2\times(K+1)}$ and $\mathbf{X}'\mathbf{X}/N$ converges to a finite matrix while, according to Theorem 1, $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)$ converges to a normal distribution, as $N \to \infty$. Hence, using fact (F2) and the probability limit in (29), we obtain

$$\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right) = (\mathbf{g}_2 \otimes \mathbf{g}_2)' \sqrt{N}\left(\text{vec}\left(\mathbf{U}_2'\mathbf{U}_2/N\right) - \widehat{\mathbf{v}}_2\right) + o_p(1). \tag{52}$$

Then, it can be shown that $\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right) = \boldsymbol{\pi}_\alpha' \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1)$, where $\boldsymbol{\pi}_\alpha$ is a suitable vector (see equation (117) in the Appendix). The proof of the following theorem, which provides the asymptotic distribution of the centered version of $\widehat{\mathcal{Q}}$, contains the details.

Theorem 8 *Let Assumptions 1, 2, 3, and 4 be in effect. Then, the asymptotic distribution of the centered version of* $\widehat{\mathcal{Q}} = \frac{1}{N}\widehat{\boldsymbol{\alpha}}'\widehat{\boldsymbol{\alpha}}$*, as* $N \to \infty$*, is given by*

$$\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right) \xrightarrow{d} N\left(0, v_\alpha\right), \tag{53}$$

where $\widehat{\mathbf{v}}_2$ *is defined in (37),*

$$v_\alpha = \boldsymbol{\pi}_\alpha' \mathbf{V}_e \boldsymbol{\pi}_\alpha, \tag{54}$$

$\boldsymbol{\pi}_\alpha$ *is defined in (117) in the Appendix, and* $\mathbf{V}_e$ *is defined in Assumption 4.*

According to Theorem 8, $\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right)$ converges to a zero-mean normal distribution with variance v_α. Hence, one could use $\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right)/\widehat{v}_\alpha$ as a statistic for testing for the magnitude of aggregate mispricing, where $\widehat{v}_\alpha$ is a suitable estimator of v_α. Building on the results of the next subsection, we construct such a statistic in subsection 2.6.

2.4.4 Joint asymptotic distribution of the ex-post risk premia estimator, the characteristics rewards estimator, and the average squared pricing error

According to Theorems 6, 7, and 8, the three test statistics $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)$, $\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)$, and $\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\right)$ asymptotically follow normal distributions. The first statistic can be used to test ex-post risk premia implications while the last two can be used to test mispricing captured by the ex-post alpha estimates. We next combine the three statistics to obtain a comprehensive statistic that can be used to test all model implications simultaneously. To this end, we define the $(K+J+2) \times 1$ vector

$$\widehat{\boldsymbol{\delta}} = \begin{bmatrix} \widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda} \\ \widehat{\boldsymbol{\phi}} - \boldsymbol{\phi} \\ \widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2 \end{bmatrix}. \tag{55}$$

It follows from the proofs of Theorems 6, 7, and 8 that $\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) = \boldsymbol{\Psi}_\lambda \boldsymbol{\Pi}_\lambda \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1)$, $\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = \boldsymbol{\Psi}_\phi \boldsymbol{\Pi}_\phi \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1)$, and $\sqrt{N}\left(\widehat{\mathcal{Q}} - (\mathbf{g}_2 \otimes \mathbf{g}_2)' \widehat{\mathbf{v}}_2\right) = \boldsymbol{\pi}_\alpha' \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1)$, where the matrices $\boldsymbol{\Psi}_\lambda$, $\boldsymbol{\Pi}_\lambda$, $\boldsymbol{\Psi}_\phi$, and $\boldsymbol{\Pi}_\phi$ are defined in equations (45), (102), (109), and (112), respectively, while the vector $\boldsymbol{\pi}_\alpha$ is defined in equation (117). Note that the $J \times (K+J+1)$ matrix $\boldsymbol{\Psi}_\phi$ can be partitioned as $\boldsymbol{\Psi}_\phi = [\ \boldsymbol{\Psi}_{\phi,1} \quad \boldsymbol{\Psi}_{\phi,2}\ ]$, where $\boldsymbol{\Psi}_{\phi,1}$ and $\boldsymbol{\Psi}_{\phi,2}$ are matrices of dimensions $J \times (K+L+1)$ and $J \times (J-L)$. Moreover, note that equation (112) implies the relationship $\boldsymbol{\Pi}_\lambda = \boldsymbol{\Pi}_{\phi,1}$. Then, combining the above expressions, we obtain that

$$\sqrt{N}\widehat{\boldsymbol{\delta}} = \boldsymbol{\Psi}_\delta \boldsymbol{\Pi}_\delta \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1), \tag{56}$$

where the $(K+J+2) \times (K+J+2)$ matrix $\boldsymbol{\Psi}_\delta$ and the $(K+J+2) \times \mathcal{T}$ matrix $\boldsymbol{\Pi}_\delta$ are given by

$$\boldsymbol{\Psi}_\delta = \begin{bmatrix} \boldsymbol{\Psi}_\lambda & \mathbf{0}_{(K+1)\times(J-L)} & \mathbf{0}_{K+1} \\ \boldsymbol{\Psi}_{\phi,1} & \boldsymbol{\Psi}_{\phi,2} & \mathbf{0}_J \\ \mathbf{0}'_{K+L+1} & \mathbf{0}'_{J-L} & 1 \end{bmatrix}, \tag{57}$$

and

$$\boldsymbol{\Pi}_\delta = \begin{bmatrix} \boldsymbol{\Pi}_\phi \\ \boldsymbol{\pi}_\alpha' \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}_{\phi,1} \\ \boldsymbol{\Pi}_{\phi,2} \\ \boldsymbol{\pi}_\alpha' \end{bmatrix}. \tag{58}$$

The following theorem summarizes the above results and provides the asymptotic distribution of the joint statistic $\widehat{\boldsymbol{\delta}}$.

Theorem 9 *Let Assumptions 1, 2, 3, and 4 be in effect. Then, the asymptotic distribution of* $\widehat{\boldsymbol{\delta}} = \left[\ \left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)' \quad \left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)' \quad \widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)' \widehat{\mathbf{v}}_2\ \right]'$ *is given by*

$$\sqrt{N}\widehat{\boldsymbol{\delta}} \xrightarrow{d} N(\mathbf{0}_{K+J+2}, \mathbf{V}_\delta), \tag{59}$$

where

$$\mathbf{V}_{\delta} = \mathbf{\Psi}_{\delta}\mathbf{\Pi}_{\delta}\mathbf{V}_{e}\mathbf{\Pi}_{\delta}'\mathbf{\Psi}_{\delta}', \tag{60}$$

$\mathbf{\Psi}_{\delta}$ and $\mathbf{\Pi}_{\delta}$ are defined in equations (57) and (58), respectively, and $\mathbf{V}_e$ is defined in Assumption 4.

To make operational the asymptotic distribution obtained in Theorem 9, and obtain feasible test statistics, we need to obtain a consistent estimator of $\mathbf{V}_{\delta}$. This is the subject of the next subsection.

2.5 Estimation of the asymptotic variance-covariance matrix $\mathbf{V}_{\delta}$

According to equation (60), the variance-covariance matrix $\mathbf{V}_{\delta}$ involves the matrix $\mathbf{V}_e$ which, according to Assumption 4, is limiting variance-covariance matrix of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{e}_i$. Hence, the structure of $\mathbf{V}_{\delta}$ depends on the structure of $\mathbf{V}_e$ which, in turn, depends on potential cross-sectional correlations of the shocks $\mathbf{e}_i$. Note that in the return generating process described by (2), the disturbance vector $\mathbf{u}_t$ could potentially exhibit cross-sectional correlation due to economic links such as industry effects. In that case, the vectors $\mathbf{e}_i$ would be correlated across firms as it follows from definition (39). To incorporate such correlations, we use a clustering approach that we describe next.[10]

We assume that there are M_N clusters and that the m-th cluster consists of N_m stocks, for $m = 1, \ldots, M_N$, so that $\sum_{m=1}^{M_N} N_m = N$. For all N, we assume that the cluster sizes N_m, $m = 1, \ldots, M_N$ are bounded. As $N \to \infty$, the number of clusters, M_N, is assumed to increase so that $\frac{N}{M_N} \to G$, where G is to be interpreted as the limiting average cluster size. For $m = 1, \ldots, M_N$, let I_m be the set of all indices i for which the i-th stock belongs to the m-th cluster, and define the aggregate cluster shocks

$$\boldsymbol{\eta}_m = \sum\nolimits_{i \in I_m} \mathbf{e}_i. \tag{61}$$

In the next assumption, we postulate that the central limit theorem applies to the random sequence $\boldsymbol{\eta}_m$, $m = 1, 2, \ldots$

Assumption 5 *The aggregate cluster shocks $\boldsymbol{\eta}_m$ are independent across clusters and, as $N \to \infty$, $\frac{1}{\sqrt{M_N}}\sum_{m=1}^{M_N}\boldsymbol{\eta}_m \xrightarrow{d} N\left(\mathbf{0}_{\mathcal{T}}, \mathbf{V}_{\eta}\right)$, where*

$$\mathbf{V}_{\eta} = p\text{-}\lim \frac{1}{M_N}\sum_{m=1}^{M_N}\boldsymbol{\eta}_m\boldsymbol{\eta}_m'. \tag{62}$$

[10] Our empirical applications, following standard economic intuition, we use an industry classification to determine the clusters. In addition, for robustness purposes, we consider clusters based on firm characteristics such as size and book-to-market ratio.

Utilizing Assumption 5, we obtain

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{e}_i = \sqrt{\frac{M_N}{N}}\frac{1}{\sqrt{M_N}}\sum_{m=1}^{M_N}\boldsymbol{\eta}_m \xrightarrow{d} N\left(\mathbf{0}_{\mathcal{T}}, \mathbf{V}_\eta/G\right),$$

and so it follows that $\mathbf{V}_e = \frac{1}{G}\mathbf{V}_\eta$. Equation (60) then yields $\mathbf{V}_\delta = \frac{1}{G}\boldsymbol{\Psi}_\delta\left(\boldsymbol{\Pi}_\delta\mathbf{V}_\eta\boldsymbol{\Pi}_\delta'\right)\boldsymbol{\Psi}_\delta'$. In light of (57), to estimate $\boldsymbol{\Psi}_\delta$, we need to estimate $\boldsymbol{\Psi}_\lambda$ and $\boldsymbol{\Psi}_\phi$, defined in (45) and (109), respectively. According to (43), $\widehat{\boldsymbol{\Psi}}_\lambda$ is a consistent estimator of $\boldsymbol{\Psi}_\lambda$. Moreover, as suggested by (109), $\boldsymbol{\Psi}_\phi$ is consistently estimated by

$$\widehat{\boldsymbol{\Psi}}_\phi = -\left(\mathbf{C}'\mathbf{C}/N\right)^{-1}\left((\mathbf{C}'\widehat{\mathbf{X}}_2/N)\widehat{\boldsymbol{\Psi}}_\lambda\boldsymbol{\mathcal{I}}_1 - \boldsymbol{\mathcal{I}}_2\right), \tag{63}$$

where $\boldsymbol{\mathcal{I}}_1$ and $\boldsymbol{\mathcal{I}}_2$ are defined in (107). The matrix $\widehat{\boldsymbol{\Psi}}_\phi$ can be partitioned as $\widehat{\boldsymbol{\Psi}}_\phi = \left[\begin{array}{cc}\widehat{\boldsymbol{\Psi}}_{\phi,1} & \widehat{\boldsymbol{\Psi}}_{\phi,2}\end{array}\right]$, where $\widehat{\boldsymbol{\Psi}}_{\phi,1}$ and $\widehat{\boldsymbol{\Psi}}_{\phi,2}$ are matrices of dimensions $J\times(K+L+1)$ and $J\times(J-L)$. Hence, it follows from (57) that $\boldsymbol{\Psi}_\lambda$ is consistently estimated by

$$\widehat{\boldsymbol{\Psi}}_\delta = \left[\begin{array}{ccc}\widehat{\boldsymbol{\Psi}}_\lambda & \mathbf{0}_{(K+1)\times(J-L)} & \mathbf{0}_{K+1}\\ \widehat{\boldsymbol{\Psi}}_{\phi,1} & \widehat{\boldsymbol{\Psi}}_{\phi,2} & \mathbf{0}_J\\ \mathbf{0}_{K+L+1}' & \mathbf{0}_{J-L}' & 1\end{array}\right]. \tag{64}$$

Therefore, to estimate the asymptotic variance-covariance matrix $\mathbf{V}_\delta$, it suffices to obtain an estimator of the $(K+J+2)\times(K+J+2)$ matrix $\boldsymbol{\Theta}$ defined by

$$\boldsymbol{\Theta} = \boldsymbol{\Pi}_\delta\mathbf{V}_\eta\boldsymbol{\Pi}_\delta'. \tag{65}$$

To construct such an estimator, we need to introduce some additional notation. Define the cluster selection $M_N\times N$ matrix $\boldsymbol{\mathcal{C}}$ with (m,i) element given by

$$\boldsymbol{\mathcal{C}}(m,i) = \mathbf{1}_{[i\in I_m]}, \quad m=1,\ldots,M_N, \quad i=1,\ldots,N, \tag{66}$$

the $N\times\mathcal{T}$ matrix of $\mathbf{E}$, stacking the firm shocks $\mathbf{e}_i$ in (39), by

$$\mathbf{E} = \left[\begin{array}{ccc}\mathbf{e}_1 & \cdots & \mathbf{e}_N\end{array}\right]', \tag{67}$$

and the $M_N\times\mathcal{T}$ matrix of $\boldsymbol{\mathcal{H}}$, stacking the cluster aggregate shocks $\boldsymbol{\eta}_m$ in (61), by

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{ccc}\boldsymbol{\eta}_1 & \cdots & \boldsymbol{\eta}_{M_N}\end{array}\right]', \tag{68}$$

so that

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{C}}\mathbf{E}. \tag{69}$$

For any $N \times 1$ vector $\mathbf{x} = (x_1, \ldots, x_N)'$, let diag$(\mathbf{x})$ be the $N \times N$ diagonal matrix with (i,i) element equal to x_i, for $i = 1, \ldots, N$. Furthermore, let $\widehat{\boldsymbol{z}}'_{1,i}$ denote the i-th row of $\widehat{\boldsymbol{\mathcal{Z}}}_1$ defined in (49) and $\omega_{2,i}$ denote the i-th element of $\boldsymbol{\omega}_2$ defined in (17). It follows from the proof of Theorem 7 (see equation (111) in the Appendix) that, for $i = 1, \ldots, N$, $\widehat{\boldsymbol{z}}_{1,i}\omega_{2,i}$ equals $\mathbf{\Pi}_\phi \mathbf{e}_i$, which implies that

$$\text{diag}(\boldsymbol{\omega}_2)\widehat{\boldsymbol{\mathcal{Z}}}_1 = \begin{bmatrix} \widehat{\boldsymbol{z}}_{1,1}\omega_{2,1} & \cdots & \widehat{\boldsymbol{z}}_{1,N}\omega_{2,N} \end{bmatrix}' = \mathbf{E}\mathbf{\Pi}'_\phi. \tag{70}$$

Moreover, note that equation (18) yields $\omega_{2,i} = \mathbf{u}'_{2,[i]}\mathbf{g}_2$ and so, using fact (F2) stated in the Appendix, we obtain $\omega^2_{2,i} = (\mathbf{g}_2 \otimes \mathbf{g}_2)' \, \text{vec}\left(\mathbf{u}_{2,[i]}\mathbf{u}'_{2,[i]}\right)$. In addition, equation (35) implies $\widehat{\mathbf{u}}_{2,[i]} = \mathbf{H}_2\mathbf{u}_{2,[i]}$ and so, using fact (F2) again, we obtain

$$\text{vec}\left(\widehat{\mathbf{u}}_{2,[i]}\widehat{\mathbf{u}}'_{2,[i]}\right) = (\mathbf{H}_2 \otimes \mathbf{H}_2)\, \text{vec}\left(\mathbf{u}_{2,[i]}\mathbf{u}'_{2,[i]}\right). \tag{71}$$

Therefore,

$$\omega^2_{2,i} - (\mathbf{g}_2 \otimes \mathbf{g}_2)' \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1} \boldsymbol{\mathcal{S}}'\right] \text{vec}\left(\widehat{\mathbf{u}}_{2,[i]}\widehat{\mathbf{u}}'_{2,[i]}\right) = \boldsymbol{\pi}'_\alpha \mathbf{e}_i, \tag{72}$$

where $\mathbf{e}_i$ is defined in (39) and $\boldsymbol{\pi}_\alpha$ is defined in (117) in the Appendix. It follows that

$$\boldsymbol{\omega}_2 \odot \boldsymbol{\omega}_2 - \widehat{\boldsymbol{\mathcal{U}}}_2 \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1} \boldsymbol{\mathcal{S}}'\right] (\mathbf{g}_2 \otimes \mathbf{g}_2) = \mathbf{E}\boldsymbol{\pi}_\alpha, \tag{73}$$

where $\widehat{\boldsymbol{\mathcal{U}}}_2$ is $N \times \tau_2^2$ matrix given by

$$\widehat{\boldsymbol{\mathcal{U}}}'_2 = \begin{bmatrix} \text{vec}\left(\widehat{\mathbf{u}}_{2,[1]}\widehat{\mathbf{u}}'_{2,[1]}\right) & \cdots & \text{vec}\left(\widehat{\mathbf{u}}_{2,[N]}\widehat{\mathbf{u}}'_{2,[N]}\right) \end{bmatrix}. \tag{74}$$

Aggregating (70) and (73), in light of (58), we obtain

$$\widetilde{\boldsymbol{\mathcal{E}}} \equiv \begin{bmatrix} \text{diag}\left(\boldsymbol{\omega}_2\right)\widehat{\boldsymbol{\mathcal{Z}}}_1 & \boldsymbol{\omega}_2 \odot \boldsymbol{\omega}_2 - \widehat{\boldsymbol{\mathcal{U}}}_2 \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1} \boldsymbol{\mathcal{S}}'\right] (\mathbf{g}_2 \otimes \mathbf{g}_2) \end{bmatrix} = \mathbf{E}\mathbf{\Pi}'_\delta. \tag{75}$$

Let $\widetilde{\varepsilon}'_i$ denote the i-th row of the matrix $\widetilde{\boldsymbol{\mathcal{E}}}$ so that

$$\widetilde{\boldsymbol{\mathcal{E}}} = \begin{bmatrix} \widetilde{\varepsilon}_1 & \cdots & \widetilde{\varepsilon}_N \end{bmatrix}'. \tag{76}$$

For each cluster $m = 1, \ldots, M_N$, we define

$$\widetilde{\boldsymbol{\theta}}_m = \sum_{i \in I_m} \widetilde{\boldsymbol{\varepsilon}}_i, \tag{77}$$

so that $\boldsymbol{\mathcal{C}}\widetilde{\boldsymbol{\mathcal{E}}} = \left[\begin{array}{ccc} \widetilde{\boldsymbol{\theta}}_1 & \cdots & \widetilde{\boldsymbol{\theta}}_{M_N} \end{array} \right]'$. Note that, since $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{C}}\mathbf{E}$, we have

$$\widetilde{\boldsymbol{\theta}}_m = \boldsymbol{\Pi}_\delta \boldsymbol{\eta}_m. \tag{78}$$

It follows that one could consistently estimate $\boldsymbol{\Theta}$ by

$$\widetilde{\boldsymbol{\Theta}} = \frac{1}{M_N}\left(\widetilde{\boldsymbol{\mathcal{E}}}'\boldsymbol{\mathcal{C}}'\boldsymbol{\mathcal{C}}\widetilde{\boldsymbol{\mathcal{E}}}\right). \tag{79}$$

Indeed, as $N \to \infty$,

$$\widetilde{\boldsymbol{\Theta}} = \frac{1}{M_N}\sum_{m=1}^{M_N} \widetilde{\boldsymbol{\theta}}_m \widetilde{\boldsymbol{\theta}}_m' = \boldsymbol{\Pi}_\delta \left(\frac{1}{M_N}\sum_{m=1}^{M_N} \boldsymbol{\eta}_m \boldsymbol{\eta}_m'\right) \boldsymbol{\Pi}_\delta' \xrightarrow{p} \boldsymbol{\Pi}_\delta \mathbf{V}_\eta \boldsymbol{\Pi}_\delta' = \boldsymbol{\Theta},$$

where, in the last step, we make use of definition (62) in Assumption 5. Note, however, that the matrix $\widetilde{\boldsymbol{\mathcal{E}}}$, and hence $\widetilde{\boldsymbol{\Theta}}$, depends on $\boldsymbol{\omega}_2$ which can be thought of as the vector of residuals in the second-pass cross-sectional regression obtained after imposing the null hypothesis. Instead of using $\boldsymbol{\omega}_2$, we proceed in the traditional fashion and define residuals without imposing the null hypothesis as follows

$$\widehat{\boldsymbol{\omega}}_2 = \bar{\mathbf{r}}_2 - \widehat{\mathbf{X}}_2 \widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}. \tag{80}$$

Replacing $\boldsymbol{\omega}_2$ by $\widehat{\boldsymbol{\omega}}_2$ in (75) and incorporating the standard degrees-of-freedom adjustment,[11] we propose estimating $\boldsymbol{\Theta}$ by

$$\widehat{\boldsymbol{\Theta}} \;\; = \;\; \frac{1}{M_N - K - 1}\widehat{\boldsymbol{\mathcal{E}}}'\boldsymbol{\mathcal{C}}'\boldsymbol{\mathcal{C}}\widehat{\boldsymbol{\mathcal{E}}}, \tag{81}$$

where $\widehat{\boldsymbol{\mathcal{E}}}$ is the $N \times (K + J + 2)$ matrix defined by

$$\widehat{\boldsymbol{\mathcal{E}}} \equiv \left[\begin{array}{cc} \mathrm{diag}\left(\widehat{\boldsymbol{\omega}}_2\right)\widehat{\boldsymbol{\mathcal{Z}}}_1 & \widehat{\boldsymbol{\omega}}_2 \odot \widehat{\boldsymbol{\omega}}_2 - \widehat{\boldsymbol{\mathcal{U}}}_2 \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2\right) \end{array} \right]. \tag{82}$$

Next, we provide a mild regularity condition under which $\widehat{\boldsymbol{\Theta}}$ is indeed an N-consistent estimator of $\boldsymbol{\Theta}$. The following theorem provides the desired consistent estimator of $\mathbf{V}_\delta$ based on which we can

[11] The degrees-of-freedom adjustment does not affect the asymptotic properties of $\widehat{\boldsymbol{\Theta}}$ but, following standard econometric practice, we use it to improve the finite sample behavior of the estimator.

construct feasible test statistics. To this end, for each $i = 1, \ldots, N$, we define the vector $\boldsymbol{\zeta}_i$ by

$$\boldsymbol{\zeta}_i = \left[\begin{array}{cc} \mathbf{e}_i' & \mathbf{h}_i' \end{array} \right]', \tag{83}$$

where $\mathbf{e}_i$ is defined by (39) and

$$\mathbf{h}_i = \left[1 \quad \mathbf{u}_{1,[i]}' \quad \boldsymbol{\beta}_i' \quad \mathbf{c}_{1,i}' \quad (\boldsymbol{\beta}_i \otimes \boldsymbol{\beta}_i)' \quad (\boldsymbol{\beta}_i \otimes \mathbf{c}_{1,i})' \quad (\boldsymbol{\beta}_i \otimes \mathbf{u}_{1,[i]})' \quad (\boldsymbol{\beta}_i \otimes \mathbf{u}_{2,[i]})' \quad (\mathbf{u}_{2,[i]} \otimes \mathbf{u}_{2,[i]})'\right]'. \tag{84}$$

Then, for each cluster $m = 1, \ldots, M_N$, we define

$$\boldsymbol{\varphi}_m = \sum\nolimits_{i \in I_m} \boldsymbol{\zeta}_i, \tag{85}$$

and make the following assumption which amounts to the existence of cross-sectional second moments of $\boldsymbol{\varphi}_m$.

Assumption 6 *As $N \to \infty$, $\frac{1}{M_N} \sum_{m=1}^{M_N} \boldsymbol{\varphi}_m \boldsymbol{\varphi}_m'$ converges in probability to a finite matrix.*

The following theorem summarizes the preceding discussion and provides an N-consistent estimator of the asymptotic variance-covariance matrix $\mathbf{V}_\delta$.

Theorem 10 *Under Assumptions 1-6, as $N \to \infty$,*

$$\widehat{\mathbf{V}}_\delta = \frac{M_N}{N} \widehat{\boldsymbol{\Psi}}_\delta \widehat{\boldsymbol{\Theta}} \widehat{\boldsymbol{\Psi}}_\delta' \xrightarrow{p} \mathbf{V}_\delta,$$

where the matrices $\widehat{\boldsymbol{\Psi}}_\delta$ and $\widehat{\boldsymbol{\Theta}}$ are defined in (64) *and* (81)*, respectively.*

In the next subsection, we put together the results obtained in the last two subsections and derive novel test statistics that can be used to test the ex-post risk premia and aggregate mispricing implications of the asset pricing model under consideration.

2.6 Test statistics

Combining Theorems 9 and 10, we can readily obtain statistics for testing the various implications of the asset pricing model. Formally, the null hypothesis we test is that the asset pricing model is correctly specified.

First, to examine the ex-post risk premia implications of the model, we propose using the quadratic form $J(\boldsymbol{\lambda}) = N(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda})' \widehat{\mathbf{V}}_\lambda^{-1} (\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda})$, where $\widehat{\mathbf{V}}_\lambda$ is the $(K+1) \times (K+1)$ upper-left block of the matrix $\widehat{\mathbf{V}}_\delta$. The statistic $J(\boldsymbol{\lambda})$ asymptotically follows a χ^2 distribution with $K+1$ degrees of freedom under the null hypothesis. Focusing on the cross-sectional intercept, which should be equal to zero under the null hypothesis, we can use the t statistic

$$t(\lambda_0) = \frac{\widehat{\lambda}_0}{\sqrt{\widehat{v}_{\lambda,0}/N}}, \tag{86}$$

where $\widehat{\lambda}_0$ denotes the first element of $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$ and $\widehat{v}_{\lambda,0}$ denotes the $(1,1)$ element of the matrix $\widehat{\mathbf{V}}_\lambda$. The t statistic $t(\lambda_0)$ asymptotically follows a standard normal distribution. Focusing on the k-th factor ex-post risk premium, for $k = 1, \ldots, K$, which should be equal to the k-th factor's average realization, we can use the t statistic

$$t(\lambda_k) = \frac{\widehat{\lambda}_k - \overline{f}_{k,2}}{\sqrt{\widehat{v}_{\lambda,k}/N}}, \tag{87}$$

where $\widehat{\lambda}_k$ denotes the $(k+1)$-th element of $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$ and $\widehat{v}_{\lambda,k}$ denotes the $(k+1, k+1)$ element of the matrix $\widehat{\mathbf{V}}_\lambda$. The asymptotic distribution of t statistic $t(\lambda_k)$ is also standard normal.

Second, to gauge the importance of characteristics in explaining mispricing at the individual stock level as captured by the characteristics rewards estimator $\widehat{\boldsymbol{\phi}}$, we propose using the quadratic form $J(\boldsymbol{\phi}) = N\widehat{\boldsymbol{\phi}}'\widehat{\mathbf{V}}_\phi^{-1}\widehat{\boldsymbol{\phi}}$, where $\widehat{\mathbf{V}}_\phi$ is the $J \times J$ matrix with (j_1, j_2) element equal to the $(K+1+j_1, K+1+j_2)$ element of the matrix $\widehat{\mathbf{V}}_\delta$, with j_1 and j_2 ranging from 1 to J. The statistic $J(\boldsymbol{\phi})$ asymptotically follows a χ^2 distribution with J degrees of freedom under the null hypothesis. To examine the importance of a particular characteristic, we can use the t statistic

$$t(\phi_j) = \frac{\widehat{\phi}_j}{\sqrt{\widehat{v}_{\phi,j}/N}}, \tag{88}$$

where $\widehat{\phi}_j$ denotes the j-th element of $\widehat{\boldsymbol{\phi}}$ and $\widehat{v}_{\phi,j}$ denotes the (j,j) element of the matrix $\widehat{\mathbf{V}}_\phi$, for $j = 1, \ldots, J$. The t statistic $t(\phi_j)$ asymptotically follows a standard normal distribution.

Third, focusing on the aggregate mispricing implications of the model, we propose using the t statistic

$$t(\boldsymbol{\alpha}) = \frac{\widehat{\mathcal{Q}} - (\mathbf{g}_2 \otimes \mathbf{g}_2)'\widehat{\mathbf{v}}_2}{\sqrt{\widehat{v}_{\delta,K+J+2}/N}}, \tag{89}$$

where $\widehat{v}_{\delta,K+J+2}$ is the $(K+J+2, K+J+2)$ element of the matrix $\widehat{\mathbf{V}}_\delta$. It follows from Theorems 8 and 10 that the test statistic $t(\boldsymbol{\alpha})$ asymptotically follows a standard normal distribution under the null hypothesis.

Both ex-post risk premia and individual stock alpha implications can be tested jointly using the quadratic form $J(\boldsymbol{\delta}) = N\widehat{\boldsymbol{\delta}}'\widehat{\mathbf{V}}_\delta^{-1}\widehat{\boldsymbol{\delta}}$ which asymptotically follows a χ^2 distribution with $K+J+2$ degrees of freedom under the null hypothesis of correct model specification. However, simulation evidence suggests that the joint test statistics $J(\boldsymbol{\delta})$ typically overreject the null hypothesis exhibiting poor performance for empirically relevant finite sample sizes. It appears that the reason is that the variance-covariance matrix estimator $\widehat{\mathbf{V}}_\delta$ can be ill-conditioned in small samples. Motivated by this observation, we propose using the following quadratic form

$$J_d(\boldsymbol{\delta}) = N\left(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right)' \widehat{\mathbf{D}}_\delta^{-1}\left(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right), \tag{90}$$

where $\widehat{\mathbf{D}}_\delta$ is the $(K+J+2) \times (K+J+2)$ diagonal matrix consisting of the diagonal elements of $\widehat{\mathbf{V}}_\delta$. While the test statistic $J_d(\boldsymbol{\delta})$ does not asymptotically follow a standard distribution, such as χ^2, under the null hypothesis, one can easily compute p-values associated with $J_d(\boldsymbol{\delta})$ using simulation. Let $\mathbf{Q}_\delta$ be the Cholesky factor of $\mathbf{V}_\delta$ so that $\mathbf{V}_\delta = \mathbf{Q}_\delta\mathbf{Q}_\delta'$. Then, the asymptotic distribution of $J_d(\boldsymbol{\delta})$ is the same as the distribution of the quadratic form $\zeta = \boldsymbol{\zeta}'\left[\mathbf{Q}_\delta'\mathbf{D}_\delta^{-1}\mathbf{Q}_\delta\right]\boldsymbol{\zeta}$, where $\mathbf{D}_\delta$ is the $(K+J+2) \times (K+J+2)$ diagonal matrix with (j,j) element equal to the (j,j) element of $\mathbf{V}_\lambda$, for $j = 1, \ldots, K+J+2$, and $\boldsymbol{\zeta}$ follows a $(K+J+2)$-dimensional standard normal distribution. The matrix $\mathbf{P}_\delta = \mathbf{Q}_\delta'\mathbf{D}_\delta^{-1}\mathbf{Q}_\delta$ can be N-consistently estimated by $\widehat{\mathbf{P}}_\delta = \widehat{\mathbf{Q}}_\delta'\widehat{\mathbf{D}}_\delta^{-1}\widehat{\mathbf{Q}}_\delta$, where $\widehat{\mathbf{Q}}_\delta$ is the Cholesky factor of $\widehat{\mathbf{V}}_\delta$ so that $\widehat{\mathbf{V}}_\delta = \widehat{\mathbf{Q}}_\delta\widehat{\mathbf{Q}}_\delta'$. Let $\{\boldsymbol{\zeta}_i : i = 1, \ldots, I\}$ be a large sample of simulated draws from $N(\mathbf{0}_{K+J+2}, \mathbf{I}_{K+J+2})$ and define $\zeta_i = \boldsymbol{\zeta}_i'\widehat{\mathbf{P}}_\delta\boldsymbol{\zeta}_i$, $i = 1, \ldots, I$. It follows by the Monte Carlo principle that the distribution function of ζ, $F_\zeta(a) = \mathrm{P}[\zeta \le a]$, can be approximated by $\frac{1}{I}\sum_{i=1}^{I} 1_{[\zeta_i \le a]}$, with the approximation becoming better as N and I increase. In our simulation exercises and empirical tests, we use $I = 100,000$.

In the same spirit, we can modify $J(\boldsymbol{\lambda})$ and $J(\boldsymbol{\phi})$ by using diagonal weighting matrices. Specifically, we propose the quadratic forms

$$J_d(\boldsymbol{\lambda}) = N\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)' \widehat{\mathbf{D}}_\lambda^{-1}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right), \tag{91}$$

$$J_d(\boldsymbol{\phi}) = N\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)' \widehat{\mathbf{D}}_\phi^{-1}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right), \tag{92}$$

where $\widehat{\mathbf{D}}_\lambda$ and $\widehat{\mathbf{D}}_\phi$ are the $(K+1) \times (K+1)$ and $J \times J$ diagonal matrices consisting of the diagonal elements of $\widehat{\mathbf{V}}_\lambda$ and $\widehat{\mathbf{V}}_\phi$, respectively. We compute p-values for the $J_d(\boldsymbol{\lambda})$ and $J_d(\boldsymbol{\phi})$ test statistics by simulation, as described above.

When we use both past betas and characteristics as instruments, ex-post risk premia are estimated by the IV-GMM estimator (with $L \ge 1$). In this case, the test statistics described above could be used for inference for any generic weighting matrix $\mathbf{W}$. However, as discussed in Subsection 2.4.1, selecting $\mathbf{W}^* = (\boldsymbol{\Pi}_\lambda\mathbf{V}_e\boldsymbol{\Pi}_\lambda')^{-1}$ yields the efficient IV-GMM estimator. We follow standard GMM practice and obtain the two-step estimator as follows. First, we set $\widehat{\mathbf{W}} = \mathbf{I}_{K+L+1}$ to obtain an initial N-consistent estimator of $\boldsymbol{\lambda}$, say $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{I}}$. Then, using $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{I}}$, we obtain an N-consistent estimator of $\boldsymbol{\Theta}$ from (81) which then provides an N-consistent estimator of $\mathbf{W}^*$, say $\widehat{\mathbf{W}}^*$. Using $\widehat{\mathbf{W}}^*$ as a weighting matrix in the second step, we obtain the two-step IV-GMM estimator of $\boldsymbol{\lambda}$ which is efficient. In addition, we obtain the iterated IV-GMM estimator by repeating the above process of successively obtaining estimators of the risk premia and the asymptotic variance-covariance matrix, in an alternate fashion, till the sequence of risk premia estimators converges. In practice, we stop the iteration when the L_1-norm of the difference between two successive risk premia estimates becomes less than 10^{-6}. We denote the two-step and iterated IV-GMM estimators by $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$, respectively, and use both of them in our simulations and empirical applications.

3 Monte Carlo Simulation Evidence

In this section, we investigate the properties of the IV-GMM ex-post risk premia estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}}$ and the various tests on ex-post implications of asset pricing models for empirically relevant finite sample sizes through a number of Monte Carlo simulation experiments. We illustrate the importance of the EIV correction offered by our IV approach in terms of bias reduction and efficiency enhancement by comparing two versions of the IV-GMM ex-post risk premia estimator with two alternative estimators that do not use an EIV correction. We report the bias as well as the root mean squared error of all estimators under consideration. Furthermore, we investigate the finite sample performance of the various test statistics in terms of size and power properties. Finally, we illustrate that the traditional Fama and MacBeth (1973) approach for computing standard errors is not suitable in our small-T context as it leads to severe underrejection. As we explain below, we use three popular asset pricing models in our calibration.

Next, we provide the details of our simulation design. We consider all individual stocks in the CRSP universe, trading between 2005 and 2014, i.e., the last ten years of the sample in our empirical exercise presented in the next section, with price above one dollar and, among those, we select the 1,000 stocks with longest time series histories. We jointly calibrate the betas and the idiosyncratic shock variances of those 1,000 stocks in order to simulate excess return data according to the data generating process (2). Our simulation is based on the following three linear asset pricing models: the single-factor CAPM, the three-factor model of Fama and French (1993) and the four-factor model of Hou, Xue, and Zhang (2015). The factors in the second model, which we refer to as the FF3 model, are the market excess return (MKT), the small size minus big size spread portfolio return (SMB), and the high book-to-market minus low book-to-market spread portfolio return (HML).[12] The factors in the third model, which we refer to as the HXZ4 model, are the market excess return (MKT), the difference between the return on a portfolio of small size stocks and the return on a portfolio of big size stocks (ME), the difference between the return on a portfolio of low investment stocks and the return on a portfolio of high investment stocks (I/A), and the difference between the return on a portfolio of high profitability (return on equity) stocks and the return on a portfolio of low profitability stocks (ROE).[13] In the case of the CAPM, we do not use any characteristics as instruments, i.e., the matrix $\mathbf{C}_f$ is empty, and therefore the IV-GMM estimator becomes the standard exactly identified IV estimator, which we denote by $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$. We use the size and book-to-market ratio characteristics averaged over the 2005–2014 period as $\mathbf{C}_a$. For the FF3 and HXZ4 models, we use the firm characteristics associated with the factors of each model as instrumental variables. Specifically, for the FF3 model, the instrumental variables are the size and book-to-market ratio averaged over

[12] The data on the three factors of the Fama and French (1993) model are obtained from Kenneth French's data library.
[13] We thank the authors for providing the data on the four factors of the Hou, Xue, and Zhang (2015) model.

the 2005–2014 period. For the HXZ4 model, the instrumental variables are the size, investment over asset, and return on equity averaged over the 2005-2014 period.[14] To mitigate the effect of pronounced skewness and extreme outliers, we apply the rank-based inverse normal transformation to the cross-sectional distribution of each characteristic at each time period. Specifically, using the raw characteristics data $\gamma_1, \ldots, \gamma_N$, we (i) compute the corresponding ranks $\xi_1, \ldots, \xi_N$, (ii) convert them into quantiles $q_1, \ldots, q_N$, where $q_i = \frac{\xi_i - 0.5}{N}$, for $i = 1, \ldots, N$, and (iii) obtain the transformed values $c_1, \ldots, c_N$, given by $c_i = \Phi^{-1}(q_i)$, for $i = 1, \ldots, N$, where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Finally, for each characteristic, we demean the transformed values to ensure the cross-sectional zero-mean condition. For both the FF3 and HXZ4 models, we consider two efficient versions of the IV-GMM estimator, namely the two-step estimator $\widehat{\boldsymbol{\lambda}}_{\text{IV}}^{\text{TS}}$ and the iterated estimator $\widehat{\boldsymbol{\lambda}}_{\text{IV}}^{\text{IT}}$.

We pay particular attention to the following two aspects of the simulation design: (i) the number of clusters in the stock universe and (ii) the correlation structure among stock returns within clusters. Due to space limitations, we only consider clusters of equal size and assume that correlations within clusters are constant. In the first part of the simulation exercise, that focuses on the bias and the mean squared error of the IV-GMM estimators, we set the number of clusters, M_N, equal to 50 and the pairwise correlation ρ, within each cluster, equal to 0.10. In the second part of the simulation exercise, that focuses on the finite sample behavior of the various asset pricing test statistics, we let the number of clusters M_N take the values 50 and 100 and the within-cluster correlation take the values 0, 0.10, and 0.20. Note that in the empirical investigation of Section 4, we consider clustering based on the 49-industry classification of Kenneth French.[15] Following this classification, we estimate an average correlation within industries around 0.10 based on an industry residual model for the shocks, in the spirit of Ang, Liu, and Schwarz (2010) (see their Appendix F.2). Hence, the range of correlation values that we employ in our simulation is empirically relevant.

First, we illustrate the importance of the IV-GMM approach in dealing with the EIV problem by comparing the two IV-GMM estimators, $\widehat{\boldsymbol{\lambda}}_{\text{IV}}^{\text{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\text{IV}}^{\text{IT}}$, with two alternative estimators, $\widehat{\boldsymbol{\lambda}}_1$ and $\widehat{\boldsymbol{\lambda}}_2$, in terms of finite sample bias. The first alternative estimator, denoted by $\widehat{\boldsymbol{\lambda}}_1$, ignores the EIV problem and regresses average excess returns over the testing period on a constant and beta estimates obtained by standard time series regression over the pretesting period, that is $\widehat{\boldsymbol{\lambda}}_1 = (\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_1)^{-1}\widehat{\mathbf{X}}_1'\bar{\mathbf{r}}_2$, where $\widehat{\mathbf{X}}_1 = [\ \mathbf{1}_N \quad \widehat{\mathbf{B}}_1\ ]$. Similarly, the second alternative estimator, denoted by $\widehat{\boldsymbol{\lambda}}_2$, also ignores the EIV problem but uses beta estimates from the testing period, that is $\widehat{\boldsymbol{\lambda}}_2 = (\widehat{\mathbf{X}}_2'\widehat{\mathbf{X}}_2)^{-1}\widehat{\mathbf{X}}_2'\bar{\mathbf{r}}_2$. We compute the bias of all four estimators as the average of estimation errors over 10,000 Monte Carlo repetitions. We consider pretesting and testing periods that consist of 60 months and, to provide a comprehensive picture, we repeat the exercise over eight testing periods from 1975–1979 to 2010–2014. In the baseline scenario, the idiosyncratic shocks are assumed to follow a normal distribution. The results, reported in Table 1 in annualized basis points (bps), clearly illustrate the bias reduction gains provided by the IV estimators for all three models. For brevity, we only comment on the results for the

[14] I/A is defined as change in inventory, property, plant and equipment (PP&E) over the previous year's total asset. ROE is defined as (IB - DVP + TXDI) over book value of equity where IB is the total earnings before extraordinary items, DVP is the preferred dividends (if available), and TXDI is the deferred taxes (if available).

[15] The classification is available at `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data-Library/det-49-ind-port.html`.

HXZ4 model. The average absolute biases of $\widehat{\boldsymbol{\lambda}}_1$ ($\widehat{\boldsymbol{\lambda}}_2$) are 757.9 (728.4), 477.8 (471.8), 434.6 (348.4), 366.6 (305.6), and 484.8 (474.0) annualized bps for λ_0, $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$, respectively. When $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ ($\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$) is used, the corresponding values are 45.8 (46.3), 22.8 (21.3), 30.9 (33.0), 20.3 (23.8), and 42.1 (46.4) annualized bps. While this evidence is based on normally distributed disturbances, additional simulation experiments show that these results are robust to the assumption of normality.[16]

Next, we compare the IV-GMM estimators to the alternative estimators in terms of mean squared error. The purpose of this exercise is to examine whether the gain in bias reduction comes at the cost of a higher variance and perhaps efficiency loss. We compute the root mean squared error (RMSE) of each estimator as the square root of the sample mean of squared estimation errors over 10,000 Monte Carlo simulations. The simulation setup is identical to the one used above to examine the finite sample bias. In Table 2, we report, in units of annualized bps, the RMSE of the IV-GMM estimators along with those of the alternative estimators $\widehat{\boldsymbol{\lambda}}_1$ and $\widehat{\boldsymbol{\lambda}}_2$. The results clearly illustrate that the IV-GMM estimators achieve much lower mean squared errors in comparison with the alternative estimators for all three models. Again, for brevity, we only comment on the results for the HZX4 model. The average RMSEs of $\widehat{\boldsymbol{\lambda}}_1$ ($\widehat{\boldsymbol{\lambda}}_2$) are 785.4 (756.4), 523.9 (515.8), 463.0 (384.6), 381.6 (321.3), and 499.4 (489.2) annualized bps for λ_0, $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$, respectively. When $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ ($\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$) is used, the corresponding values are 465.5 (466.6), 367.2 (367.9), 338.1 (339.4), 283.2 (285.4), and 448.3 (450.9) annualized bps. As in the case of bias, while this evidence is based on normally distributed disturbances, additional simulation experiments show that the results on mean squared error are robust to the assumption of normality.[17] Collectively, the simulation evidence, which is robust across different factor model specifications and distributional assumptions, illustrates that the IV-GMM estimators exhibit superior performance in terms of bias reduction without sacrificing efficiency.

In our next simulation exercise, we investigate the behavior of various tests on the ex-post asset pricing implications. Specifically, we focus on empirical rejection frequencies of (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistic $t(\lambda_k)$, $k = 1, \ldots, K$, that focuses on the k-th factor ex-post risk premium, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistic $t(\phi_j)$, $j = 1, \ldots, J$, that examines the ability of the j-th characteristic to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). We fix the factors for the pretesting and testing periods, consisting of $\tau_1 = 60$ and $\tau_2 = 60$ observations, as the historical factors over 2005 to 2009 and 2010 to 2014, respectively. Using the factor realizations and the calibrated pairs of betas and

[16] We repeat the same exercise under the assumption that the idiosyncratic disturbances follow a Student-t distribution with 6 degrees of freedom. The results, reported in Table A1 in the Online Appendix, are almost identical. Hence, our conclusions regarding the superior performance of the IV-GMM estimators in terms of bias reduction is robust to the assumption of normally distributed disturbances.

[17] We repeat the same exercise under the assumption that the idiosyncratic shocks follow a Student-t distribution with 6 degrees of freedom. Table A2 in the Online Appendix reports the results that are almost identical to the ones obtained under the assumption of normally distributed shocks.

idiosyncratic shock variances, we simulate individual stock returns using the data generating process (2) for $t = 1, \ldots, 120$. Since the asymptotic variance of the IV-GMM estimators crucially depends on the cluster structure, we consider a number of difference scenarios. Specifically, we let the number of clusters M_N take the values 50 and 100 and assume that, within each cluster, pairwise correlations are equal to ρ which takes the following three values: 0, 0.10, and 0.20. We consider three nominal levels of significance, 1%, 5%, and 10%, and compute the corresponding empirical rejection frequencies from 10,000 Monte Carlo repetitions. The simulation exercise is first performed for idiosyncratic shocks following a normal distribution and the results are reported in Table 3. We observe that the t statistics $t(\lambda_0)$, $t(\lambda_k)$, $k = 1, \ldots, K$, $t(\phi_j)$, $j = 1, \ldots, J$, and $t(\boldsymbol{\alpha})$ as well as the joint test statistics $J_d(\boldsymbol{\lambda})$, $J_d(\boldsymbol{\phi})$, and $J_d(\boldsymbol{\delta})$ all yield empirical rejection frequencies that are reasonably close to the corresponding nominal levels of significance. The simulation is repeated for shocks following a Student-t distribution with 6 degrees of freedom and the results are almost identical.[18] The conclusions hold for all three asset pricing factor models considered and under both distributional assumptions.

Next, we illustrate the importance of using the variance-covariance estimator $\widehat{\mathbf{V}}_\lambda$ for obtaining ex-post risk premia tests with good size properties.[19] We do so by examining the empirical rejection frequencies of the various tests when the variance-covariance matrix of the IV ex-post risk premia estimator is estimated using the Fama and MacBeth (1973) (FMB) procedure as suggested by Jegadeesh, Noh, Pukthuanthong, Roll, and Wang (2015), for the CAPM, the FF3 model, and the HXZ4 model. It is important to emphasize the difference between our setting and the setting in Jegadeesh, Noh, Pukthuanthong, Roll, and Wang (2015). We perform our simulations and empirical tests over short time intervals, covering 120 months, while they use much longer horizons, ranging from 480 to 720 months. The point of our simulation is to illustrate that using the FMB procedure results in misleading inference about ex-post risk premia in a small T context like ours. Table 4 presents the empirical rejection frequency results for the case of idiosyncratic shocks following a normal distribution under two scenarios. In the first scenario, the pretesting and testing periods are 2000-2004 and 2005-2009, respectively, while in the second scenario, the pretesting and testing periods are 2005-2009 and 2010-2014, respectively. The factor betas and the characteristics are calibrated as before and the factor realizations are kept fixed over simulation repetitions during both pretesting and testing periods. For the majority of the tests, we consistently find that the FMB variance-covariance matrix leads to severe underrejection under both scenarios. The simulation is repeated for shocks following a Student-t distribution with 6 degrees of freedom and the results are almost identical.[20] The conclusion that the FMB procedure leads to severe underrejection holds for all three asset pricing factor models and under both distributional assumptions considered.

[18]The results are reported in Table A3 in the Online Appendix.

[19]Recall that $\widehat{\mathbf{V}}_\lambda$ is the $(K+1) \times (K+1)$ upper-left block of the estimator $\widehat{\mathbf{V}}_\delta$ provided in Theorem 10.

[20]The results are reported in Table A4 in the Online Appendix.

4 Empirical Evidence

In this section, we use the IV-GMM approach developed in Section 2 to empirically evaluate a number of popular factor models that have been proposed in the asset pricing literature. Specifically, we focus on four models: (i) the standard single-factor CAPM, (ii) the three-factor Fama and French (1993) model (FF3), (iii) the four-factor Hou, Xue, and Zhang (2015) model (HXZ4), and (iv) the five-factor Fama and French (2015) model (FF5). The factors involved in the CAPM, FF3, and FF5 models are obtained from Kenneth French's website. In particular, the market excess return (MKT) is used by all three aforementioned models; the small size minus big size spread portfolio return (SMB) and the high book-to-market minus low book-to-market spread portfolio return (HML) are used by both the FF3 and FF5 models;[21] and, finally, the robust minus weak spread portfolio return (RMW) and conservative minus aggressive spread portfolio return (CMA) are used by FF5. The four factors involved in the HXZ4 q-factor model are the market excess return (MKT), the difference between the return on a portfolio of small size stocks and the return on a portfolio of big size stocks (ME), the difference between the return on a portfolio of low investment stocks and the return on a portfolio of high investment stocks (I/A), and the difference between the return on a portfolio of high profitability (return on equity) stocks and the return on a portfolio of low profitability stocks (ROE).[22] We use individual stock data at the monthly frequency covering the time period between 1970 and 2014 from the CRSP universe and apply the following filters: (i) we require that the share code (SHRCD) is equal to 10 or 11 to keep only ordinary common shares, (ii) we require that the exchange code (EXCHCD) is equal to 1, 2, or 3 to keep only stocks traded at NYSE, AMEX, or NASDAQ, and (iii) we keep a stock in the sample only for the months in which its price (PRC) is at least 1 dollar. When we use clustering based on the 49-industry classification of Kenneth French for estimating the variance-covariance matrix of the IV-GMM estimators, we further we require that stocks have a Standard Industry Classification (SIC) code.[23]

Our pretesting and testing periods consist of five years ($\tau_1 = \tau_2 = 60$ months) resulting in 8 non-overlapping testing periods from 1975 to 2014. The cross section of our test assets consists of all stocks with full histories over both the pretesting and testing periods. Our empirical evidence consists of (i) estimates of $\boldsymbol{\lambda} = \left[\begin{array}{cc} \lambda_0 & \boldsymbol{\lambda}_f' \end{array} \right]'$, where $\boldsymbol{\lambda}_f = [\begin{array}{ccc} \lambda_1 & \cdots & \lambda_K \end{array}]'$ is the vector of ex-post factor risk premia for the CAPM, FF3, HXZ4, and FF5 models, which they employ $K = 1$, $K = 3$, $K = 4$, and $K = 5$ factors, respectively, and (ii) various test statistics for evaluating the implications of each model and their corresponding p-values.

When estimating ex-post risk premia for the FF3, HXZ4, and FF5 models, in addition to past beta estimates, we employ as instrumental variables the firm characteristics based on which the various factors are constructed. Specifically, for the FF3 model, we use market capitalization (SIZE) and book-to-market ratio (BTM) as firm characteristics. For the HXZ4 model, we use SIZE, investment over

[21] The SMB factor used by the FF3 model is slightly different from the SMB factor used by the FF5 model. Details on how the SMB factor is constructed for each model are provided in Kenneth French's data library website.

[22] We are grateful to the authors for providing the data on the factors of the Hou, Xue, and Zhang (2015) model.

[23] SIC codes are obtained from Compustat. If the SIC code does not exist in Compustat, it is obtained from CRSP.

asset (I/A), and return on equity (ROE) as firm characteristics. Lastly, for the FF5 model, we use SIZE, BTM, operating profitability (OP) and asset growth (AG) as firm characteristics. Next, we describe how the above characteristics are computed. The SIZE characteristic for month m is defined as the ratio of the market capitalization a given firm at the end of month $m-1$ to the aggregate market capitalization at the end of month $m-1$. The BTM characteristic from July of year $y+1$ till June of year $y+2$ is defined as the ratio of book equity (BE) in the accounting data of fiscal year y to the market capitalization at the end of year y. BE is computed following the method in Kenneth French's database, i.e., BE is defined the book value of stockholders equity (SEQ), plus balance sheet deferred taxes and investment tax credit (TXDITC, if available), minus the book value of preferred stock.[24] The I/A characteristic from July of year $y+1$ till June of year $y+2$ is defined as change in inventory, property, plant and equipment (PP&E) from year $y-1$ to year y over the year $y-1$ total assets. The ROE characteristic from July of year $y+1$ till June of year $y+2$ is defined as the ratio of (IB − DVP + TXDI) for the year y over BE of year y, where IB is the total earnings before extraordinary items, DVP is the preferred dividends (if available), and TXDI is the deferred taxes (if available). The OP characteristic from July of year $y+1$ till June of year $y+2$ is defined as the ratio of (REV − COGS − XINT − XSGA) for year y over BE of year y, where REV is revenue, COGS is cost of goods sold, XINT is interest expense, and XSGA is selling, general and administrative expenses. Finally, the AG characteristic from July of year $y+1$ till June of year $y+2$ is defined as the ratio of change in the total assets from year $y-1$ to year y over the year $y-1$ total assets. The cross-sectional distributions of the majority of these characteristics are highly skewed and contain extreme outliers. To deal with this issue, as we did in our simulation exercise, we apply the rank-based inverse normal transformation to the cross-sectional distribution of each characteristic at each time period. Specifically, using the raw characteristics data $\gamma_1, \ldots, \gamma_N$, we (i) compute the corresponding ranks $\xi_1, \ldots, \xi_N$, (ii) convert them into quantiles $q_1, \ldots, q_N$ defined by $q_i = \frac{\xi_i - 0.5}{N}$, for $i = 1, \ldots, N$, and (iii) compute the transformed values $c_1, \ldots, c_N$, given by $c_i = \Phi^{-1}(q_i)$, for $i = 1, \ldots, N$, where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Finally, we demean the transformed values of each characteristic to impose the cross-sectional zero-mean condition.

The asset pricing models under examination employ factors that are constructed as differences between returns on top and bottom portfolios, or vice versa, after sorting according to a particular firm characteristic. As a result, we expect a firm characteristic to be cross-sectionally correlated with the beta with respect to the corresponding spread factor. This provides a clear rationale for using firm characteristics as instrumental variables, in addition to past beta estimates. In Table 5, we provide evidence supporting this rationale. Specifically, for each characteristic, we consider decile portfolios sorted according to that characteristic and estimate their betas with respect to the related spread factor within the context of each model that we evaluate. For each asset pricing model, the portfolio betas are estimated jointly for all factors using data from 07/1970 to 12/2014. As illustrated in Table 5, there is a clear monotonic pattern in the betas for each spread factor within each model. This evidence justifies our choice of firm characteristics as instruments in the estimation of ex-post risk

[24] For a more detailed description, the reader is referred to the definition of BE at Kenneth French's website `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/variable_definitions.html`.

premia.

We present our empirical results in Tables 6 through 9. We report results on (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistic $t(\lambda_k)$, $k = 1, \ldots, K$, that focuses on the k-th factor ex-post risk premium, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistic $t(\phi_j)$, $j = 1, \ldots, J$, that examines the ability of the j-th characteristic to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90).[25] In our discussion of the results, we consider both of the conventional 5% and 10% level of significance.

The results for the CAPM, based on the IV estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$ and using past beta estimates as instruments, are reported in Table 6. Overall, our evidence points to overwhelming rejection of the CAPM. The t statistics $t(\lambda_0)$ and $t(\lambda_{\texttt{MKT}})$ and the joint statistic $J_d(\boldsymbol{\lambda})$ reject the null hypothesis in five (five), five (seven), and five (six) out of eight periods at the 5% (10%) level of significance, respectively. The t statistics $t(\phi_{\texttt{SIZE}})$, $t(\phi_{\texttt{BTM}})$ and the joint statistic $J_d(\boldsymbol{\phi})$ reject the null hypothesis in seven (seven), one (one), and seven (eight) out of eight testing periods, respectively at the 5% (10%) level of significance. Regarding the aggregate mispricing test, the $t\,(\boldsymbol{\alpha})$ statistic rejects the null hypothesis in six and seven out of eight testing periods at the 5% and 10% levels of significance. Finally, the statistic $J_d(\boldsymbol{\delta})$, which jointly tests ex-post risk premia, characteristics rewards, and aggregate mispricing implications, rejects the null hypothesis in all of eight testing periods with p-values less than or equal to 0.01. These findings are not very surprising given that the CAPM has been frequently rejected in the literature using portfolios of stocks.

The results for the FF3 model, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}^{\texttt{TS}}_{\texttt{IV}}$ and $\widehat{\boldsymbol{\lambda}}^{\texttt{IT}}_{\texttt{IV}}$, where we use the SIZE and BTM characteristics as instruments in addition to past beta estimates, are reported in Table 7. Since the two IV-GMM estimators overall yield similar results, we only discuss the results based on the $\widehat{\boldsymbol{\lambda}}^{\texttt{IT}}_{\texttt{IV}}$ estimator. The $t(\lambda_0)$ and $t(\lambda_{\texttt{MKT}})$ statistics both reject the null hypothesis in four out of eight testing periods at the 5% level of significance. The $t(\lambda_{\texttt{SMB}})$ statistic rejects the null hypothesis in six out of eight testing periods with p-values less than or equal to 0.03, while the $t(\lambda_{\texttt{HML}})$ statistic rejects the null hypothesis in three out of eight testing periods with p-values less than or equal to 0.04. The $J_d(\boldsymbol{\lambda})$ statistic, which jointly tests risk premia implications, rejects the null hypothesis in six out of eight testing periods with the exception of the 1975–1979 and 2005-2009 periods, at the 5% level of significance. The t statistics $t(\phi_{\texttt{SIZE}})$, $t(\phi_{\texttt{BTM}})$ and the joint statistic $J_d(\boldsymbol{\phi})$ reject the null hypothesis only in one (two), two (two), and three (three) out of eight testing periods, respectively at the 5% (10%) level of significance. Moreover, the aggregate mispricing test statistic $t\,(\boldsymbol{\alpha})$ rejects the null hypothesis in four (five) out of eight testing periods at 5% (10%) level of significance. Finally, the $J_d(\boldsymbol{\delta})$ statistic, which jointly tests all model implications under consideration, rejects the null hypothesis in six out of eight testing periods at the 5% level of significance. Although

[25] As mentioned in Section 3, the joint test statistic $J(\boldsymbol{\delta})$ tends to overreject the null hypothesis in small samples. That is why we focus on the $J_d(\boldsymbol{\lambda})$, $J_d(\boldsymbol{\phi})$, and $J_d(\boldsymbol{\delta})$ joint test statistics.

overall the results for the FF3 model are slightly better than those for the CAPM, the null hypothesis of correct model specification is rejected by the majority of the tests, with the exception of the 2005-2009 period.

In Table 8, we report the results for the HZX4 model, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. The SIZE, I/A, and ROE characteristics are used as instruments in addition to past beta estimates. The two IV-GMM estimators again yield similar results, and hence we only discuss the results based on the $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$ estimator. The $t(\lambda_0)$ and $t(\lambda_{\texttt{MKT}})$ statistics reject the null hypothesis only in two and one out of eight testing periods at the 5% level of significance, respectively. The $t(\lambda_{\texttt{ME}})$ statistic rejects the null hypothesis in three out of eight testing periods with p-values less than or equal to 0.02, while the $t(\lambda_{\texttt{I/A}})$ and $t(\lambda_{\texttt{ROE}})$ statistics reject the null hypothesis only in two out of eight testing periods with p-values less than or equal to 0.03. Accordingly, the $J_d(\boldsymbol{\lambda})$ statistic, which jointly tests risk premia implications, rejects the null hypothesis in three out of eight testing periods, at the 5% level of significance. The t statistics $t(\phi_{\texttt{SIZE}})$, $t(\phi_{\texttt{I/A}})$, $t(\phi_{\texttt{ROE}})$ and the joint statistic $J_d(\boldsymbol{\phi})$ reject the null hypothesis in two, one, two, and one out of eight testing periods, respectively at the 5% level of significance. Moreover, the aggregate mispricing test statistic $t\left(\boldsymbol{\alpha}\right)$ rejects the null hypothesis in four out of eight testing periods with p-values less than or equal to 0.03. Finally, the $J_d(\boldsymbol{\delta})$ statistic, which jointly tests all model implications under consideration, rejects the null hypothesis in the last four out of eight testing periods at the 5% level of significance. While we document rejection of the implications of the HXZ4 model in several testing periods, the model clearly performs better that the CAPM and the FF3 model.

The results for the FF5 model, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$, where we use the SIZE, BTM, OP, and AG characteristics as instruments in addition to past beta estimates, are reported in Table 9. As before, the results are consistent across the two IV-GMM estimators. Hence, we focus on the results based on $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. In terms of testing the risk premia implications, the statistics $t(\lambda_0)$, $t(\lambda_{\texttt{MKT}})$, $t(\lambda_{\texttt{SMB}})$, $t(\lambda_{\texttt{HML}})$, $t(\lambda_{\texttt{RMW}})$, and $t(\lambda_{\texttt{CMA}})$ reject the null hypothesis in four, four, five, four, three, and three out of testing periods at the 5% level of significance, respectively. The $J_d(\boldsymbol{\lambda})$ statistic, which jointly tests risk premia implications, rejects the null hypothesis in seven out of eight testing periods with the exception of the 2005-2009 period, at the 5% level of significance. As far as characteristics rewards are concerned, the t statistics $t(\phi_{\texttt{SIZE}})$, $t(\phi_{\texttt{BTM}})$, $t(\phi_{\texttt{OP}})$, $t(\phi_{\texttt{AG}})$ and the joint statistic $J_d(\boldsymbol{\phi})$ reject the null hypothesis in two, one, two, one, and three out of eight testing periods, respectively at the 5% level of significance. In addition, the aggregate mispricing test statistic $t\left(\boldsymbol{\alpha}\right)$ rejects the null hypothesis in three (four) out of eight testing periods at the 5% (10%) level of significance. Finally, the $J_d(\boldsymbol{\delta})$ statistic, which jointly tests all model implications under consideration, rejects the null hypothesis in seven out of eight testing periods at the 5% level of significance, with the exception of the 2005-2009 testing period. Overall, the evidence suggests that, based on individual stock data, the performance of the FF5 model is very similar to that of the FF3 model.

Collectively, our results show that the CAPM is strongly rejected by our IV-GMM tests while the FF3 and the FF5 models are rejected by our tests in all but one testing period (2005–2009). In contrast, the HZX4 model appears to perform better, although it is supported over the first half of our overall sample period (1975-1994) while it is rejected in the second half (1995-2014). In the Online Appendix, we report the test statistics and the corresponding p-values for the four models examined above using alternative clustering schemes and price filters. Tables A5– A8 contain the results based on 49 industry clusters and a \$3 price filter. Tables A9– A12 contain the results based on 49 industry clusters and a \$5 price filter. Finally, tables A13– A16 contain the results based on 30 industry clusters and a \$1 price filter. The results of our tests remain very similar under all alternative scenarios.

5 Conclusion

A linear asset pricing factor model characterizes the average return of an asset as a linear function of its factor betas with the risk premia being the slopes. In theory, such a relationship is supposed to be valid for all individual assets. However, the majority of empirical tests of asset pricing models are based on portfolios. One of the main reasons for this practice is that individual stock beta estimates are plagued by significant sampling error giving rise to the well-known error-in-variables (EIV) problem. When the size of the cross section N is large, as is the case when individual stocks are used as test assets, while the the time series sample size T is small and fixed, the EIV problem is so severe that it renders the standard two-pass cross-sectional regression (CSR) risk premia estimator inconsistent. To deal with the EIV problem, we develop a modification of the two-pass CSR approach that employs past beta estimates and firm characteristics as instrumental variables and yields an IV-GMM ex-post risk premia estimator, which is shown to be consistent and asymptotically normal. We further contribute to the literature by developing an estimator of the asymptotic variance-covariance matrix of the risk premia estimator, based on a cluster structure for idiosyncratic shock correlations, which is used to build tests of important asset pricing implications. These include implications about (i) ex-post risk premia (ii) the ability of characteristics to explain potential mispricing at the individual stock level, and (iii) aggregate mispiricing as captured by the average squared pricing error.

The good performance of the IV-GMM estimator and the associated tests of various asset pricing implications for empirically relevant finite sample sizes is illustrated through a number of Monte Carlo simulations. Using three different asset pricing models for calibration, we show that (i) the IV-GMM approach leads to significant bias reduction in the cross-sectional regression intercept and ex-post risk premia estimates without sacrificing efficiency, and (ii) the associated asset pricing test statistics yield empirical rejection frequencies very close to the desired levels of significance. In our empirical investigation, we estimate and evaluate four popular linear asset pricing factor models: the CAPM, the three-factor of model of Fama and French (1993), the q-factor model of Hou, Xue, and Zhang (2015), and the five-factor model of Fama and French (2015). We find that the CAPM is strongly rejected in all testing periods, while the FF3 and FF5 models perform similarly and are rejected in seven out of eight testing periods, with the exception of the 2005-2009 period. Finally, according to our evidence, the HZX4 model is supported in the first half of the overall sample period, covering the years 1975-1994, while it is rejected in the second half, covering the years 1995-2004.

A Proofs

A few facts from matrix algebra are used in the main text and/or in the subsequent proofs. We collect them here for the convenience of the reader. In terms of notation, vec denotes the column-stacking operator and $\otimes$ denotes the Kronecker product.

(F1) For column vectors $\mathbf{x}$ and $\mathbf{y}$, we have $\text{vec}\,(\mathbf{x}\mathbf{y}') = \mathbf{y} \otimes \mathbf{x}$.

(F2) For conformable matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we have $\text{vec}\,(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\,\text{vec}\,(\mathbf{B})$.

(F3) For conformable matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, we have $(\mathbf{AC}) \otimes (\mathbf{BD}) = (\mathbf{A} \otimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D})$.

(F4) Let $\boldsymbol{\mathcal{S}}$ be the $\tau_2^2 \times \tau_2$ selection matrix such that the $(\tau_2\,(s-1)+s, s)$ element of $\boldsymbol{\mathcal{S}}$ is 1, for $s = 1, \ldots, \tau_2$, and all other elements are zero. Then, $\mathbf{A} \odot \mathbf{B} = \boldsymbol{\mathcal{S}}'(\mathbf{A} \otimes \mathbf{B})\boldsymbol{\mathcal{S}}$, for any $\tau_2 \times \tau_2$ matrices $\mathbf{A}$ and $\mathbf{B}$.

The facts (F1), (F2), (F3), and (F4) follow from Theorem 8.9, Theorem 8.11, Theorem 8.2, and Section 8.5 in Schott (2017), respectively.

Proof of Theorem 1: Recall from equation (22) that

$$\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{GMM}} = \boldsymbol{\lambda} + \left(\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\widehat{\boldsymbol{\Omega}}\right)^{-1}\widehat{\boldsymbol{\Omega}}'\widehat{\mathbf{W}}\left(\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2/N\right), \tag{93}$$

where $\widehat{\boldsymbol{\Omega}} = \frac{1}{N}\widehat{\mathbf{Z}}_1'\widehat{\mathbf{X}}_2$. In light of (13), it follows from Assumption 1 that, as $N \to \infty$, $\widehat{\mathbf{B}}_1'\mathbf{1}_N/N = \mathbf{B}'\mathbf{1}_N/N + \mathbf{G}_1'(\mathbf{U}_1'\mathbf{1}_N/N) \xrightarrow{p} \boldsymbol{\mu}_\beta$, $\widehat{\mathbf{B}}_2'\mathbf{1}_N/N = \mathbf{B}'\mathbf{1}_N/N + \mathbf{G}_2'(\mathbf{U}_2'\mathbf{1}_N/N) \xrightarrow{p} \boldsymbol{\mu}_\beta$, and $\widehat{\mathbf{B}}_1'\widehat{\mathbf{B}}_2/N = \mathbf{B}'\mathbf{B}/N + \mathbf{G}_1'(\mathbf{U}_1'\mathbf{B}/N) + (\mathbf{B}'\mathbf{U}_2/N)\mathbf{G}_2 + \mathbf{G}_1'(\mathbf{U}_1'\mathbf{U}_2/N)\mathbf{G}_2 \xrightarrow{p} \mathbf{M}_\beta$, where $\mathbf{M}_\beta = \mathbf{V}_\beta + \boldsymbol{\mu}_\beta\boldsymbol{\mu}_\beta'$. Moreover, using (13) again, it follows from Assumption 2(iii) that $\mathbf{C}_f'\mathbf{1}_N/N = \mathbf{0}_L$ and from Assumption 1(i) and Assumption 2(iv) that $\mathbf{C}_f'\widehat{\mathbf{B}}_2/N = \mathbf{C}_f'\mathbf{B}/N + (\mathbf{B}'\mathbf{U}_2/N)\mathbf{G}_2 \xrightarrow{p} \mathbf{V}_{c\beta}^f$. It then follows from definitions (16) and (21) and the above probability limits that

$$\widehat{\boldsymbol{\Omega}} = \widehat{\mathbf{Z}}_1'\widehat{\mathbf{X}}_2/N \xrightarrow{p} \boldsymbol{\Omega}, \tag{94}$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & \boldsymbol{\mu}_\beta' \\ \boldsymbol{\mu}_\beta & \mathbf{M}_\beta \\ \mathbf{0}_L & \mathbf{V}_{c\beta}^f \end{bmatrix}. \tag{95}$$

According to Assumption 1, $\mathbf{V}_\beta$ is positive definite. Since $\mathbf{M}_\beta = \mathbf{V}_\beta + \boldsymbol{\mu}_\beta\boldsymbol{\mu}_\beta'$, it follows that the $(K+L+1) \times (K+1)$ matrix $\boldsymbol{\Omega}$ has full rank. Thus, to prove the theorem, it suffices to show that, as $N \to \infty$, $\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2/N$ converges to a vector of zeros. Using equations (13), and (18), and invoking Assumption 1(i), Assumption 1(ii), and Assumption 2(ii), we obtain that, as $N \to \infty$, $\mathbf{1}_N'\boldsymbol{\omega}_2/N = (\mathbf{1}_N'\mathbf{U}_2/N)\mathbf{g}_2 \xrightarrow{p} 0$, $\widehat{\mathbf{B}}_1'\boldsymbol{\omega}_2/N = (\mathbf{B}'\mathbf{U}_2/N)\mathbf{g}_2 +$

$\mathbf{G}_1'(\mathbf{U}_1'\mathbf{U}_2/N)\mathbf{g}_2 \xrightarrow{p} \mathbf{0}_K$, and $\mathbf{C}_f'\boldsymbol{\omega}_2/N = (\mathbf{C}_f'\mathbf{U}_2/N)\mathbf{g}_2 \xrightarrow{p} \mathbf{0}_L$. Then, using equation (21), we obtain

$$\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2/N \xrightarrow{p} \mathbf{0}_{K+L+1}. \tag{96}$$

Combining (93) with the probability limits in (94) and (96) yields the N-consistency of $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}$. ■

Proof of Proposition 4: Using equation (35), we obtain $\mathrm{vec}\left(\widehat{\mathbf{U}}_2'\widehat{\mathbf{U}}_2/N\right) = \mathrm{vec}\left(\mathbf{H}_2\left(\mathbf{U}_2'\mathbf{U}_2/N\right)\mathbf{H}_2\right) = \left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\mathrm{vec}\left(\mathbf{U}_2'\mathbf{U}_2/\right.$ where the last equality follows from fact (F2). It then follows from (37) that

$$\widehat{\mathbf{v}}_2 = \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\mathrm{vec}\left(\mathbf{U}_2'\mathbf{U}_2/N\right). \tag{97}$$

Invoking Assumption 3, we obtain $\widehat{\mathbf{v}}_2 \xrightarrow{p} \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\mathbf{v}_2$. Let $\mathbf{d}_2$ be the $\tau_2 \times 1$ vector with s-th element equal to $v_{2,s}$, $s = 1, \ldots, \tau_2$, so that $\mathbf{v}_2 = \boldsymbol{\mathcal{S}}\mathbf{d}_2$.

It follows that $\left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\mathbf{v}_2 = \boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}[\boldsymbol{\mathcal{S}}'\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\boldsymbol{\mathcal{S}}]\mathbf{d}_2$. Finally, fact (F4) yields $\boldsymbol{\mathcal{S}}'\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\boldsymbol{\mathcal{S}} = \mathbf{H}_2 \odot \mathbf{H}_2$ and so

$$\left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\mathbf{v}_2 = \boldsymbol{\mathcal{S}}\mathbf{d}_2 = \mathbf{v}_2, \tag{98}$$

implying that $\widehat{\mathbf{v}}_2 \xrightarrow{p} \mathbf{v}_2$, as $N \to \infty$. ■

Proof of Theorem 6: Equations (42) and (43) yield

$$\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) = \widehat{\boldsymbol{\Psi}}_\lambda\left(\frac{1}{\sqrt{N}}\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2\right). \tag{99}$$

Let $\widehat{\mathbf{z}}_{1,i}'$ denote the i-th row of $\widehat{\mathbf{Z}}_1$ defined in (21), $\widehat{\boldsymbol{\beta}}_{1,i}'$ denote the i-th row of $\widehat{\mathbf{B}}_1$, and $\omega_{2,i}$ denote the i-th element of $\boldsymbol{\omega}_2$ defined in (17). Hence, we have $\widehat{\mathbf{z}}_{1,i} = \begin{bmatrix} 1 & \widehat{\boldsymbol{\beta}}_{1,i}' & \mathbf{c}_{f,i}' \end{bmatrix}'$, $\widehat{\mathbf{Z}}_1 = \begin{bmatrix} \widehat{\mathbf{z}}_{1,1} & \cdots & \widehat{\mathbf{z}}_{1,N} \end{bmatrix}'$ and $\boldsymbol{\omega}_2 = \begin{bmatrix} \omega_{2,1} & \cdots & \omega_{2,N} \end{bmatrix}'$. It follows that

$$\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2 = \sum\nolimits_{i=1}^N \widehat{\mathbf{z}}_{1,i}\omega_{2,i} = \begin{bmatrix} \sum_{i=1}^N \omega_{2,i} \\ \sum_{i=1}^N \widehat{\boldsymbol{\beta}}_{1,i}\omega_{2,i} \\ \sum_{i=1}^N \mathbf{c}_{f,i}\omega_{2,i} \end{bmatrix}. \tag{100}$$

It follows from equation (18) that $\omega_{2,i} = \mathbf{u}_{2,[i]}'\mathbf{g}_2 = \mathbf{g}_2'\mathbf{u}_{2,[i]}$. Hence, using equations (13) and (18), we obtain $\widehat{\boldsymbol{\beta}}_{1,i}\omega_{2,i} = \boldsymbol{\beta}_i\mathbf{u}_{2,[i]}'\mathbf{g}_2 + \mathbf{G}_1'\mathbf{u}_{1,[i]}\mathbf{u}_{2,[i]}'\mathbf{g}_2 = (\mathbf{g}_2' \otimes \mathbf{I}_K)(\mathbf{u}_{2,[i]} \otimes \boldsymbol{\beta}_i) + (\mathbf{g}_2' \otimes \mathbf{G}_1')(\mathbf{u}_{2,[i]} \otimes \mathbf{u}_{1,[i]})$, and $\mathbf{c}_{f,i}\omega_{2,i} = \mathbf{c}_{f,i}\mathbf{u}_{2,[i]}'\mathbf{g}_2 = (\mathbf{g}_2' \otimes \mathbf{I}_L)(\mathbf{u}_{2,[i]} \otimes \mathbf{c}_{f,i})$, where we use facts (F1), (F2), and (F3). Combining the last two equations, we obtain that

$$\widehat{\mathbf{z}}_{1,i}\omega_{2,i} = \boldsymbol{\Pi}_\lambda\mathbf{e}_i, \tag{101}$$

where $\mathbf{e}_i$ is defined by (39) and $\mathbf{\Pi}_\lambda$ is the $(K+L+1)\times\mathcal{T}$ matrix, with $\mathcal{T}=\tau_2(\tau_1+\tau_2+K+J+1)$, defined by

$$\mathbf{\Pi}_\lambda = \begin{bmatrix} \mathbf{g}_2' & \mathbf{0}_{\tau_1\tau_2}' & \mathbf{0}_{K\tau_2}' & \mathbf{0}_{L\tau_2}' & \mathbf{0}_{(J-L)\tau_2}' & \mathbf{0}_{\tau_2^2}' \\ \mathbf{0}_{K\times\tau_2} & \mathbf{g}_2'\otimes\mathbf{G}_1' & \mathbf{g}_2'\otimes\mathbf{I}_K & \mathbf{0}_{K\times(L\tau_2)} & \mathbf{0}_{K\times(J-L)\tau_2} & \mathbf{0}_{K\times\tau_2^2} \\ \mathbf{0}_{L\times\tau_2} & \mathbf{0}_{L\times(\tau_1\tau_2)} & \mathbf{0}_{L\times(K\tau_2)} & \mathbf{g}_2'\otimes\mathbf{I}_L & \mathbf{0}_{L\times(J-L)\tau_2} & \mathbf{0}_{L\times\tau_2^2} \end{bmatrix}. \tag{102}$$

It follows from (101) that

$$\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2 = \mathbf{\Pi}_\lambda\sum_{i=1}^N \mathbf{e}_i. \tag{103}$$

Combining (99), (103), and (44), we obtain that

$$\sqrt{N}\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right) = \mathbf{\Psi}_\lambda\mathbf{\Pi}_\lambda\frac{1}{\sqrt{N}}\sum_{i=1}^N \mathbf{e}_i + o_p(1). \tag{104}$$

Finally, invoking Assumption 4 yields the desired result. ■

Proof of Theorem 7: According to Assumption 2, as $N\to\infty$, we have $\mathbf{C}'\mathbf{C}/N\to\mathbf{V}_c$ and $\mathbf{C}'\mathbf{X}/N\to\mathbf{V}_{cx}$, where

$$\mathbf{V}_{cx} = [\ \mathbf{0}_J \quad \mathbf{V}_{c\beta}\ ]. \tag{105}$$

Moreover, it follows from Assumption 4 and equations (29) and (18) that $\left(\frac{\mathbf{C}'\mathbf{U}_2}{\sqrt{N}}\right)\widehat{\mathbf{g}}_2 = \frac{\mathbf{C}'\boldsymbol{\omega}_2}{\sqrt{N}} + o_p(1)$. Hence, in light of (44), equation (47) implies

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi}\right) = -\mathbf{V}_c^{-1}\left[\mathbf{V}_{cx}\mathbf{\Psi}_\lambda\left(\frac{\widehat{\mathbf{Z}}_1'\boldsymbol{\omega}_2}{\sqrt{N}}\right) - \left(\frac{\mathbf{C}'\boldsymbol{\omega}_2}{\sqrt{N}}\right)\right] + o_p(1). \tag{106}$$

It follows from (49) that $\widehat{\mathbf{Z}}_1 = \widehat{\boldsymbol{\mathcal{Z}}}_1\boldsymbol{\mathcal{I}}_1'$ and $\mathbf{C} = \widehat{\boldsymbol{\mathcal{Z}}}_1\boldsymbol{\mathcal{I}}_2'$ where

$$\boldsymbol{\mathcal{I}}_1 = [\ \mathbf{I}_{K+L+1} \quad \mathbf{0}_{(K+L+1)\times(J-L)}\ ], \quad \boldsymbol{\mathcal{I}}_2 = [\ \mathbf{0}_{J\times(K+1)} \quad \mathbf{I}_J\ ]. \tag{107}$$

Hence,

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi}\right) = \mathbf{\Psi}_\phi\left(\frac{\widehat{\boldsymbol{\mathcal{Z}}}_1'\boldsymbol{\omega}_2}{\sqrt{N}}\right) + o_p(1). \tag{108}$$

where the $J\times(K+J+1)$ matrix $\mathbf{\Psi}_\phi$ is defined by

$$\mathbf{\Psi}_\phi = -\mathbf{V}_c^{-1}\left(\mathbf{V}_{cx}\mathbf{\Psi}_\lambda\boldsymbol{\mathcal{I}}_1 - \boldsymbol{\mathcal{I}}_2\right). \tag{109}$$

Let $\widehat{\mathbf{z}}'_{1,i}$ and $\widehat{\boldsymbol{z}}'_{1,i}$ denote the i-th row of $\widehat{\mathbf{Z}}_1$ and $\widehat{\boldsymbol{\mathcal{Z}}}_1$, defined in (21) and (49), respectively, and $\omega_{2,i}$ denote the i-th element of $\boldsymbol{\omega}_2$ defined in (17). It follows that $\widehat{\boldsymbol{\mathcal{Z}}}'_1\boldsymbol{\omega}_2 = \sum_{i=1}^N \widehat{\boldsymbol{z}}_{1,i}\omega_{2,i}$ with $\widehat{\boldsymbol{z}}_{1,i} = [\ \widehat{\mathbf{z}}'_{1,i} \quad \mathbf{c}'_{a,i}\ ]'$ and

$$\widehat{\boldsymbol{z}}_{1,i}\omega_{2,i} = \begin{bmatrix} \widehat{\mathbf{z}}_{1,i}\omega_{2,i} \\ \mathbf{c}_{a,i}\omega_{2,i} \end{bmatrix}. \tag{110}$$

According to equation (101), we have $\widehat{\mathbf{z}}_{1,i}\omega_{2,i} = \mathbf{\Pi}_\lambda \mathbf{e}_i$. Also, definition (18) yields $\mathbf{c}_{a,i}\omega_{2,i} = \mathbf{c}_{a,i}\mathbf{u}'_{2,[i]}\mathbf{g}_2 = (\mathbf{g}'_2 \otimes \mathbf{I}_{J-L})(\mathbf{u}_{2,[i]} \otimes \mathbf{c}_{a,i})$, where we use facts (F1) and (F2). Combining the last two equations with the definition of $\mathbf{e}_i$ given by (39), we obtain that

$$\widehat{\boldsymbol{z}}_{1,i}\omega_{2,i} = \mathbf{\Pi}_\phi \mathbf{e}_i, \tag{111}$$

where $\mathbf{e}_i$ is defined by (39) and $\mathbf{\Pi}_\phi$ is the $(K+J+1)\times\mathcal{T}$ matrix, with $\mathcal{T} = \tau_2(\tau_1+\tau_2+K+J+1)$, defined by

$$\mathbf{\Pi}_\phi = \begin{bmatrix} & \mathbf{\Pi}_\lambda & \\ \mathbf{0}_{(J-L)\times\tau_2(1+\tau_1+K+L)} & \mathbf{g}'_2 \otimes \mathbf{I}_{J-L} & \mathbf{0}_{(J-L)\times\tau_2^2} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{\Pi}_{\phi,1} \\ \mathbf{\Pi}_{\phi,2} \end{bmatrix}. \tag{112}$$

It follows from (111) that

$$\widehat{\boldsymbol{\mathcal{Z}}}'_1\boldsymbol{\omega}_2 = \mathbf{\Pi}_\phi \sum_{i=1}^N \mathbf{e}_i. \tag{113}$$

Combining (108) and (113), we obtain that

$$\sqrt{N}\left(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = \mathbf{\Psi}_\phi \mathbf{\Pi}_\phi \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbf{e}_i + o_p(1). \tag{114}$$

Finally, invoking Assumption 4 yields the desired result. ■

<u>Proof of Theorem 8</u>: It follows from equation (52) that

$$\begin{aligned} &\sqrt{N}\left(\widehat{\mathcal{Q}} - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2)'\,\widehat{\mathbf{v}}_2\right) \\ =&(\mathbf{g}_2 \otimes \mathbf{g}_2)'\left(\sqrt{N}\text{vec}\left(\mathbf{U}'_2\mathbf{U}_2/N - \mathbf{V}_2\right) - \sqrt{N}\left(\widehat{\mathbf{v}}_2 - \mathbf{v}_2\right)\right) + o_p(1). \end{aligned} \tag{115}$$

In addition, from (37), (35), fact (F2), and the symmetry of $\mathbf{H}_2$, it follows that

$$\widehat{\mathbf{v}}_2 = \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\text{vec}\left(\widehat{\mathbf{U}}'_2\widehat{\mathbf{U}}_2/N\right) = \left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2 \odot \mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2 \otimes \mathbf{H}_2\right)\text{vec}\left(\mathbf{U}'_2\mathbf{U}_2/N\right).$$

Hence, upon using equation (98), we obtain

$$\sqrt{N}\left(\widehat{\mathbf{v}}_2-\mathbf{v}_2\right)=\left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2\odot\mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\mathbf{H}_2\otimes\mathbf{H}_2\right)\sqrt{N}\mathrm{vec}\left(\mathbf{U}_2'\mathbf{U}_2/N-\mathbf{V}_2\right). \tag{116}$$

Therefore, by substituting (116) into (115) and using the definition of $\mathbf{e}_i$ in (39) and the symmetry of $\mathbf{H}_2$, we obtain that $\sqrt{N}\left(\widehat{\mathcal{Q}}-\left(\widehat{\mathbf{g}}_2\otimes\widehat{\mathbf{g}}_2\right)'\widehat{\mathbf{v}}_2\right)=\boldsymbol{\pi}_\alpha'\frac{1}{\sqrt{N}}\sum_{i=1}^N\mathbf{e}_i+o_p(1)$, where

$$\boldsymbol{\pi}_\alpha=\begin{bmatrix}\mathbf{0}_{\tau_2^2\times\tau_2(1+K+L+\tau_1)} & \mathbf{I}_{\tau_2^2}\end{bmatrix}'\left(\mathbf{I}_{\tau_2^2}-\left(\mathbf{H}_2\otimes\mathbf{H}_2\right)\left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2\odot\mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\right)\left(\mathbf{g}_2\otimes\mathbf{g}_2\right). \tag{117}$$

Finally, invoking Assumption 4 completes the proof of the theorem. ■

<u>Proof of Theorem 10</u>: Let $\widehat{\varepsilon}_i'$ denote the i-th row of the matrix $\widehat{\boldsymbol{\mathcal{E}}}$, defined by (82), so that

$$\widehat{\boldsymbol{\mathcal{E}}}=\begin{bmatrix}\widehat{\varepsilon}_1 & \cdots & \widehat{\varepsilon}_N\end{bmatrix}'. \tag{118}$$

For each cluster $m=1,\ldots,M_N$, we define $\widehat{\boldsymbol{\theta}}_m=\sum_{i\in I_m}\widehat{\varepsilon}_i$, so that $\boldsymbol{\mathcal{C}}\widehat{\boldsymbol{\mathcal{E}}}=\begin{bmatrix}\widehat{\boldsymbol{\theta}}_1 & \cdots & \widehat{\boldsymbol{\theta}}_{M_N}\end{bmatrix}'$. Definition (81) then yields

$$\widehat{\boldsymbol{\Theta}}=\frac{1}{M_N-K-1}\sum_{m=1}^{M_N}\widehat{\boldsymbol{\theta}}_m\widehat{\boldsymbol{\theta}}_m'. \tag{119}$$

Moreover, it follows from definitions (75) and (82) that

$$\widehat{\boldsymbol{\mathcal{E}}}=\widetilde{\boldsymbol{\mathcal{E}}}+\begin{bmatrix}\boldsymbol{\mathcal{D}}_1 & \boldsymbol{\mathcal{D}}_2\end{bmatrix}, \tag{120}$$

where

$$\boldsymbol{\mathcal{D}}_1=\mathrm{diag}\left(\widehat{\boldsymbol{\omega}}_2-\boldsymbol{\omega}_2\right)\widehat{\boldsymbol{\mathcal{Z}}}_1, \tag{121}$$

$$\boldsymbol{\mathcal{D}}_2=\widehat{\boldsymbol{\omega}}_2\odot\widehat{\boldsymbol{\omega}}_2-\boldsymbol{\omega}_2\odot\boldsymbol{\omega}_2-\widehat{\boldsymbol{\mathcal{U}}}_2\left[\boldsymbol{\mathcal{S}}\left(\mathbf{H}_2\odot\mathbf{H}_2\right)^{-1}\boldsymbol{\mathcal{S}}'\right]\left(\widehat{\mathbf{g}}_2\otimes\widehat{\mathbf{g}}_2-\mathbf{g}_2\otimes\mathbf{g}_2\right). \tag{122}$$

It follows from definition (74) and fact (F1) that the i-th column of the matrix $\widehat{\boldsymbol{\mathcal{U}}}_2'$, for $i=1,\ldots,N$, is

$$\mathrm{vec}\left(\widehat{\mathbf{u}}_{2,[i]}\widehat{\mathbf{u}}_{2,[i]}'\right)=\left(\mathbf{H}_2\otimes\mathbf{H}_2\right)\mathrm{vec}\left(\mathbf{u}_{2,[i]}\mathbf{u}_{2,[i]}'\right)=\left(\mathbf{H}_2\otimes\mathbf{H}_2\right)\left(\mathbf{u}_{2,[i]}\otimes\mathbf{u}_{2,[i]}\right).$$

Therefore, in light of equations (76) and (118), it follows from equations (120), (121), and (122) that, for $i=1,\ldots,N$,

$$\widehat{\varepsilon}_i=\widetilde{\varepsilon}_i+\begin{bmatrix}\widehat{\boldsymbol{z}}_{1,i}\left(\widehat{\omega}_{2,i}-\omega_{2,i}\right)\\ \widehat{\omega}_{2,i}^2-\omega_{2,i}^2-\left(\widehat{\mathbf{g}}_2\otimes\widehat{\mathbf{g}}_2-\mathbf{g}_2\otimes\mathbf{g}_2\right)'\boldsymbol{\Phi}_g\boldsymbol{\zeta}_i\end{bmatrix}, \tag{123}$$

where $\boldsymbol{\zeta}_i$ is defined in (83) and $\boldsymbol{\Phi}_g$ is a suitable matrix that depends on $\mathbf{H}_2$.

Equations (6), (17), and (80) imply $\widehat{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_2 - \widehat{\mathbf{X}}_2 \left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)$ and so

$$\widehat{\omega}_{2,i} = \omega_{2,i} - \widehat{\mathbf{x}}_{2,i}'(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}), \tag{124}$$

which then yields $\widehat{\boldsymbol{z}}_{1,i}\left(\widehat{\omega}_{2,i} - \omega_{2,i}\right) = -\widehat{\boldsymbol{z}}_{1,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)$. Using fact (F2) and then fact (F1), we obtain

$$\widehat{\boldsymbol{z}}_{1,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) = \left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) \otimes \mathbf{I}_{K+L+1}\right)'\left(\widehat{\mathbf{x}}_{2,i} \otimes \widehat{\boldsymbol{z}}_{1,i}\right). \tag{125}$$

It follows from (13) that $\widehat{\mathbf{X}}_2 = [\begin{array}{cc} \mathbf{1}_N & \mathbf{B} + \mathbf{U}_2\mathbf{G}_2 \end{array}]$ and $\widehat{\boldsymbol{\mathcal{Z}}}_1 = [\begin{array}{ccc} \mathbf{1}_N & \mathbf{B} + \mathbf{U}_1\mathbf{G}_1 & \mathbf{C} \end{array}]$ and so

$$\widehat{\mathbf{x}}_{2,i} = \begin{bmatrix} 1 \\ \boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]} \end{bmatrix}, \quad \widehat{\boldsymbol{z}}_{1,i} = \begin{bmatrix} 1 \\ \boldsymbol{\beta}_i + \mathbf{G}_1'\mathbf{u}_{1,[i]} \\ \mathbf{c}_i \end{bmatrix}.$$

Hence,

$$\widehat{\mathbf{x}}_{2,i} \otimes \widehat{\boldsymbol{z}}_{1,i} = \boldsymbol{\mathcal{K}}_{xz} \begin{bmatrix} 1 \\ \boldsymbol{\beta}_i + \mathbf{G}_1'\mathbf{u}_{1,[i]} \\ \mathbf{c}_i \\ \boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]} \\ (\boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]}) \otimes (\boldsymbol{\beta}_i + \mathbf{G}_1'\mathbf{u}_{1,[i]}) \\ (\boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]}) \otimes \mathbf{c}_i \end{bmatrix},$$

where $\boldsymbol{\mathcal{K}}_{xz}$ is a suitable $(K+1)(K+J+1) \times (K+1)(K+J+1)$ matrix with elements equal to 0 or 1 (see Theorem 8.26(e) in Schott (2017)). Fact (F3) implies

$$\begin{aligned}(\boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]}) \otimes (\boldsymbol{\beta}_i + \mathbf{G}_1'\mathbf{u}_{1,[i]}) =& \boldsymbol{\beta}_i \otimes \boldsymbol{\beta}_i + (\mathbf{G}_2' \otimes \mathbf{I}_K)(\mathbf{u}_{2,[i]} \otimes \boldsymbol{\beta}_i) \\ &+ (\mathbf{I}_K \otimes \mathbf{G}_1')(\boldsymbol{\beta}_i \otimes \mathbf{u}_{1,[i]}) + (\mathbf{G}_2' \otimes \mathbf{G}_1')(\mathbf{u}_{2,[i]} \otimes \mathbf{u}_{1,[i]})\end{aligned}$$

and

$$(\boldsymbol{\beta}_i + \mathbf{G}_2'\mathbf{u}_{2,[i]}) \otimes \mathbf{c}_i = \boldsymbol{\beta}_i \otimes \mathbf{c}_i + (\mathbf{G}_2' \otimes \mathbf{I}_L)(\mathbf{u}_{2,[i]} \otimes \mathbf{c}_i).$$

It follows from the last three equations that

$$\widehat{\mathbf{x}}_{2,i} \otimes \widehat{\boldsymbol{z}}_{1,i} = \boldsymbol{\Phi}_{xz}\boldsymbol{\zeta}_i, \tag{126}$$

where $\boldsymbol{\zeta}_i$ is defined in (83) and $\boldsymbol{\Phi}_{xz}$ is a suitable matrix that depends on $\mathbf{G}_1$ and $\mathbf{G}_2$. Combining equations

(125) and (126) then yields

$$\widehat{z}_{1,i}\left(\widehat{\omega}_{2,i}-\omega_{2,i}\right)=-\widehat{z}_{1,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)=-\left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)\otimes\mathbf{I}_{K+J+1}\right)'\boldsymbol{\Phi}_{xz}\boldsymbol{\zeta}_i. \tag{127}$$

Next, we investigate the term $\widehat{\omega}_{2,i}^2-\omega_{2,i}^2-(\widehat{\mathbf{g}}_2\otimes\widehat{\mathbf{g}}_2-\mathbf{g}_2\otimes\mathbf{g}_2)'\boldsymbol{\Phi}_g\boldsymbol{\zeta}_i$. Note that equation (124) implies

$$\widehat{\omega}_{2,i}^2-\omega_{2,i}^2=-2\omega_{2,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)+\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)'\widehat{\mathbf{x}}_{2,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right). \tag{128}$$

The first term in the right hand side of equation (128) depends on

$$\omega_{2,i}\widehat{\mathbf{x}}_{2,i}=\left(\mathbf{g}_2'\mathbf{u}_{2,[i]}\right)\begin{bmatrix}1\\ \boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\end{bmatrix}=\begin{bmatrix}\mathbf{g}_2'\mathbf{u}_{2,[i]}\\ \boldsymbol{\beta}_i\mathbf{u}_{2,[i]}'\mathbf{g}_2+\mathbf{G}_2'\mathbf{u}_{2,[i]}\mathbf{u}_{2,[i]}'\mathbf{g}_2\end{bmatrix}.$$

Note that $\boldsymbol{\beta}_i\mathbf{u}_{2,[i]}'\mathbf{g}_2=(\mathbf{g}_2'\otimes\mathbf{I}_K)\left(\mathbf{u}_{2,[i]}\otimes\boldsymbol{\beta}_i\right)$ and $\mathbf{G}_2'\mathbf{u}_{2,[i]}\mathbf{u}_{2,[i]}'\mathbf{g}_2=(\mathbf{g}_2'\otimes\mathbf{G}_2')\left(\mathbf{u}_{2,[i]}\otimes\mathbf{u}_{2,[i]}\right)$, as it follows from facts (F1) and (F2). Hence,

$$\omega_{2,i}\widehat{\mathbf{x}}_{2,i}=\boldsymbol{\Phi}_{\omega x}\boldsymbol{\zeta}_i, \tag{129}$$

where $\boldsymbol{\zeta}_i$ is defined in (83) and $\boldsymbol{\Phi}_{\omega x}$ is a suitable matrix that depends on $\mathbf{g}_2$ and $\mathbf{G}_2$. Using fact (F2) and then fact (F1), we can express the second term in the right hand side of equation (128) as

$$\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)'\widehat{\mathbf{x}}_{2,i}\widehat{\mathbf{x}}_{2,i}'\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)=\left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)\otimes\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}}-\boldsymbol{\lambda}\right)\right)'\left(\widehat{\mathbf{x}}_{2,i}\otimes\widehat{\mathbf{x}}_{2,i}\right). \tag{130}$$

Since

$$\widehat{\mathbf{x}}_{2,i}=\begin{bmatrix}1\\ \boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\end{bmatrix}, \tag{131}$$

we obtain

$$\widehat{\mathbf{x}}_{2,i}\otimes\widehat{\mathbf{x}}_{2,i}=\boldsymbol{\mathcal{K}}_{xx}\begin{bmatrix}1\\ \boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\\ \boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\\ \left(\boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\right)\otimes\left(\boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]}\right)\end{bmatrix},$$

where $\boldsymbol{\mathcal{K}}_{xx}$ is a suitable $(K+1)^2\times(K+1)^2$ matrix with elements equal to 0 or 1 (see Theorem 8.26(e) in Schott (2017)). Moreover, using fact (F3), we obtain

$$\begin{aligned}(\boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]})\otimes(\boldsymbol{\beta}_i+\mathbf{G}_2'\mathbf{u}_{2,[i]})=&\boldsymbol{\beta}_i\otimes\boldsymbol{\beta}_i+(\mathbf{G}_2'\otimes\mathbf{I}_K)(\mathbf{u}_{2,[i]}\otimes\boldsymbol{\beta}_i)\\&+(\mathbf{I}_K\otimes\mathbf{G}_2')(\boldsymbol{\beta}_i\otimes\mathbf{u}_{2,[i]})+(\mathbf{G}_2'\otimes\mathbf{G}_2')(\mathbf{u}_{2,[i]}\otimes\mathbf{u}_{2,[i]}).\end{aligned}$$

It follows that

$$\widehat{\mathbf{x}}_{2,i} \otimes \widehat{\mathbf{x}}_{2,i} = \mathbf{\Phi}_{xx}\boldsymbol{\zeta}_i, \tag{132}$$

where $\boldsymbol{\zeta}_i$ is defined in (83) and $\mathbf{\Phi}_{xx}$ is a suitable matrix that depends on $\mathbf{G}_2$. In light of equations (129), (130), and (132), it follows from equation (128) that

$$\widehat{\omega}_{2,i}^2 - \omega_{2,i}^2 = -2\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)' \mathbf{\Phi}_{\omega x}\boldsymbol{\zeta}_i + \left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) \otimes \left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)\right)' \mathbf{\Phi}_{xx}\boldsymbol{\zeta}_i. \tag{133}$$

Hence, combining equations (123), (127), and (133) yields

$$\widehat{\boldsymbol{\varepsilon}}_i = \widetilde{\boldsymbol{\varepsilon}}_i + \begin{bmatrix} \widehat{\boldsymbol{z}}_{1,i}\left(\widehat{\omega}_{2,i} - \omega_{2,i}\right) \\ \widehat{\omega}_{2,i}^2 - \omega_{2,i}^2 - (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2 - \mathbf{g}_2 \otimes \mathbf{g}_2)'\mathbf{\Phi}_g\boldsymbol{\zeta}_i \end{bmatrix} = \widetilde{\boldsymbol{\varepsilon}}_i - \mathbf{\Upsilon}\boldsymbol{\zeta}_i, \tag{134}$$

where

$$\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{\Upsilon}_1 \\ \mathbf{\Upsilon}_2 \end{bmatrix}, \tag{135}$$

with

$$\mathbf{\Upsilon}_1 = \left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) \otimes \mathbf{I}_{K+J+1}\right)' \mathbf{\Phi}_{xz}, \tag{136}$$

$$\mathbf{\Upsilon}_2 = 2\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)' \mathbf{\Phi}_{\omega x} - \left(\left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right) \otimes \left(\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{GMM}} - \boldsymbol{\lambda}\right)\right)' \mathbf{\Phi}_{xx} + (\widehat{\mathbf{g}}_2 \otimes \widehat{\mathbf{g}}_2 - \mathbf{g}_2 \otimes \mathbf{g}_2)'\mathbf{\Phi}_g. \tag{137}$$

In light of equations (77), (78) and (85), equation (134) yields that $\widehat{\boldsymbol{\theta}}_m = \sum_{i \in I_m} \widehat{\boldsymbol{\varepsilon}}_i = \mathbf{\Pi}_\delta \boldsymbol{\eta}_m - \mathbf{\Upsilon}\boldsymbol{\varphi}_m$. It then follows from equation (119) that

$$\begin{aligned} \frac{M_N - K - 1}{M_N}\widehat{\mathbf{\Theta}} =& \mathbf{\Pi}_\delta \left(\frac{1}{M_N}\sum\nolimits_{m=1}^{M_N} \boldsymbol{\eta}_m\boldsymbol{\eta}_m'\right)\mathbf{\Pi}_\delta' + \mathbf{\Upsilon}\left(\frac{1}{M_N}\sum\nolimits_{m=1}^{M_N} \boldsymbol{\varphi}_m\boldsymbol{\varphi}_m'\right)\mathbf{\Upsilon}' \\ & - \mathbf{\Upsilon}\left(\frac{1}{M_N}\sum\nolimits_{m=1}^{M_N} \boldsymbol{\varphi}_m\boldsymbol{\eta}_m'\right)\mathbf{\Pi}_\delta' - \mathbf{\Pi}_\delta\left(\frac{1}{M_N}\sum\nolimits_{m=1}^{M_N} \boldsymbol{\varphi}_m\boldsymbol{\eta}_m'\right)' \mathbf{\Upsilon}'. \end{aligned} \tag{138}$$

Inspection of definitions (39), (61), (83), (84), and (85) reveals that $\boldsymbol{\eta}_m$ is a subvector of $\boldsymbol{\varphi}_m$, and so Assumption 6 implies that, as $N \to \infty$, both $\frac{1}{M_N}\sum_{m=1}^{M_N} \boldsymbol{\varphi}_m\boldsymbol{\varphi}_m'$ and $\frac{1}{M_N}\sum_{m=1}^{M_N} \boldsymbol{\varphi}_m\boldsymbol{\eta}_m'$ converge in probability to some finite matrices. Moreover, in light of Theorem 1 and the probability limit in (29), it follows from definitions (135), (136), and (137), that $\mathbf{\Upsilon}$ converges in probability to a matrix of zeros. Hence, in light of equation (62), it follows from equation (138) that $\widehat{\mathbf{\Theta}} \xrightarrow{p} \mathbf{\Pi}_\delta\mathbf{V}_\eta\mathbf{\Pi}_\delta' = \mathbf{\Theta}$ and thus the proof of the theorem is complete. ■

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Table 1: **Bias in the estimation of $\boldsymbol{\lambda}$ with normally distributed shocks: the role of the EIV correction through the IV-GMM approach.** This table presents simulation results on the absolute bias, in annualized basis points, in the estimation of $\boldsymbol{\lambda} = [\ \lambda_0 \ \ \boldsymbol{\lambda}_f'\ ]'$, where $\boldsymbol{\lambda}_f = [\ \lambda_1 \ \ \cdots \ \ \lambda_K\ ]'$ is the vector of ex-post risk premia and K is the number of factors. The shocks u_{it} are assumed to follow a normal distribution and the factor realizations are kept fixed throughout. The number of individual stocks, N, is equal to 1,000 and the number of clusters, M_N, is set equal to 50. The pairwise correlation of shocks, assumed to be constant within each cluster, is set equal to 0.10. The simulation is calibrated to the following three linear asset pricing models: the single-factor CAPM, the three-factor Fama and French (1993) model (FF3), and the four-factor Hou, Xue, and Zhang (2015) model (HXZ4). For the CAPM, $K = 1$ and $\lambda_{\texttt{MKT}}$ is the ex-post risk premia of MKT. For FF3, $K = 3$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{SMB}}$, and $\lambda_{\texttt{HML}}$ are the ex-post risk premia of MKT, SMB, and HML, respectively. For HXZ4, $K = 4$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$ are the ex-post risk premia of MKT, ME, I/A, and ROE, respectively. For the CAPM, we consider the IV estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$, while for the FF3 and HXZ4 models, we consider both the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. In addition, we consider two alternative estimators $\widehat{\boldsymbol{\lambda}}_1 = (\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_1)^{-1}\widehat{\mathbf{X}}_1'\bar{\mathbf{r}}_2$ and $\widehat{\boldsymbol{\lambda}}_2 = (\widehat{\mathbf{X}}_2'\widehat{\mathbf{X}}_2)^{-1}\widehat{\mathbf{X}}_2'\bar{\mathbf{r}}_2$ that ignore the EIV problem. The results are based on 10,000 Monte Carlo repetitions.

CAPM								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
Average Factor Realizations								
$\bar{f}_{\texttt{MKT}}$	1118.2	453.0	1248.8	483.0	2069.4	-310.0	-43.4	1552.4
$\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$								
λ_0	6.0	3.2	1.5	2.6	0.9	1.1	0.8	4.0
$\lambda_{\texttt{MKT}}$	6.0	3.6	2.6	1.5	2.0	1.3	0.2	5.1
$\widehat{\boldsymbol{\lambda}}_1$								
λ_0	356.2	180.9	467.1	159.1	1001.9	131.6	14.2	571.0
$\lambda_{\texttt{MKT}}$	361.2	183.8	472.0	162.3	1013.8	133.2	15.2	578.1
$\widehat{\boldsymbol{\lambda}}_2$								
λ_0	441.6	170.3	415.4	233.7	882.0	109.0	15.0	711.8
$\lambda_{\texttt{MKT}}$	447.7	172.9	419.8	237.9	892.7	110.3	15.9	720.6

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FF3 Model								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
Average Factor Realizations								
$\overline{f}_{\texttt{MKT}}$	1118.2	453.0	1248.8	483.0	2069.4	-310.0	-43.4	1552.4
$\overline{f}_{\texttt{SMB}}$	1576.0	470.2	-451.6	158.0	-369.4	1002.4	131.4	149.2
$\overline{f}_{\texttt{HML}}$	651.0	931.4	251.8	269.4	-549.4	1510.8	210.8	-88.0
$\widehat{\boldsymbol{\lambda}}^{\texttt{TS}}_{\texttt{IV}}$								
λ_0	4.6	6.4	14.0	3.6	16.3	14.7	6.3	2.0
$\lambda_{\texttt{MKT}}$	3.4	4.6	10.7	4.0	14.5	11.6	5.7	6.1
$\lambda_{\texttt{SMB}}$	3.4	0.4	0.3	0.5	0.6	0.4	2.5	4.0
$\lambda_{\texttt{HML}}$	2.4	1.1	7.2	1.3	4.5	7.4	0.0	6.1
$\widehat{\boldsymbol{\lambda}}^{\texttt{IT}}_{\texttt{IV}}$								
λ_0	4.5	3.2	12.2	2.7	13.0	14.4	6.3	0.4
$\lambda_{\texttt{MKT}}$	3.2	1.8	9.0	2.9	11.7	11.1	5.7	4.1
$\lambda_{\texttt{SMB}}$	3.8	0.5	0.1	0.6	0.6	1.0	2.6	4.3
$\lambda_{\texttt{HML}}$	2.7	0.7	7.4	1.6	3.7	7.9	0.1	6.7
$\widehat{\boldsymbol{\lambda}}_1$								
λ_0	779.9	389.9	633.3	368.3	834.8	756.2	77.3	573.6
$\lambda_{\texttt{MKT}}$	301.4	122.0	741.5	230.2	1140.7	78.1	3.7	683.9
$\lambda_{\texttt{SMB}}$	595.3	147.1	339.5	106.9	374.3	657.0	76.7	50.9
$\lambda_{\texttt{HML}}$	498.5	508.9	208.4	238.8	227.9	912.9	97.5	170.4
$\widehat{\boldsymbol{\lambda}}_2$								
λ_0	601.4	684.8	555.2	382.6	788.7	596.3	6.8	695.0
$\lambda_{\texttt{MKT}}$	158.2	332.8	623.7	278.4	1013.8	37.0	77.8	798.7
$\lambda_{\texttt{SMB}}$	684.7	305.5	291.8	77.7	308.4	567.5	82.6	73.7
$\lambda_{\texttt{HML}}$	282.4	538.6	240.7	192.4	111.8	717.6	112.1	115.9

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HXZ4 Model								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
	Average Factor Realizations							
$\overline{f}_{\mathtt{MKT}}$	1115.2	404.8	1212.8	455.0	2014.4	-244.8	37.6	1420.2
$\overline{f}_{\mathtt{ME}}$	1726.6	420.2	-391.4	225.4	-436.2	1435.6	258.2	188.4
$\overline{f}_{\mathtt{I/A}}$	302.8	875.4	657.4	364.4	75.4	1113.4	-56.6	362.6
$\overline{f}_{\mathtt{ROE}}$	359.8	1022.8	921.0	1116.4	731.8	593.2	348.0	186.4
	$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{TS}}$							
λ_0	34.5	37.3	56.9	60.4	73.6	39.1	21.0	43.7
$\lambda_{\mathtt{MKT}}$	13.3	12.4	26.2	16.4	44.9	24.5	15.6	29.1
$\lambda_{\mathtt{ME}}$	39.2	27.1	36.9	60.8	35.3	20.3	9.3	18.0
$\lambda_{\mathtt{I/A}}$	5.4	36.3	38.2	35.4	23.9	8.4	4.9	10.0
$\lambda_{\mathtt{ROE}}$	48.5	30.7	41.9	89.8	52.0	20.5	18.3	35.4
	$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{IT}}$							
λ_0	36.9	36.0	57.9	63.2	70.1	40.9	22.2	43.4
$\lambda_{\mathtt{MKT}}$	13.4	8.6	24.5	15.2	40.1	24.7	16.3	27.6
$\lambda_{\mathtt{ME}}$	43.0	28.6	38.5	65.2	36.2	22.5	10.4	19.3
$\lambda_{\mathtt{I/A}}$	7.6	42.4	44.6	41.4	27.4	10.5	4.8	11.8
$\lambda_{\mathtt{ROE}}$	54.0	33.5	45.4	98.9	56.4	23.4	19.9	39.4
	$\widehat{\boldsymbol{\lambda}}_1$							
λ_0	1067.7	580.1	721.7	514.1	1147.5	946.5	175.2	910.5
$\lambda_{\mathtt{MKT}}$	465.5	129.8	669.9	232.7	1252.1	175.8	107.1	789.7
$\lambda_{\mathtt{ME}}$	946.4	465.5	163.8	322.4	276.5	1087.9	139.0	75.0
$\lambda_{\mathtt{I/A}}$	323.3	695.6	487.8	306.8	109.0	674.5	57.3	278.4
$\lambda_{\mathtt{ROE}}$	445.8	618.8	480.7	848.8	460.9	476.5	217.5	329.4
	$\widehat{\boldsymbol{\lambda}}_2$							
λ_0	860.0	667.8	768.7	548.5	1272.5	699.6	169.9	840.2
$\lambda_{\mathtt{MKT}}$	273.6	256.3	659.3	288.2	1271.0	199.0	59.2	768.2
$\lambda_{\mathtt{ME}}$	958.7	429.5	86.5	299.1	74.1	709.4	206.6	23.3
$\lambda_{\mathtt{I/A}}$	233.8	618.1	527.6	272.7	90.0	449.5	39.6	213.3
$\lambda_{\mathtt{ROE}}$	440.3	642.3	643.7	852.4	547.5	193.9	271.5	200.6

Table 2: **Root mean squared error in the estimation of $\boldsymbol{\lambda}$ with normally distributed shocks: the role of the EIV correction through the IV-GMM approach.** This table presents simulation results on the root mean squared error (RMSE), in annualized basis points, in the estimation of $\boldsymbol{\lambda} = [\ \lambda_0 \quad \boldsymbol{\lambda}_f'\ ]'$, where $\boldsymbol{\lambda}_f = [\ \lambda_1 \quad \cdots \quad \lambda_K\ ]'$ is the vector of ex-post risk premia and K is the number of factors. The shocks u_{it} are assumed to follow a normal distribution and the factor realizations are kept fixed throughout. The number of individual stocks, N, is equal to 1,000 and the number of clusters, M_N, is set equal to 50. The pairwise correlation of shocks, assumed to be constant within each cluster, is set equal to 0.10. The simulation is calibrated to the following three linear asset pricing models: the single-factor CAPM, the three-factor Fama and French (1993) model (FF3), and the four-factor Hou, Xue, and Zhang (2015) model (HXZ4). For the CAPM, $K = 1$ and $\lambda_{\texttt{MKT}}$ is the ex-post risk premia of MKT. For FF3, $K = 3$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{SMB}}$, and $\lambda_{\texttt{HML}}$ are the ex-post risk premia of MKT, SMB, and HML, respectively. For HXZ4, $K = 4$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$ are the ex-post risk premia of MKT, ME, I/A, and ROE, respectively. For the CAPM, we consider the IV estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$, while for the FF3 and HXZ4 models, we consider the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. In addition, we consider two alternative estimators $\widehat{\boldsymbol{\lambda}}_1 = (\widehat{\mathbf{X}}_1'\widehat{\mathbf{X}}_1)^{-1}\widehat{\mathbf{X}}_1'\bar{\mathbf{r}}_2$ and $\widehat{\boldsymbol{\lambda}}_2 = (\widehat{\mathbf{X}}_2'\widehat{\mathbf{X}}_2)^{-1}\widehat{\mathbf{X}}_2'\bar{\mathbf{r}}_2$ that ignore the EIV problem. The results are based on 10,000 Monte Carlo repetitions.

CAPM								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
Average Factor Realizations								
$\bar{f}_{\texttt{MKT}}$	1118.2	453.0	1248.8	483.0	2069.4	-310.0	-43.4	1552.4
$\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$								
λ_0	216.7	231.8	226.6	214.9	272.4	238.0	217.5	234.0
$\lambda_{\texttt{MKT}}$	225.7	241.2	235.9	224.1	280.5	248.2	225.9	242.8
$\widehat{\boldsymbol{\lambda}}_1$								
λ_0	386.7	233.3	489.7	217.7	1012.5	195.2	148.5	590.4
$\lambda_{\texttt{MKT}}$	390.7	233.4	493.6	219.3	1022.9	193.6	146.5	596.3
$\widehat{\boldsymbol{\lambda}}_2$								
λ_0	464.6	224.7	441.3	272.3	894.5	184.1	145.7	726.2
$\lambda_{\texttt{MKT}}$	469.1	224.4	444.9	272.3	904.0	184.0	142.6	733.6

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FF3 Model								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
Average Factor Realizations								
$\overline{f}_{\texttt{MKT}}$	1118.2	453.0	1248.8	483.0	2069.4	-310.0	-43.4	1552.4
$\overline{f}_{\texttt{SMB}}$	1576.0	470.2	-451.6	158.0	-369.4	1002.4	131.4	149.2
$\overline{f}_{\texttt{HML}}$	651.0	931.4	251.8	269.4	-549.4	1510.8	210.8	-88.0
$\widehat{\boldsymbol{\lambda}}^{\texttt{TS}}_{\texttt{IV}}$								
λ_0	311.5	285.5	346.4	335.0	364.3	414.9	307.5	270.7
$\lambda_{\texttt{MKT}}$	278.5	267.3	296.4	274.9	320.1	347.1	269.1	264.9
$\lambda_{\texttt{SMB}}$	190.2	194.0	197.2	194.3	199.9	202.8	173.7	195.0
$\lambda_{\texttt{HML}}$	224.7	231.8	227.3	248.5	244.3	256.1	196.1	218.9
$\widehat{\boldsymbol{\lambda}}^{\texttt{IT}}_{\texttt{IV}}$								
λ_0	312.6	286.1	346.7	335.4	365.1	415.4	308.1	271.2
$\lambda_{\texttt{MKT}}$	279.1	268.0	296.8	275.5	320.7	347.6	269.6	265.4
$\lambda_{\texttt{SMB}}$	190.9	194.7	198.0	195.0	200.8	203.7	174.6	195.6
$\lambda_{\texttt{HML}}$	225.8	233.1	227.9	249.1	245.1	256.8	196.8	219.7
$\widehat{\boldsymbol{\lambda}}_1$								
λ_0	798.4	424.9	653.9	402.9	850.3	772.7	180.4	600.2
$\lambda_{\texttt{MKT}}$	345.4	195.7	758.3	282.4	1150.7	176.6	160.5	701.5
$\lambda_{\texttt{SMB}}$	608.6	183.7	355.6	149.2	389.9	668.9	153.5	115.6
$\lambda_{\texttt{HML}}$	512.5	520.0	240.9	258.2	253.3	920.1	159.9	204.1
$\widehat{\boldsymbol{\lambda}}_2$								
λ_0	625.9	704.1	579.1	414.4	804.1	620.2	171.7	714.9
$\lambda_{\texttt{MKT}}$	225.0	367.7	645.7	315.2	1025.7	171.0	171.7	811.9
$\lambda_{\texttt{SMB}}$	693.8	323.3	309.9	131.9	332.8	584.0	130.8	121.3
$\lambda_{\texttt{HML}}$	303.1	551.6	260.7	220.4	159.0	729.6	158.2	148.2

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Table 2 – continued from previous page

HXZ4 Model								
Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
Average Factor Realizations								
$\overline{f}_{\mathtt{MKT}}$	1115.2	404.8	1212.8	455.0	2014.4	-244.8	37.6	1420.2
$\overline{f}_{\mathtt{ME}}$	1726.6	420.2	-391.4	225.4	-436.2	1435.6	258.2	188.4
$\overline{f}_{\mathtt{I/A}}$	302.8	875.4	657.4	364.4	75.4	1113.4	-56.6	362.6
$\overline{f}_{\mathtt{ROE}}$	359.8	1022.8	921.0	1116.4	731.8	593.2	348.0	186.4
$\widehat{\boldsymbol{\lambda}}^{\mathtt{TS}}_{\mathtt{IV}}$								
λ_0	435.7	406.7	471.7	482.7	522.3	508.4	425.9	470.6
$\lambda_{\mathtt{MKT}}$	337.3	353.8	385.4	357.7	420.5	390.2	344.6	348.2
$\lambda_{\mathtt{ME}}$	329.7	345.7	352.0	384.5	359.9	331.9	256.9	344.4
$\lambda_{\mathtt{I/A}}$	246.4	372.5	313.8	317.2	309.6	239.5	184.6	281.8
$\lambda_{\mathtt{ROE}}$	418.1	488.0	461.4	493.9	503.1	463.9	317.6	440.7
$\widehat{\boldsymbol{\lambda}}^{\mathtt{IT}}_{\mathtt{IV}}$								
λ_0	437.3	406.5	472.8	485.0	523.2	510.0	426.9	471.3
$\lambda_{\mathtt{MKT}}$	338.4	353.7	386.2	358.1	421.2	391.3	345.3	348.7
$\lambda_{\mathtt{ME}}$	331.8	346.2	353.1	385.7	362.0	333.0	258.3	345.1
$\lambda_{\mathtt{I/A}}$	248.3	375.6	317.1	319.7	312.0	241.0	185.7	283.6
$\lambda_{\mathtt{ROE}}$	422.5	488.6	464.1	497.4	506.9	465.7	319.1	443.2
$\widehat{\boldsymbol{\lambda}}_1$								
λ_0	1083.8	612.6	748.2	547.4	1161.3	962.5	240.3	927.0
$\lambda_{\mathtt{MKT}}$	497.2	214.8	693.1	291.6	1262.0	241.1	186.2	805.4
$\lambda_{\mathtt{ME}}$	955.6	481.0	200.0	338.7	299.0	1095.7	200.1	134.1
$\lambda_{\mathtt{I/A}}$	334.6	698.4	494.0	314.7	132.1	680.2	111.4	287.8
$\lambda_{\mathtt{ROE}}$	460.8	628.6	494.3	854.5	471.3	486.5	252.6	346.2
$\widehat{\boldsymbol{\lambda}}_2$								
λ_0	881.3	695.4	792.8	576.2	1283.9	720.2	240.9	860.5
$\lambda_{\mathtt{MKT}}$	319.5	310.7	683.7	327.6	1281.1	254.0	166.9	783.2
$\lambda_{\mathtt{ME}}$	966.0	444.5	137.9	319.6	148.2	724.3	233.8	102.7
$\lambda_{\mathtt{I/A}}$	241.6	622.9	532.6	282.2	126.2	460.4	80.5	223.9
$\lambda_{\mathtt{ROE}}$	453.8	652.8	651.6	857.9	556.0	235.6	290.2	215.4

Table 3: **Empirical rejection frequencies of various tests of ex-post asset pricing implications with normally distributed shocks.** This table presents simulation results on the finite-sample performance of the asset pricing tests based on the IV-GMM estimators of $\boldsymbol{\lambda} = [\ \lambda_0 \ \ \boldsymbol{\lambda}_f'\ ]'$, where $\boldsymbol{\lambda}_f = [\ \lambda_1 \ \cdots \ \lambda_K\ ]'$ is the vector of ex-post risk premia and K is the number of factors. For the CAPM, we consider the IV estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$, while for the FF3 and HXZ4 models, we consider both the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. Reported are the empirical rejection frequencies, in percentages, of (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistic $t(\lambda_k)$, $k = 1, \ldots, K$, that focuses on the k-th factor ex-post risk premium, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistic $t(\phi_j)$, $j = 1, \ldots, J$, that examines the ability of the j-th characteristic to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). The shocks u_{it} are assumed to follow a normal distribution and the factor realizations are kept fixed throughout. We set the number of individual stocks, N, equal to 1,000 and consider two possible values for the number of clusters, M_N, namely 50 and 100. The pairwise correlation of shocks, assumed to be constant within each cluster, is denoted by ρ and takes three values, namely 0, 0.10, and 0.20. Three nominal levels of significance are considered: 1%, 5%, and 10%. The simulation is calibrated to the following three linear asset pricing models: the single-factor CAPM, the three-factor Fama and French (1993) model (FF3), and the four-factor Hou, Xue, and Zhang (2015) model (HXZ4). For the CAPM, $K = 1$ and $\lambda_{\texttt{MKT}}$ is the ex-post risk premia of MKT. For the FF3 model, $K = 3$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{SMB}}$, and $\lambda_{\texttt{HML}}$ are the ex-post risk premia of MKT, SMB, and HML, respectively. For the HXZ4 model, $K = 4$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$ are the ex-post risk premia of MKT, ME, I/A, and ROE, respectively. The pretesting and testing periods, both consisting of 60 months, are from 2005 to 2009 and from 2010 to 2014, respectively. The results are based on 10,000 Monte Carlo repetitions.

	CAPM																	
ρ	0						0.10						0.20					
M_N	50			100			50			100			50			100		
Nominal Size (%)	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
	$\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$																	
$t\,(\lambda_0)$	1.2	5.9	11.1	1.2	5.0	10.0	1.3	5.6	10.7	1.1	5.5	10.7	1.2	5.2	10.1	1.0	5.0	10.0
$t\,(\lambda_{\texttt{MKT}})$	1.4	5.9	10.9	1.2	5.4	9.9	1.3	5.7	10.8	1.2	5.6	10.9	1.3	5.3	10.6	1.1	4.9	9.8
$J_d(\boldsymbol{\lambda})$	1.3	5.9	11.1	1.2	5.3	10.1	1.4	5.7	10.9	1.2	5.6	10.9	1.2	5.3	10.6	1.0	4.9	10.2
$t\,(\phi_{\texttt{SIZE}})$	0.7	4.3	9.5	0.8	4.7	9.6	0.7	4.4	9.4	1.0	4.5	9.5	0.9	4.7	9.9	0.6	4.6	9.3
$t\,(\phi_{\texttt{BTM}})$	0.7	4.5	9.4	0.8	4.5	9.6	0.8	4.2	9.2	0.9	4.7	9.8	0.7	4.0	9.0	0.7	4.5	9.3
$J_d(\boldsymbol{\phi})$	0.9	5.3	10.3	1.0	5.2	9.9	0.9	4.9	10.1	1.0	5.1	10.1	1.1	5.2	10.2	0.8	4.8	9.6
$t\,(\boldsymbol{\alpha})$	1.1	5.1	10.3	1.4	5.5	10.2	1.4	5.5	10.1	1.5	5.7	10.8	1.7	6.1	11.3	1.5	6.0	10.9
$J_d(\boldsymbol{\delta})$	1.0	4.8	10.0	1.1	5.2	9.8	0.9	4.8	10.1	1.1	5.5	10.6	0.9	4.8	10.2	0.8	4.7	9.8

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Table 3 – continued from previous page

FF3 Model																		
ρ	0						0.10						0.20					
M_N	50			100			50			100			50			100		
Nominal Size (%)	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{TS}}$																		
$t\,(\lambda_0)$	1.4	5.7	11.2	1.2	5.5	10.6	1.3	5.6	11.1	1.2	5.7	10.7	1.4	6.2	11.2	1.2	5.6	11.2
$t\,(\lambda_{\mathtt{MKT}})$	1.5	5.6	11.0	1.3	5.3	10.7	1.3	6.0	11.2	1.0	5.3	11.0	1.4	6.1	11.1	1.1	5.6	11.0
$t\,(\lambda_{\mathtt{SMB}})$	1.2	5.2	10.3	1.1	5.0	10.0	1.3	5.8	11.2	1.0	5.3	10.6	1.4	5.9	10.8	1.1	5.0	9.9
$t\,(\lambda_{\mathtt{HML}})$	1.4	6.0	11.1	1.2	5.6	10.8	1.2	5.7	11.2	1.1	5.4	10.6	1.3	6.0	11.7	1.2	5.6	11.0
$J_d(\boldsymbol{\lambda})$	1.3	5.8	11.4	1.0	5.6	10.8	1.2	5.8	11.5	1.1	5.4	10.9	1.4	5.9	11.9	1.1	5.5	11.2
$t\,(\phi_{\mathtt{SIZE}})$	0.5	3.9	8.7	0.7	4.5	9.5	0.4	3.7	8.4	0.6	4.1	9.1	0.6	3.6	8.4	0.7	4.2	9.4
$t\,(\phi_{\mathtt{BTM}})$	0.6	3.8	8.0	0.8	4.6	9.4	0.5	3.8	8.6	0.6	4.1	9.2	0.5	3.5	8.2	0.7	4.3	9.3
$J_d(\boldsymbol{\phi})$	1.1	4.8	10.0	1.0	4.9	9.8	0.8	4.6	9.6	0.8	5.1	10.0	1.0	5.0	10.1	1.0	4.9	9.8
$t\,(\boldsymbol{\alpha})$	1.0	4.7	9.9	1.3	5.3	10.2	1.1	5.0	10.2	1.4	5.4	10.6	1.3	5.6	11.0	1.5	5.7	10.7
$J_d(\boldsymbol{\delta})$	0.9	4.6	9.8	1.0	4.9	9.9	0.8	4.7	9.7	1.0	4.6	9.8	1.1	4.7	10.0	0.8	5.0	10.5
$\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{IT}}$																		
$t\,(\lambda_0)$	1.5	5.8	11.2	1.2	5.6	10.5	1.3	5.6	11.2	1.2	5.6	10.6	1.4	6.2	11.2	1.2	5.5	11.2
$t\,(\lambda_{\mathtt{MKT}})$	1.6	5.6	10.9	1.2	5.3	10.7	1.4	6.1	11.3	1.1	5.4	10.9	1.4	6.1	11.2	1.1	5.5	10.9
$t\,(\lambda_{\mathtt{SMB}})$	1.2	5.3	10.3	1.1	5.0	10.1	1.4	5.8	11.3	1.0	5.4	10.6	1.4	5.9	11.0	1.1	5.0	9.9
$t\,(\lambda_{\mathtt{HML}})$	1.4	6.0	11.3	1.2	5.7	10.7	1.3	5.9	11.4	1.1	5.5	10.6	1.4	6.1	11.9	1.2	5.6	11.0
$J_d(\boldsymbol{\lambda})$	1.3	5.9	11.4	1.0	5.7	10.9	1.3	6.0	11.7	1.1	5.4	10.9	1.5	6.1	12.0	1.0	5.5	11.2
$t\,(\phi_{\mathtt{SIZE}})$	0.5	3.8	8.4	0.7	4.4	9.3	0.4	3.6	8.2	0.6	4.0	8.9	0.6	3.4	8.2	0.6	4.2	9.4
$t\,(\phi_{\mathtt{BTM}})$	0.5	3.7	7.8	0.8	4.4	9.3	0.5	3.7	8.4	0.7	4.0	9.1	0.5	3.4	8.0	0.7	4.3	9.2
$J_d(\boldsymbol{\phi})$	1.1	4.9	10.0	1.0	4.9	9.7	0.8	4.8	9.7	0.8	4.9	9.9	1.1	4.9	10.0	1.0	4.8	9.8
$t\,(\boldsymbol{\alpha})$	1.0	4.7	9.9	1.3	5.3	10.2	1.1	5.1	10.2	1.3	5.4	10.5	1.3	5.6	10.9	1.5	5.7	10.7
$J_d(\boldsymbol{\delta})$	0.9	4.7	9.8	0.9	4.8	9.8	0.9	4.8	9.7	1.0	4.6	9.8	1.2	4.8	10.1	0.8	4.8	10.5

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Table 3 – continued from previous page

	HXZ4 Model																	
ρ	0						0.10						0.20					
M_N	50			100			50			100			50			100		
Nominal Size (%)	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
$\widehat{\boldsymbol{\lambda}}^{\mathtt{TS}}_{\mathtt{IV}}$																		
$t(\lambda_0)$	1.1	5.1	10.1	0.8	4.4	9.0	1.0	4.8	9.5	0.9	4.7	9.2	1.0	4.7	9.9	0.8	4.3	9.2
$t(\lambda_{\mathtt{MKT}})$	1.1	5.0	10.1	0.8	4.4	8.8	1.1	4.9	10.0	0.9	4.8	9.3	1.0	5.0	9.8	0.9	4.3	9.3
$t(\lambda_{\mathtt{ME}})$	0.7	3.8	8.6	0.4	3.8	8.4	0.8	4.0	8.9	0.5	3.7	8.3	0.6	4.2	9.0	0.5	3.7	8.3
$t(\lambda_{\mathtt{I/A}})$	1.1	4.9	10.0	0.8	4.7	9.6	0.9	4.7	9.6	0.7	4.1	8.9	0.8	4.4	9.2	0.7	4.1	9.1
$t(\lambda_{\mathtt{ROE}})$	0.7	4.3	9.1	0.7	4.0	8.9	0.7	4.5	9.7	0.7	4.1	8.7	0.8	4.3	9.3	0.6	4.0	8.9
$J_d(\boldsymbol{\lambda})$	0.8	4.2	9.3	0.6	3.7	8.2	0.8	4.3	9.2	0.7	3.8	8.1	0.9	4.0	8.8	0.6	3.5	7.9
$t(\phi_{\mathtt{SIZE}})$	0.3	3.0	7.2	0.5	3.5	7.8	0.3	2.6	6.7	0.5	3.3	7.6	0.3	2.8	6.6	0.5	3.3	7.9
$t(\phi_{\mathtt{I/A}})$	0.4	3.1	7.1	0.5	3.6	8.1	0.4	2.7	6.8	0.5	3.3	7.3	0.3	2.9	7.4	0.5	3.6	7.7
$t(\phi_{\mathtt{ROE}})$	0.3	3.0	7.2	0.5	3.5	8.0	0.3	2.7	6.5	0.5	3.6	7.7	0.3	2.4	6.6	0.6	3.2	7.9
$J_d(\boldsymbol{\phi})$	0.3	2.4	6.4	0.4	3.0	6.8	0.4	2.4	6.1	0.3	2.8	6.7	0.3	2.5	5.9	0.3	2.5	6.4
$t(\boldsymbol{\alpha})$	0.9	4.6	10.0	1.0	5.4	10.6	0.8	4.6	9.7	1.1	5.3	11.0	1.1	5.0	10.2	1.3	5.9	11.2
$J_d(\boldsymbol{\delta})$	0.6	3.1	6.8	0.5	2.9	7.1	0.5	2.9	6.5	0.4	2.9	7.0	0.5	2.9	6.9	0.5	2.9	6.6
$\widehat{\boldsymbol{\lambda}}^{\mathtt{IT}}_{\mathtt{IV}}$																		
$t(\lambda_0)$	1.1	5.0	10.2	0.8	4.4	9.1	1.1	5.0	9.6	0.9	4.7	9.2	1.0	4.9	9.7	0.8	4.3	9.4
$t(\lambda_{\mathtt{MKT}})$	1.2	5.1	10.3	0.8	4.4	8.9	1.1	5.0	10.1	0.9	4.9	9.4	1.1	5.2	9.9	0.9	4.4	9.5
$t(\lambda_{\mathtt{ME}})$	0.7	4.0	8.7	0.4	3.8	8.4	0.8	4.1	9.1	0.5	3.6	8.2	0.7	4.3	9.1	0.6	3.9	8.1
$t(\lambda_{\mathtt{I/A}})$	1.2	5.2	10.3	0.8	4.8	9.8	1.0	4.9	9.9	0.7	4.2	8.9	0.8	4.6	9.7	0.7	4.2	9.2
$t(\lambda_{\mathtt{ROE}})$	0.8	4.5	9.4	0.8	4.1	8.9	0.8	4.8	9.9	0.7	4.1	8.6	0.8	4.5	9.5	0.5	4.1	9.0
$J_d(\boldsymbol{\lambda})$	0.9	4.4	9.6	0.6	3.8	8.2	0.8	4.5	9.5	0.7	3.9	8.1	0.9	4.2	9.0	0.6	3.6	8.0
$t(\phi_{\mathtt{SIZE}})$	0.3	3.1	7.3	0.5	3.4	7.7	0.3	2.5	6.6	0.5	3.3	7.6	0.2	2.8	6.8	0.5	3.2	7.6
$t(\phi_{\mathtt{I/A}})$	0.3	2.9	7.0	0.5	3.6	8.0	0.3	2.7	6.6	0.5	3.3	7.3	0.3	2.9	7.3	0.5	3.5	7.7
$t(\phi_{\mathtt{ROE}})$	0.2	2.9	7.0	0.5	3.4	8.0	0.3	2.7	6.4	0.5	3.4	7.6	0.3	2.4	6.4	0.5	3.2	7.6
$J_d(\boldsymbol{\phi})$	0.4	2.6	6.7	0.4	2.9	6.8	0.4	2.6	6.1	0.3	2.9	6.7	0.3	2.6	6.0	0.3	2.5	6.5
$t(\boldsymbol{\alpha})$	0.9	4.5	9.9	1.0	5.4	10.5	0.8	4.6	9.7	1.1	5.3	11.0	1.1	5.0	9.9	1.3	5.8	11.0
$J_d(\boldsymbol{\delta})$	0.6	3.2	7.2	0.6	2.9	7.0	0.5	3.1	6.9	0.4	3.0	7.1	0.5	3.0	6.9	0.5	3.0	6.6

Table 4: **Empirical rejection frequencies of IV asset pricing tests with normally distributed shocks based on the Fama-MacBeth variance-covariance estimator.** This table presents simulation results on the finite-sample performance of the asset pricing tests based on the IV-GMM estimators of $\boldsymbol{\lambda} = [\ \lambda_0 \ \ \boldsymbol{\lambda}_f'\ ]'$, where $\boldsymbol{\lambda}_f = [\ \lambda_1 \ \ \cdots \ \ \lambda_K\ ]'$ is the vector of ex-post risk premia and K is the number of factors. The weighting matrix used is the identity matrix ($\mathbf{W} = \mathbf{I}_{K+L+1}$) and the variance-covariance of the ex-post risk premia estimator is estimated using the approach of Fama and MacBeth (1973). Reported are the empirical rejection frequencies, in percentages, of (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistic $t(\lambda_k)$, $k = 1, \ldots, K$, that focuses on the k-th factor ex-post risk premium, given in (87); and (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91). The shocks u_{it} are assumed to follow a normal distribution and the factor realizations are kept fixed throughout. We set the number of individual stocks, N, equal to 1,000 and consider two possible values for the number of clusters, M_N, namely 50 and 100. The pairwise correlation of shocks, assumed to be constant within each cluster, is denoted by ρ and takes three values, namely 0, 0.10, and 0.20. Three nominal levels of significance are considered: 1%, 5%, and 10%. The simulation is calibrated to the following three linear asset pricing models: the single-factor CAPM, the three-factor Fama and French (1993) model (FF3), and the four-factor Hou, Xue, and Zhang (2015) model (HXZ4). For the CAPM, $K = 1$ and $\lambda_{\texttt{MKT}}$ is the ex-post risk premia of MKT. For the FF3 model, $K = 3$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{SMB}}$, and $\lambda_{\texttt{HML}}$ are the ex-post risk premia of MKT, SMB, and HML, respectively. For the HXZ4 model, $K = 4$ and $\lambda_{\texttt{MKT}}$, $\lambda_{\texttt{ME}}$, $\lambda_{\texttt{I/A}}$, and $\lambda_{\texttt{ROE}}$ are the ex-post risk premia of MKT, ME, I/A, and ROE, respectively. The pretesting and testing periods, both consisting of 60 months, are from from 2000 to 2004 and from 2005 to 2009, respectively, in Panels A-1, B-1, and C-1 and from 2005 to 2009 and from 2010 to 2014, respectively, in Panels A-2, B-2, and C-2. The results are based on 10,000 Monte Carlo repetitions.

ρ	0						0.10						0.20					
M_N	50			100			50			100			50			100		
Nominal Size (%)	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10	1	5	10
Panel A: CAPM																		
A-1: Pretesting Period of 2000-2004 and Testing Period of 2005-2009																		
$t(\lambda_0)$	1.2	5.6	10.5	1.3	5.9	11.3	1.5	5.7	10.8	1.3	5.5	10.7	1.4	5.9	11.3	1.4	5.7	10.6
$t(\lambda_{\texttt{MKT}})$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$J_d(\boldsymbol{\lambda})$	0.4	1.9	4.3	0.5	2.3	4.6	0.4	2.2	4.4	0.4	2.0	4.4	0.4	2.3	4.7	0.5	2.5	4.7
A-2: Pretesting Period of 2005-2009 and Testing Period of 2010-2014																		
$t(\lambda_0)$	2.1	7.0	12.6	1.9	7.1	13.2	1.7	6.8	12.5	1.6	6.6	12.6	1.9	7.2	12.8	1.8	7.4	13.1
$t(\lambda_{\texttt{MKT}})$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$J_d(\boldsymbol{\lambda})$	0.7	3.3	5.7	0.7	3.0	5.9	0.8	3.1	5.8	0.7	2.8	5.5	0.7	3.0	6.1	0.7	3.4	6.3

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Table 4 – continued from previous page

Panel B: FF3 Model																		
B-1: Pretesting Period of 2000-2004 and Testing Period of 2005-2009																		
$t(\lambda_0)$	1.4	6.3	11.4	1.7	6.8	12.0	1.6	6.1	11.5	1.5	6.3	11.5	1.6	6.0	11.1	1.6	6.0	11.8
$t(\lambda_{\mathtt{MKT}})$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$t(\lambda_{\mathtt{SMB}})$	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.1	0.1	0.0	0.0	0.1
$t(\lambda_{\mathtt{HML}})$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$J_d(\boldsymbol{\lambda})$	0.2	0.7	1.5	0.1	0.6	1.7	0.1	0.7	1.6	0.1	0.6	1.5	0.1	0.7	1.7	0.1	0.7	1.7
B-2: Pretesting Period of 2005-2009 and Testing Period of 2010-2014																		
$t(\lambda_0)$	2.1	8.3	14.4	2.3	7.9	13.9	2.4	8.1	13.7	2.5	8.6	14.4	2.1	8.1	14.0	2.3	8.0	14.3
$t(\lambda_{\mathtt{MKT}})$	0.0	0.0	0.2	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.2	0.0	0.0	0.2
$t(\lambda_{\mathtt{SMB}})$	0.0	0.8	2.2	0.1	0.7	2.2	0.1	0.6	1.8	0.0	0.7	2.0	0.1	0.9	2.7	0.1	0.8	2.2
$t(\lambda_{\mathtt{HML}})$	0.0	0.5	1.5	0.0	0.4	1.5	0.0	0.4	1.5	0.0	0.4	1.6	0.0	0.5	1.9	0.1	0.4	1.7
$J_d(\boldsymbol{\lambda})$	0.3	1.6	3.3	0.4	1.8	3.5	0.3	1.8	3.7	0.4	1.9	3.9	0.4	2.0	3.7	0.4	1.9	3.6
Panel C: HXZ4 Model																		
C-1: Pretesting Period of 2000-2004 and Testing Period of 2005-2009																		
$t(\lambda_0)$	1.6	6.8	12.4	1.7	6.9	12.4	1.7	6.4	12.1	1.7	6.7	12.0	1.7	6.7	12.1	1.9	6.9	12.3
$t(\lambda_{\mathtt{MKT}})$	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.1	0.2	0.0	0.1	0.3	0.0	0.1	0.3
$t(\lambda_{\mathtt{ME}})$	0.0	0.1	0.3	0.0	0.0	0.3	0.0	0.1	0.4	0.0	0.1	0.3	0.0	0.1	0.5	0.0	0.1	0.5
$t(\lambda_{\mathtt{I/A}})$	0.0	0.3	1.3	0.0	0.3	1.1	0.1	0.4	1.5	0.1	0.4	1.3	0.1	0.5	1.6	0.0	0.4	1.3
$t(\lambda_{\mathtt{ROE}})$	0.0	0.3	1.0	0.0	0.2	0.9	0.0	0.3	1.0	0.0	0.3	0.9	0.0	0.3	1.3	0.0	0.4	1.2
$J_d(\boldsymbol{\lambda})$	0.1	0.9	2.0	0.1	0.9	2.1	0.2	0.9	2.3	0.2	0.9	2.0	0.2	1.3	2.5	0.2	1.2	2.4
C-2: Pretesting Period of 2005-2009 and Testing Period of 2010-2014																		
$t(\lambda_0)$	2.8	8.5	14.6	2.6	8.2	14.3	2.4	8.5	14.8	2.5	8.5	14.0	2.5	8.3	14.2	2.6	8.5	14.5
$t(\lambda_{\mathtt{MKT}})$	0.0	0.3	0.9	0.0	0.1	0.8	0.0	0.3	1.0	0.0	0.2	0.9	0.0	0.4	1.4	0.0	0.3	1.0
$t(\lambda_{\mathtt{ME}})$	0.2	1.5	3.8	0.2	1.4	3.8	0.2	1.7	4.0	0.3	1.6	4.1	0.2	1.8	4.7	0.2	2.0	4.8
$t(\lambda_{\mathtt{I/A}})$	0.3	2.2	5.4	0.4	2.4	5.5	0.4	2.5	5.8	0.4	2.4	5.4	0.5	3.1	6.6	0.5	2.9	6.3
$t(\lambda_{\mathtt{ROE}})$	0.7	4.0	8.1	0.7	3.9	8.4	0.8	3.9	8.2	0.8	3.8	8.1	0.9	4.4	9.0	1.0	4.6	9.2
$J_d(\boldsymbol{\lambda})$	1.0	3.9	6.9	0.9	3.7	6.8	0.8	3.7	6.7	0.9	3.7	6.9	1.2	4.3	7.8	0.9	4.2	7.6

Table 5: **Betas of decile portfolios sorted by characteristics.** In this table, we consider decile portfolios sorted on a characteristic and present the beta estimates of these portfolios with respect to the corresponding spread factor within the context of the three asset pricing models we empirically examine: the FF3 model, the HXZ4 model, and the FF5 model. For each asset pricing model, the decile portfolio betas are estimated jointly for all factors using data from 07/1970 to 12/2014.

			Decile Portfolios Sorted by Characteristic									
			LOW									HIGH
Model	Factor	Characteristic	1	2	3	4	5	6	7	8	9	10
FF3	SMB	SIZE	1.19	1.10	0.92	0.81	0.69	0.49	0.38	0.28	0.07	-0.29
	HML	BTM	-0.50	-0.10	0.04	0.28	0.33	0.36	0.52	0.69	0.69	0.96
HXZ4	ME	SIZE	1.04	1.00	0.86	0.79	0.66	0.48	0.37	0.26	0.08	-0.27
	I/A	I/A	0.37	0.45	0.29	0.14	0.11	-0.03	-0.12	-0.30	-0.56	-0.42
	ROE	ROE	-0.33	-0.25	0.01	0.01	-0.15	0.06	0.09	0.10	0.23	0.32
FF5	SMB	SIZE	1.12	1.06	0.92	0.83	0.69	0.50	0.38	0.25	0.06	-0.28
	HML	BTM	-0.43	-0.19	-0.06	0.22	0.24	0.35	0.44	0.70	0.69	0.97
	RMW	OP	-0.90	-0.41	-0.27	-0.16	0.01	-0.05	0.07	0.21	0.35	0.43
	CMA	AG	0.63	0.67	0.68	0.27	0.22	0.10	0.04	-0.14	-0.62	-0.53

Table 6: **Testing the CAPM: empirical results for 5-year testing periods from 1975 to 2014 using 49 industry clusters.** This table presents the point estimates of $\boldsymbol{\lambda} = [\ \lambda_0\ \ \lambda_{\texttt{MKT}}\ ]'$ and the various statistics along with the corresponding p-values for testing the implications of the CAPM. We consider 8 non-overlapping 5-year testing periods from 1975 to 2014. For each testing period, the pretesting period consists of the preceding five years. We report point estimates, in annualized percentages, based on the IV estimator $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$. We further report (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistic $t(\lambda_{\texttt{MKT}})$, that focuses on the MKT ex-post risk premium, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistics $t\left(\phi_{\texttt{SIZE}}\right)$ and $t\left(\phi_{\texttt{BTM}}\right)$, that examine the ability of the SIZE and BTM characteristics to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of the SIZE and BTM characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). The corresponding p-values are reported in square brackets below the test statistics.

Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Number of Test Assets				
N	1204	1733	1704	1941	1959	2090	1994	1920
				Average Factor Realizations				
$\overline{f}_{\texttt{MKT}}$	11.18	4.53	12.49	4.83	20.69	-3.10	-0.43	15.52
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}$				
λ_0	-3.22	4.86	25.40	0.23	4.13	18.72	-8.59	13.48
$\lambda_{\texttt{MKT}}$	20.33	4.70	-14.90	9.18	11.99	0.06	11.20	4.05
				Test Statistics				
$t\left(\lambda_0\right)$	-1.48	2.21	4.79	0.15	2.19	10.85	-0.97	9.59
p-value	[0.14]	[0.03]	[0.00]	[0.88]	[0.03]	[0.00]	[0.33]	[0.00]
$t\left(\lambda_{\texttt{MKT}}\right)$	4.85	0.09	-5.34	2.46	-3.77	1.79	1.87	-8.59
p-value	[0.00]	[0.93]	[0.00]	[0.01]	[0.00]	[0.07]	[0.06]	[0.00]
$J_d(\boldsymbol{\lambda})$	25.68	4.88	51.45	6.08	18.98	120.94	4.42	165.78
p-value	[0.00]	[0.11]	[0.00]	[0.08]	[0.00]	[0.00]	[0.13]	[0.00]
$t\left(\phi_{\texttt{SIZE}}\right)$	-4.60	-2.94	2.39	-3.83	-2.14	-4.37	-1.40	-1.99
p-value	[0.00]	[0.00]	[0.02]	[0.00]	[0.03]	[0.00]	[0.16]	[0.05]
$t\left(\phi_{\texttt{BTM}}\right)$	1.60	2.24	-0.32	-0.41	-0.48	0.04	-0.18	0.02
p-value	[0.11]	[0.03]	[0.75]	[0.68]	[0.63]	[0.97]	[0.86]	[0.98]
$J_d(\boldsymbol{\phi})$	44.62	8.63	34.20	20.71	18.77	22.31	5.44	77.69
p-value	[0.00]	[0.01]	[0.00]	[0.00]	[0.00]	[0.00]	[0.07]	[0.00]
$t\left(\boldsymbol{\alpha}\right)$	-2.84	-1.66	1.57	-4.49	-3.50	-4.10	-3.99	-2.44
p-value	[0.00]	[0.10]	[0.12]	[0.00]	[0.00]	[0.00]	[0.00]	[0.01]
$J_d(\boldsymbol{\delta})$	57.45	21.27	59.71	41.07	36.06	156.82	22.33	175.69
p-value	[0.00]	[0.01]	[0.00]	[0.00]	[0.00]	[0.00]	[0.01]	[0.00]

Table 7: **Testing the FF3 model: empirical results for 5-year testing periods from 1975 to 2014 using 49 industry clusters.** This table presents the point estimates of $\boldsymbol{\lambda} = [\ \lambda_0 \quad \lambda_{\texttt{MKT}} \quad \lambda_{\texttt{SMB}} \quad \lambda_{\texttt{HML}}\ ]'$ and the various statistics along with the corresponding p-values for testing the implications of the FF3 model. We consider 8 non-overlapping 5-year testing periods from 1975 to 2014. For each testing period, the pretesting period consists of the preceding five years. We report point estimates, in annualized percentages, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. We further report (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistics $t(\lambda_{\texttt{MKT}})$, $t(\lambda_{\texttt{SMB}})$, and $t(\lambda_{\texttt{HML}})$, that focus on the MKT, SMB, and HML ex-post risk premia, respectively, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistics $t\,(\phi_{\texttt{SIZE}})$ and $t\,(\phi_{\texttt{BTM}})$, that examine the ability of the SIZE and BTM characteristics to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of the SIZE and BTM characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). The corresponding p-values are reported in square brackets below the test statistics.

Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Number of Test Assets				
N	1204	1733	1704	1941	1959	2090	1994	1920
				Average Factor Realizations				
$\overline{f}_{\texttt{MKT}}$	11.18	4.53	12.49	4.83	20.69	-3.10	-0.43	15.52
$\overline{f}_{\texttt{SMB}}$	15.76	4.70	-4.52	1.58	-3.69	10.02	1.31	1.49
$\overline{f}_{\texttt{HML}}$	6.51	9.31	2.52	2.69	-5.49	15.11	2.11	-0.88
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$				
λ_0	5.86	2.00	13.85	3.54	4.70	17.08	-10.93	11.96
$\lambda_{\texttt{MKT}}$	6.62	0.79	-1.15	1.85	9.64	-11.99	5.83	2.08
$\lambda_{\texttt{SMB}}$	15.61	8.46	-3.78	5.37	4.89	23.34	12.11	5.85
$\lambda_{\texttt{HML}}$	1.49	11.83	3.06	-2.27	-5.27	8.70	7.53	2.23
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$				
λ_0	5.56	1.10	18.49	2.61	4.78	18.94	-10.93	11.82
$\lambda_{\texttt{MKT}}$	6.89	2.28	-2.56	2.43	9.52	-13.53	5.79	2.23
$\lambda_{\texttt{SMB}}$	15.60	8.63	-3.48	4.98	4.96	24.06	12.17	5.81
$\lambda_{\texttt{HML}}$	1.56	10.89	-6.81	-1.03	-5.32	8.28	7.46	2.16

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Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
					Test Statistics: $\widehat{\boldsymbol{\lambda}}^{\mathtt{TS}}_{\mathtt{IV}}$			
$t\,(\lambda_0)$	2.08	0.46	2.64	2.11	1.16	5.47	-1.62	10.76
p-value	[0.04]	[0.65]	[0.01]	[0.03]	[0.25]	[0.00]	[0.11]	[0.00]
$t\,(\lambda_{\mathtt{MKT}})$	-1.71	-0.96	-2.94	-1.98	-3.68	-3.87	0.91	-9.34
p-value	[0.09]	[0.34]	[0.00]	[0.05]	[0.00]	[0.00]	[0.36]	[0.00]
$t\,(\lambda_{\mathtt{SMB}})$	-0.12	3.54	0.52	4.55	5.45	5.63	2.11	2.51
p-value	[0.90]	[0.00]	[0.60]	[0.00]	[0.00]	[0.00]	[0.03]	[0.01]
$t\,(\lambda_{\mathtt{HML}})$	-2.08	0.60	0.14	-2.22	0.09	-2.36	0.99	1.28
p-value	[0.04]	[0.55]	[0.89]	[0.03]	[0.93]	[0.02]	[0.32]	[0.20]
$J_d(\boldsymbol{\lambda})$	11.57	14.03	15.88	34.04	44.63	82.12	8.87	211.02
p-value	[0.06]	[0.03]	[0.02]	[0.00]	[0.00]	[0.00]	[0.10]	[0.00]
$t\,(\phi_{\mathtt{SIZE}})$	0.27	-1.30	0.51	-2.77	-0.87	0.44	0.72	0.77
p-value	[0.79]	[0.20]	[0.61]	[0.01]	[0.39]	[0.66]	[0.47]	[0.44]
$t\,(\phi_{\mathtt{BTM}})$	1.28	-0.97	0.58	1.33	0.20	-1.01	0.41	-0.43
p-value	[0.20]	[0.33]	[0.56]	[0.18]	[0.84]	[0.31]	[0.68]	[0.66]
$J_d(\boldsymbol{\phi})$	4.39	2.04	0.28	12.59	0.76	5.77	1.50	2.24
p-value	[0.11]	[0.36]	[0.87]	[0.00]	[0.68]	[0.06]	[0.47]	[0.33]
$t\,(\boldsymbol{\alpha})$	-2.66	-0.90	0.21	-4.28	-2.08	-4.16	0.84	-1.77
p-value	[0.01]	[0.37]	[0.84]	[0.00]	[0.04]	[0.00]	[0.40]	[0.08]
$J_d(\boldsymbol{\delta})$	20.35	17.47	16.52	61.78	49.74	100.62	10.27	214.94
p-value	[0.03]	[0.05]	[0.06]	[0.00]	[0.00]	[0.00]	[0.21]	[0.00]

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Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Test Statistics: $\widehat{\boldsymbol{\lambda}}^{\mathtt{IT}}_{\mathtt{IV}}$				
$t\,(\lambda_0)$	1.98	0.24	3.86	1.57	1.18	5.42	-1.62	10.73
p-value	[0.05]	[0.81]	[0.00]	[0.12]	[0.24]	[0.00]	[0.11]	[0.00]
$t\,(\lambda_{\mathtt{MKT}})$	-1.62	-0.58	-3.40	-1.64	-3.71	-3.96	0.91	-9.25
p-value	[0.11]	[0.56]	[0.00]	[0.10]	[0.00]	[0.00]	[0.36]	[0.00]
$t\,(\lambda_{\mathtt{SMB}})$	-0.14	3.56	1.07	4.06	5.50	5.63	2.12	2.49
p-value	[0.89]	[0.00]	[0.28]	[0.00]	[0.00]	[0.00]	[0.03]	[0.01]
$t\,(\lambda_{\mathtt{HML}})$	-2.04	0.36	-2.20	-1.73	0.07	-2.42	0.97	1.26
p-value	[0.04]	[0.72]	[0.03]	[0.08]	[0.94]	[0.02]	[0.33]	[0.21]
$J_d(\boldsymbol{\lambda})$	10.73	13.20	32.45	24.65	45.42	82.61	8.88	208.52
p-value	[0.07]	[0.04]	[0.00]	[0.00]	[0.00]	[0.00]	[0.10]	[0.00]
$t\,(\phi_{\mathtt{SIZE}})$	0.43	-0.40	1.67	-2.66	-0.65	1.34	0.74	0.74
p-value	[0.67]	[0.69]	[0.09]	[0.01]	[0.51]	[0.18]	[0.46]	[0.46]
$t\,(\phi_{\mathtt{BTM}})$	1.28	-0.50	2.51	-2.21	0.26	-0.91	0.43	-0.41
p-value	[0.20]	[0.62]	[0.01]	[0.03]	[0.79]	[0.36]	[0.67]	[0.68]
$J_d(\boldsymbol{\phi})$	4.36	0.29	7.63	10.05	0.43	7.67	1.49	2.13
p-value	[0.11]	[0.86]	[0.02]	[0.01]	[0.80]	[0.02]	[0.47]	[0.35]
$t\,(\boldsymbol{\alpha})$	-2.65	-0.94	-0.57	-4.30	-2.08	-4.08	0.84	-1.77
p-value	[0.01]	[0.35]	[0.57]	[0.00]	[0.04]	[0.00]	[0.40]	[0.08]
$J_d(\boldsymbol{\delta})$	19.59	14.48	41.86	55.07	50.24	101.85	10.31	212.39
p-value	[0.03]	[0.09]	[0.00]	[0.00]	[0.00]	[0.00]	[0.21]	[0.00]

Table 8: **Testing the HXZ4 model: empirical results for 5-year testing periods from 1975 to 2014 using 49 industry clusters.** This table presents the point estimates of $\boldsymbol{\lambda} = [\ \lambda_0 \ \ \lambda_{\texttt{MKT}} \ \ \lambda_{\texttt{ME}} \ \ \lambda_{\texttt{I/A}} \ \ \lambda_{\texttt{ROE}}\ ]'$ and the various statistics along with the corresponding p-values for testing the implications of the HXZ4 model. We consider 8 non-overlapping 5-year testing periods from 1975 to 2014. For each testing period, the pretesting period consists of the preceding five years. We report point estimates, in annualized percentages, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$. We further report (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistics $t(\lambda_{\texttt{MKT}})$, $t(\lambda_{\texttt{ME}})$, $t(\lambda_{\texttt{I/A}})$, and $t(\lambda_{\texttt{ROE}})$, that focus on the MKT, ME, I/A, and ROE ex-post risk premia, respectively, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistics $t\,(\phi_{\texttt{SIZE}})$, $t\,(\phi_{\texttt{I/A}})$, and $t\,(\phi_{\texttt{ROE}})$, that examine the ability of the SIZE, I/A, and ROE characteristics to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of the SIZE, I/A, and ROE characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). The corresponding p-values are reported in square brackets below the test statistics.

Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Number of Test Assets				
N	1059	1597	1526	1709	1627	1737	1609	1559
				Average Factor Realizations				
$\overline{f}_{\texttt{MKT}}$	11.15	4.05	12.13	4.55	20.14	-2.45	0.38	14.20
$\overline{f}_{\texttt{ME}}$	17.27	4.20	-3.91	2.25	-4.36	14.36	2.58	1.88
$\overline{f}_{\texttt{I/A}}$	3.03	8.75	6.57	3.64	0.75	11.13	-0.57	3.63
$\overline{f}_{\texttt{ROE}}$	3.60	10.23	9.21	11.16	7.32	5.93	3.48	1.86
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{TS}}$				
λ_0	4.52	2.11	8.41	7.18	6.13	21.07	0.58	14.47
$\lambda_{\texttt{MKT}}$	7.90	2.32	5.47	-4.58	7.51	-7.48	1.43	2.03
$\lambda_{\texttt{ME}}$	18.02	7.30	-1.02	-0.53	4.65	27.46	9.63	0.42
$\lambda_{\texttt{I/A}}$	-1.64	12.83	0.95	-10.07	-4.94	-1.81	1.05	14.21
$\lambda_{\texttt{ROE}}$	-0.54	-3.81	10.66	-18.10	-3.42	12.22	3.31	-0.15
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$				
λ_0	3.01	7.22	10.21	10.07	-0.96	19.34	1.83	14.35
$\lambda_{\texttt{MKT}}$	10.03	-4.50	3.54	-6.48	11.34	-5.96	-1.93	2.10
$\lambda_{\texttt{ME}}$	17.74	8.76	-1.36	-0.73	7.27	35.52	11.22	0.23
$\lambda_{\texttt{I/A}}$	0.61	7.12	0.29	-9.39	-1.27	-3.30	0.34	14.69
$\lambda_{\texttt{ROE}}$	3.50	-8.04	8.83	-15.65	-4.12	17.34	-1.35	-0.87

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Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Test Statistics: $\widehat{\boldsymbol{\lambda}}^{\mathtt{TS}}_{\mathtt{IV}}$				
$t\,(\lambda_0)$	1.13	0.20	0.63	0.55	0.86	5.63	0.13	5.34
p-value	[0.26]	[0.84]	[0.53]	[0.59]	[0.39]	[0.00]	[0.90]	[0.00]
$t\,(\lambda_{\mathtt{MKT}})$	-0.78	-0.13	-0.51	-0.66	-2.20	-1.66	0.18	-6.72
p-value	[0.43]	[0.90]	[0.61]	[0.51]	[0.03]	[0.10]	[0.86]	[0.00]
$t\,(\lambda_{\mathtt{ME}})$	0.46	1.16	1.04	-0.17	3.96	1.76	2.26	-0.47
p-value	[0.65]	[0.24]	[0.30]	[0.86]	[0.00]	[0.08]	[0.02]	[0.64]
$t\,(\lambda_{\mathtt{I/A}})$	-1.29	0.32	-1.44	-1.05	-2.10	-2.42	0.63	2.43
p-value	[0.20]	[0.75]	[0.15]	[0.29]	[0.04]	[0.02]	[0.53]	[0.02]
$t\,(\lambda_{\mathtt{ROE}})$	-0.49	-1.33	0.25	-0.90	-3.79	0.94	-0.04	-0.35
p-value	[0.63]	[0.18]	[0.80]	[0.37]	[0.00]	[0.35]	[0.97]	[0.72]
$J_d(\boldsymbol{\lambda})$	4.01	3.29	3.87	2.69	40.03	44.25	5.56	79.98
p-value	[0.41]	[0.43]	[0.40]	[0.47]	[0.00]	[0.00]	[0.32]	[0.00]
$t\,(\phi_{\mathtt{SIZE}})$	1.43	-1.36	1.72	0.96	1.32	-1.46	-2.27	0.39
p-value	[0.15]	[0.17]	[0.09]	[0.34]	[0.19]	[0.14]	[0.02]	[0.69]
$t\,(\phi_{\mathtt{I/A}})$	-1.43	-0.25	-1.09	1.40	-2.31	1.81	0.67	1.01
p-value	[0.15]	[0.80]	[0.28]	[0.16]	[0.02]	[0.07]	[0.50]	[0.31]
$t\,(\phi_{\mathtt{ROE}})$	-0.83	1.50	-0.55	-0.92	-0.72	0.39	-1.56	-0.92
p-value	[0.41]	[0.13]	[0.58]	[0.36]	[0.47]	[0.70]	[0.12]	[0.36]
$J_d(\boldsymbol{\phi})$	4.34	3.69	4.20	3.70	21.41	6.29	5.60	1.31
p-value	[0.22]	[0.28]	[0.24]	[0.29]	[0.00]	[0.10]	[0.14]	[0.72]
$t\,(\boldsymbol{\alpha})$	-3.71	-1.74	-2.36	1.83	-0.84	-3.91	-4.05	0.53
p-value	[0.00]	[0.08]	[0.02]	[0.07]	[0.40]	[0.00]	[0.00]	[0.60]
$J_d(\boldsymbol{\delta})$	22.57	10.48	13.87	9.78	48.32	65.06	29.94	82.29
p-value	[0.05]	[0.29]	[0.18]	[0.33]	[0.00]	[0.00]	[0.01]	[0.00]

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Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Test Statistics: $\widehat{\boldsymbol{\lambda}}^{\mathtt{IT}}_{\mathtt{IV}}$				
$t\,(\lambda_0)$	0.80	0.88	0.77	0.86	-0.12	4.42	0.44	5.18
p-value	[0.43]	[0.38]	[0.44]	[0.39]	[0.90]	[0.00]	[0.66]	[0.00]
$t\,(\lambda_{\mathtt{MKT}})$	-0.28	-0.80	-0.67	-0.87	-1.48	-1.06	-0.46	-6.58
p-value	[0.78]	[0.43]	[0.51]	[0.38]	[0.14]	[0.29]	[0.64]	[0.00]
$t\,(\lambda_{\mathtt{ME}})$	0.29	2.01	0.94	-0.20	5.01	2.51	2.35	-0.52
p-value	[0.77]	[0.04]	[0.35]	[0.84]	[0.00]	[0.01]	[0.02]	[0.60]
$t\,(\lambda_{\mathtt{I/A}})$	-0.74	-0.16	-1.56	-1.07	-0.68	-2.16	0.35	2.49
p-value	[0.46]	[0.88]	[0.12]	[0.28]	[0.50]	[0.03]	[0.73]	[0.01]
$t\,(\lambda_{\mathtt{ROE}})$	-0.01	-2.18	-0.07	-0.89	-3.82	1.62	-1.22	-0.47
p-value	[0.99]	[0.03]	[0.95]	[0.37]	[0.00]	[0.11]	[0.22]	[0.64]
$J_d(\boldsymbol{\lambda})$	1.34	10.20	4.35	3.49	42.35	34.25	7.57	76.78
p-value	[0.79]	[0.14]	[0.36]	[0.40]	[0.00]	[0.00]	[0.20]	[0.00]
$t\,(\phi_{\mathtt{SIZE}})$	1.89	-0.75	1.77	0.21	2.28	1.98	1.08	0.51
p-value	[0.06]	[0.45]	[0.08]	[0.84]	[0.02]	[0.05]	[0.28]	[0.61]
$t\,(\phi_{\mathtt{I/A}})$	-1.08	-0.70	-1.23	1.48	-2.19	1.11	0.12	0.96
p-value	[0.28]	[0.48]	[0.22]	[0.14]	[0.03]	[0.27]	[0.90]	[0.34]
$t\,(\phi_{\mathtt{ROE}})$	-1.18	1.96	-0.70	-0.96	-0.39	0.42	-1.97	-0.75
p-value	[0.24]	[0.05]	[0.49]	[0.34]	[0.70]	[0.67]	[0.05]	[0.45]
$J_d(\boldsymbol{\phi})$	4.74	5.79	4.66	3.04	24.55	7.77	2.68	1.40
p-value	[0.19]	[0.14]	[0.20]	[0.38]	[0.00]	[0.06]	[0.41]	[0.70]
$t\,(\boldsymbol{\alpha})$	-2.84	-0.53	-2.16	1.39	0.77	-2.64	-2.84	0.66
p-value	[0.00]	[0.60]	[0.03]	[0.17]	[0.44]	[0.01]	[0.00]	[0.51]
$J_d(\boldsymbol{\delta})$	15.55	15.40	14.17	8.57	53.08	46.55	20.74	78.96
p-value	[0.13]	[0.15]	[0.17]	[0.39]	[0.00]	[0.00]	[0.05]	[0.00]

Table 9: **Testing the FF5 model: empirical results for 5-year testing periods from 1975 to 2014 using 49 industry clusters.** This table presents the point estimates of $\boldsymbol{\lambda} = [\ \lambda_0\ \ \lambda_{\mathtt{MKT}}\ \ \lambda_{\mathtt{SMB}}\ \ \lambda_{\mathtt{HML}}\ \ \lambda_{\mathtt{RMW}}\ \ \lambda_{\mathtt{CMA}}\ ]'$ and the various statistics along with the corresponding p-values for testing the implications of the FF5 model. We consider 8 non-overlapping 5-year testing periods from 1975 to 2014. For each testing period, the pretesting period consists of the preceding five years. We report point estimates, in annualized percentages, based on the two-step and iterated IV-GMM estimators, i.e., $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{TS}}$ and $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{IT}}$. We further report (i) the t statistic $t(\lambda_0)$, that focuses on the cross-sectional intercept, given in (86); (ii) the t statistics $t(\lambda_{\mathtt{MKT}})$, $t(\lambda_{\mathtt{SMB}})$, $t(\lambda_{\mathtt{HML}})$, $t(\lambda_{\mathtt{RMW}})$, and $t(\lambda_{\mathtt{CMA}})$, that focus on the MKT, SMB, HML, RMW, and CMA ex-post risk premia, respectively, given in (87); (iii) the statistic $J_d(\boldsymbol{\lambda})$, that jointly tests the ex-post risk premia model implications, given in (91); (iv) the t statistics $t\,(\phi_{\mathtt{SIZE}})$, $t\,(\phi_{\mathtt{BTM}})$, $t\,(\phi_{\mathtt{OP}})$, and $t\,(\phi_{\mathtt{AG}})$, that examine the ability of the SIZE, BTM, OP, and AG characteristics to explain mispricing at the individual stock level, respectively, given in (88); (v) the statistic $J_d(\boldsymbol{\phi})$, that examines the ability of the SIZE, BTM, OP, and AG characteristics to jointly explain mispricing at the individual stock level, given in (92); (vi) the t statistic $t(\boldsymbol{\alpha})$, that focuses on aggregate mispricing, given in (89); and (vii) the statistic $J_d(\boldsymbol{\delta})$, that jointly tests all ex-post risk premia and individual stock alpha model implications, given in (90). The corresponding p-values are reported in square brackets below the test statistics.

Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Number of Test Assets				
N	1193	1717	1692	1879	1804	2042	1946	1875
				Average Factor Realizations				
$\overline{f}_{\mathtt{MKT}}$	11.18	4.53	12.49	4.83	20.69	-3.10	-0.43	15.52
$\overline{f}_{\mathtt{SMB}}$	17.33	4.14	-4.82	1.57	-5.30	12.78	1.63	1.54
$\overline{f}_{\mathtt{HML}}$	6.53	9.35	2.57	2.70	-5.51	15.00	2.08	-0.93
$\overline{f}_{\mathtt{RMW}}$	0.10	4.04	5.51	4.70	0.94	10.23	4.73	0.71
$\overline{f}_{\mathtt{CMA}}$	1.72	6.24	5.22	1.74	-1.69	13.93	-0.28	2.90
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{TS}}$				
λ_0	4.23	10.43	13.29	2.70	0.38	10.86	-2.91	9.41
$\lambda_{\mathtt{MKT}}$	6.64	-6.63	0.28	0.62	12.80	-8.70	6.57	5.20
$\lambda_{\mathtt{SMB}}$	18.87	5.34	-4.53	7.24	4.65	31.89	1.41	6.20
$\lambda_{\mathtt{HML}}$	0.41	3.49	3.00	0.37	-4.40	2.65	7.63	0.43
$\lambda_{\mathtt{RMW}}$	-0.70	-1.97	-1.81	-4.63	-2.42	5.69	-4.68	-5.17
$\lambda_{\mathtt{CMA}}$	-5.23	6.65	1.01	-4.57	-2.60	8.22	-8.56	7.03
				Estimates of $\boldsymbol{\lambda}$: $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{IT}}$				
λ_0	3.76	1.53	14.96	3.81	-1.74	10.70	2.48	8.65
$\lambda_{\mathtt{MKT}}$	7.60	0.04	0.51	-0.79	13.81	-9.12	1.13	6.28
$\lambda_{\mathtt{SMB}}$	18.49	8.77	-3.49	6.20	5.52	32.27	3.65	6.40
$\lambda_{\mathtt{HML}}$	-0.17	-1.12	-4.82	1.90	-4.32	3.02	10.47	-0.45
$\lambda_{\mathtt{RMW}}$	0.09	2.28	-0.50	-3.44	-1.70	5.42	-4.21	-6.72
$\lambda_{\mathtt{CMA}}$	-5.17	8.59	-3.16	-4.30	-1.68	9.37	-6.09	6.79

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Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Test Statistics: $\widehat{\boldsymbol{\lambda}}_{\mathtt{IV}}^{\mathtt{TS}}$				
$t\,(\lambda_0)$	0.90	1.64	2.10	1.35	0.07	4.19	-0.62	7.20
p-value	[0.37]	[0.10]	[0.04]	[0.18]	[0.94]	[0.00]	[0.54]	[0.00]
$t\,(\lambda_{\mathtt{MKT}})$	-1.06	-1.68	-2.00	-1.96	-2.08	-2.17	1.46	-6.38
p-value	[0.29]	[0.09]	[0.05]	[0.05]	[0.04]	[0.03]	[0.15]	[0.00]
$t\,(\lambda_{\mathtt{SMB}})$	1.05	0.93	0.15	4.46	4.78	3.09	-0.06	2.94
p-value	[0.29]	[0.35]	[0.88]	[0.00]	[0.00]	[0.00]	[0.95]	[0.00]
$t\,(\lambda_{\mathtt{HML}})$	-2.59	-1.28	0.13	-0.93	0.39	-3.16	0.90	0.78
p-value	[0.01]	[0.20]	[0.89]	[0.35]	[0.69]	[0.00]	[0.37]	[0.43]
$t\,(\lambda_{\mathtt{RMW}})$	-0.23	-1.70	-3.16	-3.01	-1.50	-1.40	-1.86	-3.11
p-value	[0.82]	[0.09]	[0.00]	[0.00]	[0.13]	[0.16]	[0.06]	[0.00]
$t\,(\lambda_{\mathtt{CMA}})$	-1.77	0.16	-1.77	-5.41	-0.20	-0.52	-1.62	2.69
p-value	[0.08]	[0.87]	[0.08]	[0.00]	[0.84]	[0.60]	[0.11]	[0.01]
$J_d(\boldsymbol{\lambda})$	12.92	10.95	21.58	64.76	29.59	44.07	9.41	118.76
p-value	[0.11]	[0.14]	[0.01]	[0.00]	[0.00]	[0.00]	[0.19]	[0.00]
$t\,(\phi_{\mathtt{SIZE}})$	1.84	0.33	-0.73	-1.65	-0.26	0.60	1.46	-0.76
p-value	[0.07]	[0.74]	[0.47]	[0.10]	[0.79]	[0.55]	[0.14]	[0.45]
$t\,(\phi_{\mathtt{BTM}})$	0.73	2.58	-0.17	0.52	0.83	-0.56	-0.85	1.04
p-value	[0.46]	[0.01]	[0.86]	[0.60]	[0.41]	[0.57]	[0.39]	[0.30]
$t\,(\phi_{\mathtt{OP}})$	-0.35	2.52	2.05	2.03	0.34	-1.45	-0.11	1.90
p-value	[0.72]	[0.01]	[0.04]	[0.04]	[0.73]	[0.15]	[0.91]	[0.06]
$t\,(\phi_{\mathtt{AG}})$	-0.90	-0.70	-2.72	-0.60	0.95	-0.45	-1.18	1.52
p-value	[0.37]	[0.48]	[0.01]	[0.55]	[0.34]	[0.66]	[0.24]	[0.13]
$J_d(\boldsymbol{\phi})$	7.15	13.12	7.89	36.32	0.91	3.07	5.49	12.50
p-value	[0.14]	[0.02]	[0.11]	[0.00]	[0.87]	[0.52]	[0.24]	[0.02]
$t\,(\boldsymbol{\alpha})$	-4.03	-2.09	0.62	-3.73	-1.01	-3.80	-0.75	-2.03
p-value	[0.00]	[0.04]	[0.54]	[0.00]	[0.31]	[0.00]	[0.45]	[0.04]
$J_d(\boldsymbol{\delta})$	34.03	28.91	34.11	86.14	32.39	61.53	14.24	130.44
p-value	[0.02]	[0.03]	[0.01]	[0.00]	[0.01]	[0.00]	[0.25]	[0.00]

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Table 9 – continued from previous page

Pretesting Period	70-74	75-79	80-84	85-89	90-94	95-99	00-04	05-09
Testing Period	75-79	80-84	85-89	90-94	95-99	00-04	05-09	10-14
				Test Statistics: $\widehat{\boldsymbol{\lambda}}_{\texttt{IV}}^{\texttt{IT}}$				
$t\,(\lambda_0)$	0.87	0.24	2.63	2.03	-0.35	4.13	0.52	6.09
p-value	[0.39]	[0.81]	[0.01]	[0.04]	[0.72]	[0.00]	[0.60]	[0.00]
$t\,(\lambda_{\texttt{MKT}})$	-0.88	-0.69	-2.19	-2.79	-1.84	-2.34	0.33	-5.16
p-value	[0.38]	[0.49]	[0.03]	[0.01]	[0.07]	[0.02]	[0.74]	[0.00]
$t\,(\lambda_{\texttt{SMB}})$	0.95	3.28	1.02	3.76	5.09	3.14	0.59	2.96
p-value	[0.34]	[0.00]	[0.31]	[0.00]	[0.00]	[0.00]	[0.56]	[0.00]
$t\,(\lambda_{\texttt{HML}})$	-3.58	-2.35	-2.31	-0.32	0.40	-3.06	1.48	0.28
p-value	[0.00]	[0.02]	[0.02]	[0.75]	[0.69]	[0.00]	[0.14]	[0.78]
$t\,(\lambda_{\texttt{RMW}})$	0.00	-0.40	-3.27	-2.66	-1.12	-1.47	-1.92	-3.54
p-value	[1.00]	[0.69]	[0.00]	[0.01]	[0.26]	[0.14]	[0.05]	[0.00]
$t\,(\lambda_{\texttt{CMA}})$	-1.90	0.86	-3.51	-5.26	0.00	-0.41	-1.24	2.57
p-value	[0.06]	[0.39]	[0.00]	[0.00]	[1.00]	[0.68]	[0.21]	[0.01]
$J_d(\boldsymbol{\lambda})$	18.91	17.72	41.05	60.85	30.88	44.08	8.16	91.68
p-value	[0.05]	[0.04]	[0.00]	[0.00]	[0.00]	[0.00]	[0.24]	[0.00]
$t\,(\phi_{\texttt{SIZE}})$	2.14	1.70	0.39	-2.16	1.07	0.60	1.22	-0.88
p-value	[0.03]	[0.09]	[0.69]	[0.03]	[0.29]	[0.55]	[0.22]	[0.38]
$t\,(\phi_{\texttt{BTM}})$	1.04	1.58	1.96	-0.96	0.75	-0.55	-1.05	1.41
p-value	[0.30]	[0.11]	[0.05]	[0.34]	[0.46]	[0.58]	[0.29]	[0.16]
$t\,(\phi_{\texttt{OP}})$	-0.56	1.73	2.41	0.69	0.19	-1.40	-0.59	1.97
p-value	[0.58]	[0.08]	[0.02]	[0.49]	[0.85]	[0.16]	[0.55]	[0.05]
$t\,(\phi_{\texttt{AG}})$	-0.99	-0.80	-2.33	-0.53	1.11	-0.35	-1.07	1.42
p-value	[0.32]	[0.42]	[0.02]	[0.60]	[0.27]	[0.73]	[0.29]	[0.15]
$J_d(\boldsymbol{\phi})$	9.58	9.14	22.10	33.67	1.73	2.80	4.49	13.23
p-value	[0.06]	[0.09]	[0.00]	[0.00]	[0.73]	[0.56]	[0.31]	[0.02]
$t\,(\boldsymbol{\alpha})$	-4.14	-0.90	-0.15	-3.72	-0.75	-3.76	-0.23	-1.77
p-value	[0.00]	[0.37]	[0.88]	[0.00]	[0.45]	[0.00]	[0.82]	[0.08]
$J_d(\boldsymbol{\delta})$	43.03	27.57	56.25	81.01	34.41	60.98	12.29	103.48
p-value	[0.01]	[0.03]	[0.00]	[0.00]	[0.01]	[0.00]	[0.33]	[0.00]