On the Factor Structure in Observation-driven Closed-form Copula Models *

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Abstract

This paper develops multi-factor copula models to capture the time-varying dependence across a large panel of financial assets. The factor structures are based on group characteristics of financial assets, which allows for different patterns of within and between group dependencies. We build score-driven dynamics for the factor loadings, possibly driven by exogenous variables. The factor copula model retains computational tractable as the copula density is available in closed form, which proves beneficial for parameter estimation. We apply our new approach to daily equity returns, realized variances and realized equi-correlations of 100 stocks of the S&P 500 index over the period 2001 to 2014. One-step ahead copula-density forecasts of the whole support and in the joint lower tail based on multi-factor copula models significantly improve upon one-factor models and recently developed benchmarks. Finally, including realized measures into the factor copula specification statistically improves the density forecasts, although the influence vanishes when the factor structure enriches.

Keywords: factor copulas, factor structure, realized correlation, score-driven dynamics

JEL: C32, C58, G17.

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1 Introduction

Copulas are an important tool in financial econometrics for risk management and asset allocation decisions of financial institutions as they are able to measure the dependence between two or more random variables. They offer a great flexibility in building multivariate stochastic models (see for example Patton (2009); Cherubini *et al.* (2011); Fan and Patton (2014) and McNeil *et al.* (2015) for an overview). While initially the copula parameters where assumed to be fixed, the literature has developed time-varying copulas (Patton, 2006; Hafner and Manner, 2012), such that the dependencies may change over time. Patton et al. (2012) reviews various copula based approaches that are suitable for modeling and forecasting risk when the dimension of the assets is relatively small.

The recently developed (dynamic) factor copula models (Oh and Patton, 2017a,b; Creal and Tsay, 2015) provide a general approach to model the time-varying dependence among (financial) variables for *large* dimensions. Oh and Patton (2017b) apply their copula to model time-varying systemic risk of 100 assets, whereas Creal and Tsay (2015) model the dynamic correlations between 200 stock returns. The crucial advantage of factor copulas is their ability to avoid the 'curse of dimensionality': the rapidly growing number of model parameters to be estimated when the dimension increases. This leads in general to huge computational costs. Moreover, parameter estimation could be infeasible. The copula approach disentangles the multivariate distribution into marginals and a dependence structure that can be estimated separately and the *factor* copula structure models the dependency across all assets by a couple of latent variables, with possibly time-varying factor loadings. In this way the number of estimated parameters stays reasonably small while (the inverse of) large covariance matrices cain be computed straightforwardly.

Although the dynamic factor copulas of Oh and Patton (2017b) and Creal and Tsay (2015) (henceforth OP2017b and CT2015) are a big step forward towards modeling high-dimensional dependence, they are limited with respect to their factor structure: they both consider just *one* factor. This seems restrictive when modeling the dependencies between a large number of financial asset returns. From an asset-pricing point of view, we nowadays use models with multiple factors, such as three- or five-factor model of Fama and French (1993, 2016), and (affine) term structure models. Adding more factors is possible in both aforementioned studies, but this will considerably increase the computational burden. OP2017b propose a skewed-t distribution for the latent factor such that the copula density is not available in closed form, hence numerical methods are required to estimate the parameters. This requires considerable computational effort, and will even become worse if one consider more factors. With respect to the second study, CT2015 proposes a stochastic copula (see also Hafner and Manner, 2012) where the factor loadings evolve by a stochastic transition equation. Hence massive Bayesian simulation techniques are used as again the implied copula density is unknown in closed form. Introducing more factors will therefore cost additional computing power.

This paper extends the factor copulas of OP2017b and CT2015 with respect to the factor structure. More specifically, this structure ranges from a simple one-factor model to copula models with multi group-specific factors and/or time-varying group-specific factor loadings. In addition, we propose score-driven dynamics for these factor loadings, by adopting the Generalized Autoregressive Score (GAS) framework of Creal *et al.* (2013). This recent general framework provides intuitive parameter updates via one-step improvements of the local likelihood. Moreover, the tractability is achieved by ensuring that the copula density is available in closed form. Hence, parameter estimation is possible quite straightforwardly by means of Maximum Likelihood. This contrasts with the stochastic copula of CT2015 and the skew-factor copula of OP2017b.

A further extension of our multi-factor copulas is the easy allowance to incorporate exogenous variables into the specification of the factor loadings. We exploit this advantage by evaluating the possible influence of 'realized measures" in the factor copula specification. These high-frequency based measures of the variance and correlations lead in general to an improvement in volatility modeling, estimation and prediction (Andersen *et al.*, 2003; Hansen and Lunde, 2011). Salvatierra and Patton (2015) show the added value of introducing realized measures into a bivariate copula. We extend this approach by considering an (group) equi-based realized correlation that feeds the dynamics of the factor loadings. An important implication of our proposed dynamics is that we do not require to pre-construct a large realized covariance matrix which should be positive definite, which might become troublesome if the number of assets gets large. Instead, we retain the added value of realized volatilities in the specification of the marginal specification while using the equi-correlation into the factor copula specification.[1]. Hence this paper can also be seen as an extension of multivariate realized volatility models for large dimensions.

We apply our factor structures with new score-driven dynamics on a panel of daily returns, realized variances and realized equi-correlations of 100 stocks listed at the S&P 500 index over the period 2001-2014 and benchmark our models against the cDCC model of Engle (2002); Aielli (2013) and the (Block) DECO model of Engle and Kelly (2012). We find in-sample that our multi-factor t-copula models produce a better fit than one-factor models and the benchmarks. Allowing for industry-specific factors increases the fit considerably compared to one- or two-factor copula models with possible industry-specific loadings. Further, including the realized equi-correlation has a positive effect on the fit of factor copula models, although this effect declines when the model specification (i.e. the factor structure) enriches. Out-of-sample we compare one-step ahead copula density forecasts using the density forecast accuracy test for copulas proposed by Diks *et al.* (2014), and the Model Confidence Set approach of Hansen *et al.* (2011). We compare the dynamic copula models both in terms of their fit across the entire support, but also on the left tail of the distribution. The results suggest that our multi-factor copula is superior against the one-factor copula models, cDCC and (Block) DECO models in case of the whole support, and the joint 5%, 10% and 25% lower tails. At the 1% lower tail, the predictive accuracy is at par with the cDCC model. Finally, we show that including realized equi-correlations into the factor copula specification improves one-step ahead density forecasts. This confirms the result of Salvatierra and Patton (2015) in their setting of bivariate copula's. However, this effect

[1]If the number of assets exceed the number of intra-day observations (typically 78 5-minute returns), then the realized covariance matrix is not positive definite by definition.

vanishes when the factor structure becomes richer.

This paper touches various strands of the literature on factor (copula) models, observation driven models and multivariate volatility models. First of all, there is an extensive literature on factor models and the computation of large covariance matrices, see for example Fan *et al.* (2008) and Fan *et al.* (2011). Engle *et al.* (1990) develop factor ARCH models with an application in asset pricing with many assets. These models are related to our approach, however the benefit of our factor copula approach is the flexibility in choosing the models and distributional assumptions, both with respect to the marginals as well as to the copula specification. Second, factor copula's has been recently introduced by Krupskii and Joe (2013); Oh and Patton (2017a), among others. Oh and Patton (2017b) introduce the GAS framework of Creal *et al.* (2013) within factor copulas. This recently developed framework provides a new intuitive way to update time-varying parameters within observation driven models, and has been applied in various fields with promising results in for example credit risk modeling (Creal *et al.*, 2014) and systemic risk modeling (Lucas *et al.*, 2014, 2017). Third, we refer to Creal and Tsay (2015) for references of Bayesian analysis of (factor) copula's. Galeano *et al.* (2017) apply Bayesian inference in different specifications of the class of dynamic one-factor copula models. Examples of other studies using dynamic copula models in high dimensions are Christoffersen *et al.* (2012, 2014), where the latter study combines a skew-t copula with DCC models to study the diversification benefits of a panel of more than 200 asset returns. This lead us finally to the relation between factor copula's and multivariate volatility models with possible incorporation of realized measures (e.g. the DCC model of Engle (2002) or the Multivariate HEAVY model of Noureldin *et al.* (2012)). These models suffers in general from the aforementioned curse of dimensionality when the dimension gets large. In addition, (large) covariance/correlation matrices need to be inverted many times during parameter estimation, which becomes computationally and numerically problematic.

The rest of this paper is set up as follows. In Section 2, we introduce the factor copula model with various factor structures for the multivariate distribution of returns, while allowing for possible incorporation of realized measures into the dependency specification. We provide a simulation experiment in Section 3 to show the performance of the model and estimation procedure. In Section 4, we apply the model to a panel of 100 daily equity returns, realized variances and equi-correlations from the S&P 500 index. We conclude in Section 5.

2 Observation driven dynamic factor Copulas

Let $\mathbf{y}_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})^\top \in \mathbb{R}^N$ denote the vector of asset returns over day t, $t = 1, \ldots, T$. We aim to model the conditional joint distribution of $\mathbf{y}_t$, which we decompose into N marginals and a conditional copula (Patton, 2006):

$$\mathbf{y}_t|\mathcal{F}_{t-1} \sim \mathbf{F}_t = \mathbf{C}_t\big(F_{1t}(y_{1t} \mid \boldsymbol{\theta}_M), F_{2t}(y_{2t} \mid \boldsymbol{\theta}_M), \ldots, F_{Nt}(y_{Nt} \mid \boldsymbol{\theta}_M); \boldsymbol{\theta}_C\big) \tag{1}$$

with $\mathcal{F}_{t-1}$ the information set containing all information up to and including time $t-1$, $\mathbf{C}_t(\cdot|\boldsymbol{\theta}_C)$ the conditional copula given the copula parameter vector $\boldsymbol{\theta}_C$ and $F_{it}(y_{it}|\boldsymbol{\theta}_M)$ denotes the marginal distribution of asset i, given the marginal parameter vector $\boldsymbol{\theta}_M$. We will elaborate about the model and distribution choice of the marginals later on in this paper. Note that the conditional copula $\mathbf{C}_t$ can also be interpreted as the conditional distribution of the probability integral transforms (PITs) of y_{it}. Put differently, define for $i = 1, \ldots, N$ the PIT as

$$U_{it} \equiv F_{it}(y_{it} \mid \boldsymbol{\theta}_M), \tag{2}$$

then it holds that

$$\mathbf{U}_t|\mathcal{F}_{t-1} \sim \mathbf{C}_t(\boldsymbol{\theta}_C) \tag{3}$$

The advantage of decomposing the multivariate (conditional) distribution into marginals and a copula compared to immediately modeling the conditional joint distribution $\mathbf{F}_t$ is twofold: (1) when the dimension is high, multi-stage estimation reduces the number of parameter to be estimated, and (2), modeling a univariate distribution is relatively simple, hence the problem of modeling $\mathbf{F}_t$ is reduced to modeling $\mathbf{C}_t$.

The challenging task is to model the conditional Copula $\mathbf{C}_t$, given the estimated PITs from a large number of marginals. There is an extensive literature about copulas (see for example Patton, 2009; Fan and Patton, 2014, for an overview), although literature about copulas in large dimensions is rather scarce. This paper builds upon a recently developed class of 'Factor Copulas' with time-varying factor loadings (Oh and Patton, 2017a,b; Creal and Tsay, 2015). These models aim to reduce the dimension by making use of a factor structure for the density $\mathbf{C}_t$. We build upon the factor structure of Creal and Tsay (2015) by assuming the following specification:

$$\begin{aligned} u_{it} &= P(x_{it}|\boldsymbol{\theta}_C), \qquad i = 1, \ldots N, & (4) \\ x_{it} &= \tilde{\boldsymbol{\lambda}}_{it}^{\top}\mathbf{z}_t + \sigma_{it}\epsilon_{it} \qquad \mathbf{z}_t \sim p(\mathbf{z}_t|\boldsymbol{\theta}_C), \epsilon_{it} \sim p(\epsilon_{it}|\boldsymbol{\theta}_C) \end{aligned}$$

where $\tilde{\boldsymbol{\lambda}}_{it}$ is a $p \times 1$ vector of scaled factor loadings, $\mathbf{z}_t$ is a $p \times 1$ vector of common latent factors with a zero mean vector, $\text{Var}(\mathbf{z}_t) = \mathbf{I_p}$ and ϵ_{it} are idiosyncratic shocks. Further, $P(x_{it}|\boldsymbol{\theta}_C)$, $p(\mathbf{z}_t|\boldsymbol{\theta}_C)$ and $p(\epsilon_{it}|\boldsymbol{\theta}_C)$ denote the marginal distribution of x_{it} and the distributions for the common factors and the idiosyncratic vector respectively. We assume that there is no correlation between the common factors and the idiosyncratic shocks. The scaled factor loading vector $\tilde{\boldsymbol{\lambda}}_{it}$ and σ_{it} are defined in such a way that they are positive and imply a unit variance of x_{it}:

$$\tilde{\boldsymbol{\lambda}}_{it} = \frac{\exp(\boldsymbol{\lambda}_{it})}{\sqrt{1 + \exp(\boldsymbol{\lambda}_{it})^{\top}\exp(\boldsymbol{\lambda}_{it})}} \qquad (5)$$

$$\sigma_{it}^2 = \frac{1}{1 + \exp(\boldsymbol{\lambda}_{it})^{\top}\exp(\boldsymbol{\lambda}_{it})} \qquad (6)$$

Within our choice of distributions, which we will elaborate later on in this paper, the associated correlation matrix of $\mathbf{x}_t$ equals

$$\mathbf{R}_t = \tilde{\boldsymbol{L}}_t^\top \tilde{\boldsymbol{L}}_t + \mathbf{D}_t \tag{7}$$

with $\tilde{\boldsymbol{L}}_t^\top = (\tilde{\boldsymbol{\lambda}}_{1t}^\top, \tilde{\boldsymbol{\lambda}}_{2t}^\top, \ldots, \tilde{\boldsymbol{\lambda}}_{Nt}^\top)$ a $N \times p$ matrix of scaled loadings and $\mathbf{D}_t$ a $N \times N$ diagonal matrix with entries σ_{it}^2. A huge computational advantage of these factor copulas is that the inverse and determinant of $\mathbf{R}_t$ are known in closed form (Creal and Tsay, 2015):

$$\mathbf{R}_t^{-1} = \mathbf{D}_t^{-1} - \mathbf{D}_t^{-1}\tilde{\boldsymbol{L}}_t^\top \left(\mathbf{I}_p + \tilde{\boldsymbol{L}}_t^\top \mathbf{D}_t^{-1}\tilde{\boldsymbol{L}}_t\right)^{-1} \tilde{\boldsymbol{L}}_t \mathbf{D}_t^{-1}, \qquad |\mathbf{R}_t| = \left|\mathbf{I}_p + \tilde{\boldsymbol{L}}_t^\top \mathbf{D}_t^{-1}\tilde{\boldsymbol{L}}_t\right| |\mathbf{D}_t|. \tag{8}$$

We only have to compute the inverse of the $p \times p$ matrix $\mathbf{I}_p + \tilde{\boldsymbol{L}}_t^\top \mathbf{D}_t^{-1}\tilde{\boldsymbol{L}}_t$. This reduces the computational costs dramatically compared to inverting an $N \times N$ matrix at each time t as $p << N$. The above class of copula is very flexible, depending on the number of factors, the distributional assumptions of the common factors and ideosyncratic shocks and the proposed time-varying dynamics for the factor loadings. The following subsections discuss these choices in more detail.

2.1 The factor structure

The main goal of this paper is to exploit the factor structure within our general specification of (4). We extend both OP2017b and CT2015 by considering more than one factor, as this seems too restrictive when modeling a large number of assets. This subsection will present different structures for $\mathbf{z}_t$ and the associated loading matrix $\tilde{\boldsymbol{L}}_t^\top$. In addition, we will show the consequences for the correlation matrix by showing the upper triangular part of the matrix $\mathbf{R}_t$ in an simplified example of four assets, divided into two equal groups of two.

Let us start with the one equi-factor copula model, implying that $\mathbf{z}_t$ is a univariate random variable with a $N \times 1$ loading vector $\tilde{\boldsymbol{L}}_t^\top = \tilde{\lambda}_t \iota$ with ι a $N \times 1$ vector of ones. This one-factor model is related to the DECO model of Engle and Kelly (2012), where each pairwise correlation is assumed to be the same. From an asset pricing point of view, this factor can be seen as the market factor, with a identical 'beta' (factor loading) for all assets. We will denote this factor copula (FC) model as 'FC-1f-Equi'.

A first building block to bring more flexibility in the factor loadings of the FC-1f-Equi model is to make them group specific, see also OP2017b. The 'beta' could have a different loading across region, country, industry or other characteristics that are of interest of the researcher. In this paper, we consider a panel of stocks, which are classified into G different industries. Hence we have now different 'betas' of the market factor with respect to each industry. To provide an toy-example, suppose we have $G = 2$ groups and $N = 4$ assets, with 2 assets per group. $\tilde{\boldsymbol{L}}_t^\top$ is now given by

$$\tilde{\boldsymbol{L}}_t^\top = \begin{bmatrix} \tilde{\lambda}_{1t} \\ \tilde{\lambda}_{1t} \\ \tilde{\lambda}_{2t} \\ \tilde{\lambda}_{2t} \end{bmatrix}, \tag{9}$$

where the order of the loadings is arbitrarily chosem. The corresponding upper triangular part $\mathbf{R}_t$ is denoted as

$$\mathbf{R}_t = \begin{bmatrix} 1 & \tilde{\lambda}_{1t}^2 & \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} & \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} \\ & 1 & \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} & \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} \\ & & 1 & \tilde{\lambda}_{2t}^2 \\ & & & 1 \end{bmatrix}. \tag{10}$$

Hence *within* a group i, we have a group specific correlation $\tilde{\lambda}_{it}^2$, whereas the correlation *between* group i and j is given by $\tilde{\lambda}_{it}\tilde{\lambda}_{jt}$. We will label this one-factor copula with group-specific loadings as 'FC-1f-Group'.

Apart from varying the loadings (betas) of *one* factor per group, we can also add a different group factor as a second building block. In the above example of four assets and two groups, we have a 4×2 matrix of (scaled) factor loadings

$$\tilde{\boldsymbol{L}}_t^\top = \begin{bmatrix} \tilde{\lambda}_{1t} & 0 \\ \tilde{\lambda}_{1t} & 0 \\ 0 & \tilde{\lambda}_{2t} \\ 0 & \tilde{\lambda}_{2t} \end{bmatrix}. \tag{11}$$

In general, we would have G different factors, while each group (industry) factor has is own factor loading. It can easily be seen from the example above that here the correlation differs *within* a group, while the correlation between groups equals zero.

The above building blocks of industry specific loadings and group factors can easily be combined with a equi-factor to create a multi-factor copula model. To save space, we will only show the combination of an equi-factor with group-specific factors, although we also consider a two-factor model with an equi-loading and group-specific loadings, denoted as 'FC-2f-Group-Equi'. Combining an equi-factor with group-specific factors leads in our example to the following loading matrix:

$$\tilde{\boldsymbol{L}}_t^\top = \begin{bmatrix} \tilde{\lambda}_{0t} & \tilde{\lambda}_{1t} & 0 \\ \tilde{\lambda}_{0t} & \tilde{\lambda}_{1t} & 0 \\ \tilde{\lambda}_{0t} & 0 & \tilde{\lambda}_{2t} \\ \tilde{\lambda}_{0t} & 0 & \tilde{\lambda}_{2t} \end{bmatrix}, \tag{12}$$

such that the implied correlation matrix is given by

$$\mathbf{R}_t = \begin{bmatrix} 1 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}^2 & \tilde{\lambda}_{0t}^2 & \tilde{\lambda}_{0t}^2 \\ & 1 & \tilde{\lambda}_{0t}^2 & \tilde{\lambda}_{0t}^2 \\ & & 1 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{2t}^2 \\ & & & 1 \end{bmatrix}. \tag{13}$$

The resulting correlation matrix differs from (10) in an important way: the between group correlation contains only the parameter $\tilde{\lambda}_{0t}$, hence $\tilde{\lambda}_{1t}$ and $\tilde{\lambda}_{2t}$ have much more freedom for the within correlation, while in the group-specific loading case, an increase in $\tilde{\lambda}_{1t}$ will also effect the correlation between group 1 and 2. This factor copula model with multi factors (Mf) is denoted as 'FC-Mf-Equi-Group'.

We can even go a step further and also add another factor with industry-specific loadings to the aforementioned specification, such that (12) changes into

$$\tilde{\boldsymbol{L}}_t^\top = \begin{bmatrix} \tilde{\lambda}_{0t} & \tilde{\lambda}_{1t} & \tilde{\lambda}_{3t} & 0 \\ \tilde{\lambda}_{0t} & \tilde{\lambda}_{1t} & \tilde{\lambda}_{3t} & 0 \\ \tilde{\lambda}_{0t} & \tilde{\lambda}_{2t} & 0 & \tilde{\lambda}_{4t} \\ \tilde{\lambda}_{0t} & \tilde{\lambda}_{2t} & 0 & \tilde{\lambda}_{4t} \end{bmatrix}, \tag{14}$$

with corresponding dependence matrix

$$\mathbf{R}_t = \begin{bmatrix} 1 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}^2 + \tilde{\lambda}_{3t}^2 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} \\ & 1 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}\tilde{\lambda}_{2t} \\ & & 1 & \tilde{\lambda}_{0t}^2 + \tilde{\lambda}_{1t}^2 + \tilde{\lambda}_{4t}^2 \\ & & & 1 \end{bmatrix}. \tag{15}$$

Hence the between group correlation is more flexible compared to (12), as they are now also affected by $\tilde{\lambda}_{1t}$ and $\tilde{\lambda}_{2t}$, while $\tilde{\lambda}_{3t}$ and $\tilde{\lambda}_{4t}$ only appear in the within correlation block. Hence this model, denoted as 'FC-Mf-Full', contains two market factors with fixed and time-varying betas per group (industry), and G industry factors.

To summarize this subsection, given that the assets belong to a certain group based on some characteristic, we are able to create various factor structures to the basic one-factor model by introducing group-specific loadings and/or group-specific factors. Table 1 lists the resulting factor structures with their properties, such as the number of factors, number of different $\tilde{\lambda}'s$, and the associated dimension of $\tilde{\boldsymbol{L}}_t^\top$.

[insert Table 1]

2.2 Distributional assumptions

Given the various factor structures proposed in the previous subsection, the next step is to specify a distribution for the common factors and the idiosyncratic term of (4). OP2017b assume a skewed-t and a Student's-t density for the common factor z_t and the idiosyncratic shock ϵ_{it} respectively. This implies that the copula density of x_{it} is analytically unknown. Hence parameter estimation is computationally involved as numerical techniques are needed in order to evaluate the copula density. Likewise, CT2015 propose a stochastic transition equation for the factor loading λ_{it}. Again, the copula density is analytically unavailable in closed form. Bayesian (numerical) techniques are required to estimate the parameters, which is computationally costly for increasing dimensions.

We will retain the tractability of the model by choosing two particular choices for $p(\mathbf{z}_t|\boldsymbol{\theta}_C)$ and $p(\epsilon_{it}|\boldsymbol{\theta}_C)$ such that the implied copula density is available in closed form, at the cost of possible skewness. Although skewed copulas exists (by means of the Generalized Hyperbolic copula or the skewed-t distribution of Azzalini and Capitanio (2003)), it is not possible to maintain the factor copula structure as given in (4) combined with a closed form copula density. Besides, we also lose the analytic expressions for the inverse and determinant of $\mathbf{R}_t$ (see (7)-(8)) which is again computationally inconvenient. We therefore stick to two elliptical copulas, the Gaussian factor copula and the t-factor copula.[2]

The conditional Gaussian factor copula reads

$$\begin{aligned} u_{it} &= \Phi(x_{it}), & & i = 1, \dots N, \\ x_{it} &= \tilde{\boldsymbol{\lambda}}_{it}^{\top}\mathbf{z}_t + \sigma_{it}\epsilon_{it} & & \mathbf{z}_t \sim N(\mathbf{0}, \mathbf{I_p}), \quad \epsilon_{it} \sim N(0,1), \end{aligned} \tag{16}$$

with $\Phi(\cdot)$ the cumulative distribution function (cdf) of the Gaussian distribution function. The Gaussian copula density does not contain any additional parameters beyond the parameters associated with the dynamics of the factor loadings. Although this factor copula

[2] We have estimated a DECO model of Engle and Kelly (2012) with the skewed-t distribution of Azzalini and Capitanio (2003). Initial results showed that the maximized log-likelihood of the skewed-t distribution is only marginally higher than the log-likelihood of the (nested) Student's t distribution.

is easy to estimate (as we only have to compute the inverse $x_{it} = \Phi^{-1}(u_{it})$ once), it does not contain any tail dependence (see Embrechts *et al.*, 2005). Therefore, our main focus lies in the conditional Student's t factor copula, which is given by

$$\begin{aligned} u_{it} &= T(x_{it} \mid \nu_C), \qquad\qquad i = 1, \ldots N, \\ x_{it} &= \sqrt{\zeta_t}\left(\tilde{\boldsymbol{\lambda}}_{it}^{\top}\mathbf{z}_t + \sigma_{it}\epsilon_{it}\right) \qquad \mathbf{z}_t \sim N(\mathbf{0}, \mathbf{I_p}), \quad \epsilon_{it} \sim N(0,1) \\ \zeta_t &\sim \text{Inv-Gamma}\left(\frac{\nu_C}{2}, \frac{\nu_C}{2}\right). \end{aligned} \tag{17}$$

where $T(\cdot \mid \nu_C)$ denotes the cdf of the univariate Student's t distribution with ν_C degrees of freedom and ζ_t denotes a Inverse-Gamma distribution. Note that our proposed factor structures of the previous subsection easily fit into both assumed distributions without considerably computational costs, as the copula density remains analytically tractable. This contrast to the stochastic copulas of CT2015 and the skewed-t factor copula of OP2017b, as adding more factors would considerably increase the computational burden.

2.3 Dynamics of factor loadings

The final step to complete our factor copula specification is to impose dynamics on the factor loadings λ_{it}. In general, there are two approaches to model time-varying factor loadings. The first approach is parameter driven (Hafner and Manner, 2012; Creal and Tsay, 2015, e.g.) where λ_{it} evolves as in stochastic volatility models. This leads to so-called stochastic copula models. The second approach is observation driven with loadings depending on past observable variables. This paper considers the second method, avoiding a computational burden involved on stochastic copula models as they typically require to integrate out the random innovation term apparent in the process of the loadings. In particular, we follow OP2017b and adapt the generalized autoregressive score framework of Creal *et al.* (2013), see also Harvey (2013). The GAS framework uses the score of the conditional density function to drive the dynamics of a time-varying parameter by adjusting it in the direction of steepest ascent of the local log likelihood function. Blasques *et al.* (2015) show that score driven dynamics possess information theoretic optimality properties even if the model is mis-specified. The framework is very general and has recently been applied to various areas, such as multivariate volatility modeling (Opschoor *et al.*, 2017), credit risk management (Creal *et al.*, 2014), and systemic risk management (Lucas *et al.*, 2014).

Let us start with the simple FC-1f-Equi model. In this case, we have $\tilde{\boldsymbol{L}}_t^\top = \tilde{\lambda}_t \iota$, which depends on the unscaled scalar parameter λ_t. The GAS dynamics for λ are given by

$$\lambda_{t+1} = \omega + A\, s_t + B\, \lambda_t \tag{18}$$

$$s_t = S_t \cdot \Big(\frac{\partial \log \mathbf{c}_t(\mathbf{x}_t \mid \lambda_t, \theta_C)}{\partial \lambda_t} \Big), \tag{19}$$

with ω, A and B scalars, s_t the scaled score and S_t a scaling factor. We follow OP2017b and put all scaling factors to one for computational reasons. The score is defined as the partial derivative of the log Gaussian or Student's t copula density with respect to λ_t. Put differently, λ_t will be updated in that direction such that the one-step improvement of the local likelihood is maximized.

As indicated by Table 1, the more richer the copula structure, the more $\lambda's$ we have. Theoretically, we could impose for each λ a different ω, A and B. However, in order to keep a tractable model with a reasonable amount of parameters to be estimated, we assume that B is the same across all factors. In addition, when considering group-specific loadings, we allow for different intercepts ω_g but keep A and B the same. Finally, we allow for three different values of A in case of the FC-MF-Full copula model: one for the equi-factor, one for the industry-specific loadings and one for all industry factors.

We further extend the GAS model of (18) by allowing for the influence of exogenous variables, such as realized correlations. Realized measures use intra-day data to estimate a variance or correlation, which improves modeling volatility and correlations (e.g. see Andersen *et al.* (2003)). Salvatierra and Patton (2015) show the influence of realized correlations by specifying bivariate GRAS copula models. Given our various factor structures, we can easily insert a realized correlation by extending (18) into

$$\lambda_{t+1} = \omega + A\, s_t + B\, \lambda_t + \gamma_{RC} \log \sqrt{RCORR}_{eq,t}, \tag{20}$$

with $RCORR_{eq,t}$ the average of all pairwise realized correlations between all assets:

$$RCORR_{eq,t} = \frac{1}{N(N-1)} \sum_{i<j} RCORR_{ij,t}. \tag{21}$$

The non-linear transformation of $RCORR_{ij,t}$ serves for easy parameter interpretation of γ_{RC}: a value of one implies that the correlation of x_{it} exactly equals the realized equi-correlation. Note that in case of different group factor(s) (loadings), we could also compute a 'heterogenous' realized equi-correlation by taking the average of all pairwise correlations $RCORR_{ij,t}$ *within* one particular group or industry.

We would like to emphasize the difference between the approach of this paper and a multivariate volatility model that inserts realized covariance matrix, such as the the Multivariate HEAVY model of Noureldin *et al.* (2012). Such a model requires an $N \times N$ positive definite matrix. As noted earlier, this may become troublesome if the number of assets increases, hence shrinkage techniques are needed to retain a positive definite matrix (Ledoit and Wolf, 2003). We avoid these problems in our factor copula by easily computing realized variances first, which will be inserted in the marginals. For the correlations, we will we average out all pairwise correlations in (21). The cost is that we only deal with (heterogenous) realized equi-correlations.

Finally, note that beyond including the realized correlation, equation (18) allows us also to include any other possible exogenous variable X_t into the model specification that might influence the factor loadings and hence the correlations. Besides, it is important to realize that the GAS dynamics of (18) does not require any restrictions on $RCORR_{eq,t}$ or γ_{RC} due to the rescaling of λ as given in (5). This contrasts with any multivariate volatility model that aims to connect the covariance matrix with past realized covariances or any other exogenous variable. In that case, one should always pay attention to keep the covariance matrix positive definite.

2.4 Estimation

We estimate the parameters of the marginals and the factor copula by Maximum Likelihood. The decomposition of the joint distribution into the marginals and the copula enables us to employ a two-step estimation approach: we first estimate the marginals (separately) and then the copula parameters conditional on the marginals. This approach follows directly from decomposing the joint likelihood as:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\theta}) &\equiv \sum_{t=1}^{T} \log \mathbf{f}_t(\mathbf{y}_t; \boldsymbol{\theta}) = \sum_{i=1}^{N} \sum_{t=1}^{T} \log f_{i,t}(y_{it}; \boldsymbol{\theta}_M) \\
&\quad + \sum_{t=1}^{T} \log \mathbf{c}_t \left(F_{1,t}(y_{1t}; \boldsymbol{\theta}_M), \ldots, F_{N,t}(y_{Nt}; \boldsymbol{\theta}_M); \boldsymbol{\theta}_C \right)
\end{aligned} \tag{22}
$$

with $\boldsymbol{\theta} = \{\boldsymbol{\theta}_M, \boldsymbol{\theta}_C\}$. According to Salvatierra and Patton (2015), the implied efficiency loss is small compared to estimating the full likelihood in one step.

3 Simulation experiment

We perform a Monte Carlo study to investigate the finite sample properties of maximum likelihood estimation of θ_C for our factor copulas with different factor structures. We simulate time series of T daily returns of dimension $k = 100$, which corresponds with our empirical application. T is set equal to 500 and 1000 respectively. For brevity, we only consider the multi-factor class of copula models as the true data-generating processs (DGP). The models within this class are denoted as FC-Mf-Equi-Group and FC-Mf-Full, where the latter is represented by

$$
\begin{aligned}
x_{it} &= \sqrt{\zeta_t}\left(\tilde{\boldsymbol{\lambda}}_{it}^{\top}\mathbf{z}_t + \sigma_{it}\epsilon_{it}\right), && (23)\\
\tilde{\boldsymbol{\lambda}}_{it}^{\top} &= \left[\lambda_{1,t}^{eq}\,\lambda_{21,t}^{gr,f}\,\lambda_{22,t}^{gr,f}\,\ldots,\lambda_{2G,t}^{gr,f}\,\lambda_{3g,t}^{gr,l}\right]^{\top}, &&\\
\lambda_{1,t+1}^{eq} &= \omega_{eq} + A_{eq}\,s_t^{eq} + B\,\lambda_{1,t}^{eq}, && (24)\\
\lambda_{2g,t+1}^{gr,f} &= \omega_g + A_{gr,f}\,s_{g,t}^{gr,f} + B\,\lambda_{2g,t}^{gr,f}, \qquad g = 1,\ldots,G && (25)\\
\lambda_{3g,t+1}^{gr,l} &= \omega_{gr,l} + A_{gr,l}\,s_{g,t}^{gr,l} + B\,\lambda_{3g,t}^{gr,l}, \qquad g = 1,\ldots,G && (26)
\end{aligned}
$$

with $\mathbf{z}_t \sim N(\mathbf{0}, \mathbf{I_{G+2}})$, $\epsilon_{it} \sim N(0,1)$ and $\zeta_t \sim \text{Inv-Gamma}\left(\frac{\nu_C}{2}, \frac{\nu_C}{2}\right)$. Hence the Factor-Copula Multi-factor Full (FC-Mf-Full) model consists of three different types of $\lambda's$ which has their own GAS dynamics: one lambda for the equi factor (λ^{eq}), G factor loadings for the group specific factors $\lambda^{gr,f}$ and G factor loadings of one additional factor. Note that $\lambda_{2g,t}^{gr,f}$ is zero if asset i does not belong to group g. Each type of lambda has is own value of A and $\omega_{\cdot}$, whereas within the group specific factor, each factor has also its own intercept ω_g ($g = 1, \ldots, G$). The other considered multi-factor model is obtained by dropping equation (26). Guided by the empirical application, we consider $G = 10$ groups, where each group consists of 10 assets, and put $\omega_{eq} = \omega_{gr,l} = 0.01$, ω_g is equally spaced in the interval $[-0.05\,, 0.01]$. Further, we set $\nu_C = [30, \infty]$, where the latter corresponds with the Gaussian factor copulas. Finally, the values of $A_{eq}, A_{gr,f}, A_{gr,l}$ and B vary slightly across the two distributions but are all around 0.02 and 0.95 respectively.

[insert Table 2]

Table 2 presents the results based on 1000 replications. Panel A denotes the results of the multi-factor model with one equi-factor and 10 group-specific factors, while Panel B presents results of one-equi, 10 group-specific and one additional factor with group-specific loadings (the FC-Mf-Full model). All parameters are estimated near their true values, and the standard deviation decreases in general when the sample size T increases. In Panel A, there is a small downward bias in the group specific intercepts ω_i ($i = 1, 2, 3$) when the sample size is 500, which is more severe for the Gaussian factor copulas than for the t-factor copulas. Note that the standard errors are also higher for these intercepts compared to the remaining intercepts. This bias shrinks when the simple size increases to 1000. Panel B shows that also in case of the FC-Mf-Full-N model there is a downward bias for ω_i $(i = 1, 2, 3)$ when the sample size is small. Again the standard errors are also relatively high. Based on these two different DGPs, we conclude that the finite sample properties of the t-factor copulas are slightly better than the statistical properties of the Gaussian factor copulas.

4 Empirical application

4.1 Data

The data consist of daily open-to-close returns and daily realized covariances measures for 100 U.S. equities. Table 3 provides an overview of the Tickers of each company, grouped into 10 different industries. The data spans the period January 2, 2001 until December 31, 2014 and contains $T = 3521$ trading days. The Financial industry covers the most companies (i.e. 19), followed by Consumer Services and Energy respectively. Each industry covers at least four companies.

[insert Table 3]

We retrieve consolidated trades (transaction prices) from the Trade and Quote (TAQ) database from 9:30 until 16:00 with a time-stamp precision of one second. After cleaning the high-frequency data following the guidelines of Barndorff-Nielsen *et al.* (2009) and Brownlees and Gallo (2006), we construct realized variances as well as pairwise realized covariances based on 5-minute returns. Both quantities are used to back out pairwise realized correlations. Figure 1 shows $RCORR_{eq,t}$ of (21), the average of all pairwise realized correlations between the 100 stock returns. The figure shows that the average correlation is quite noisy over time. Note that the average correlation is relatively high during the global financial crisis (around 2008/2009), but even higher in 2011. This could represent the fears of contagion of the European sovereign debt crisis to Spain and Italy.

[insert Figure 1]

4.2 Marginals

Using the full sample of 3521 trading days, we first estimate the parameters of the marginal distributions. We model these distributions using univariate volatility models, using daily returns and daily realized variances. Typically, both quantities are fat-tailed. Opschoor *et al.* (2017) argue that this should be taken into account when proposing a statistical model for the (co)variance of returns as large returns and realized (co)variances may potentially disrupt the time series of (co)variances. They therefore adopt the GAS framework of Creal *et al.* (2013), that uses the score of the conditional density function to drive the dynamics of the latent covariance matrix. More specifically, they propose a fat-tailed Student's-t for the returns and a matrix-F distribution the realized covariance matrix respectively, where both densities depends on a latent covariance matrix. The score is then defined as the sum of the partial derivative of both individual observation densities with respect to the time-varying covariance matrix. Applying this to the univariate setting, we assume a Student's t distribution for the individual returns $y_{i,t}$ with ν_{0i} degrees of freedom, and an F-distribution with ν_{1i} and ν_{2i} degrees of freedom for the realized variance $RV_{i,t}$. The univariate HEAVY GAS tF model is then given by:

$$\begin{aligned}
y_{i,t} &\sim t(y_{i,t} \mid \mu, h_{i,t}, \nu_{0i}) \qquad RV_{i,t} \sim F(RV_{i,t} \mid h_{i,t}, \nu_{1i}, \nu_{2i}) \\
y_{i,t} &= \phi_{0,i} + \sum_{q=1}^{Q} \phi_{q,i} y_{i,t-q} + \sqrt{h_{it}}\eta_{i,t} \\
h_{i,t+1} &= \omega_i + \alpha_i\, s_{i,t} + \beta_i\, h_{i,t} \\
s_{i,t} &= S_{i,t} \nabla_{i,t} \\
\nabla_{it} &= \nabla_{y,it} + \nabla_{RV,it} = \frac{\partial \log t(y_{i,t}|\phi_0, h_{i,t}, \nu_i)}{\partial h_t} + \frac{\partial \log F(RV_{i,t}, \mid h_{i,t}, \nu_{1i}, \nu_{2i})}{\partial h_{i,t}},
\end{aligned} \tag{27}$$

with $h_{i,t}$ the conditional variance of asset i at time t, and ∇_{it} the score at time t. We follow Opschoor *et al.* (2017) by scaling the score $s_{i,t}$ with $2h_{i,t}^2$. The interpretation of the scaled score is intuitive here: large values of $y_{i,t}$ and $RV_{i,t}$ will be downweighted since the possible outlier (jump) might just appear as an result of the assumed fat-tailedness of the returns of realized variances.

We estimate the univariate HEAVY GAS tF model of (27) on the 100 time series. For the conditional mean model, we find some significance of the first two AR lags. Table 4 shows the mean and several quantiles of the estimated parameters in the cross-section. The table shows the fat-tailed nature of both the stock returns and realized variances, as the mean of ν_0 and ν_2 are equal to 8.47 and 14.68 respectively. We follow CT2015 and evaluate the fit of the marginal distributions by transforming the PIT $\hat{u}_{it}$ into Gaussian variables $\bar{x}_{it} = \Phi^{-1}$. Then we test for each series i, $(i = 1, \ldots, 100)$ on normality by the Kolmogorov-Smirnov test. Across the 100 firms, only in case of 11 models the null-hypothesis of normality is rejected. Although the size exceeds the nominal test level of 5%, we restrict ourself to the current marginal distribution for the sake of parsimony and comparability.

[insert Table 4]

4.3 Factor copula results

After estimating the parameters of the marginal distributions, we estimate the parameters of the factor copula models with various factor structures on the full sample of 3521 observations. Beyond differentiating across the factor structure, we also pay attention to the distribution (Gaussian vs. Student's t) and the inclusion of the realized equi-correlation. The groups are formed according to the specific industry of a particular stock (see Table 3).

We benchmark our factor copula models with new factor structures against the cDCC model (Engle, 2002) (with the correction of Aielli (2013)) and the (Block-) DECO model of

Engle and Kelly (2012). These models are governed by

$$
\begin{aligned}
\mathbf{Q}_{t+1} &= \mathbf{\Omega} + A\,\mathbf{Q}_t^{*}\mathbf{x}_t\mathbf{x}_t^{\top}\mathbf{Q}_t^{*} + B\,\mathbf{Q}_t \\
\mathbf{R}_t^{cDCC} &= \mathbf{Q}_t^{*-1}\mathbf{Q}_t\mathbf{Q}_t^{*-1} \\
\mathbf{R}_t^{DECO} &= \frac{1}{N(N-1)}(\boldsymbol{\iota}^{\top}\mathbf{R}_t^{cDCC}\boldsymbol{\iota} - N)
\end{aligned} \tag{28}
$$

with $\mathbf{Q}_t^*$ a diagonal matrix with entries $q_{ii,t}$, A and B scalars and $\mathbf{\Omega}$ a $N \times N$ matrix. In addition, we consider the Block-DECO model where each block represents a specific industry and could have a potentially different correlation. See Engle and Kelly (2012) for more details about this model specification.

In order to compare our models with the benchmarks, we put them in a copula framework. That is, $x_{it} = P^{-1}(u_{it})$ with u_{it} estimated in a first step by the marginals, and P^{-1} the inverse pdf of the copula at hand. For the DECO models, we assume both a Gaussian and Student's t copula for the dependencies. We estimate the cDCC model by means of the Composite Likelihood method of Engle *et al.* (2008). Further, the Block DECO model parameters can only be estimated by assuming a Gaussian distribution.[3]

[insert Table 5]

Table 5 shows the parameter estimates and maximized log-likelihoods from Factor Copula GAS (FC GAS) models with one factor (with homogenous or group-specific loadings), two factors (one equi factor with fixed and one factor with group-specific loadings), 11 factors (one equi factor and 10 group factors, denoted as Mf-Equi-Group) and $10 + 2$ factors (the Mf-Equi-Group model plus a factor with group-specific loadings). To save space, we do not report the intercepts ω_g that varies per group of each factor copula that contains group factors or group-specific loadings. Panel A.1 and A.2 lists the results from the Gaussian en t-factor copulas respectively, Panel B shows parameter estimates from the benchmark models.

[3] In this model, the multivariate Gaussian density is divided into a sum of overlapping bivariate Gaussian densities, see Engle and Kelly (2012). This can not be done in case of a Student's t distribution with ν degrees of freedom.

The main result from the table is that multi-factor models provide a much better fit than the one-factor copula models. For example, the log-likelihood difference between the Mf-Full-t model and the 1f-Equi-t model exceeds the striking amount of 15,000 points. The most gain with respect to the factor structure is obtained by including industry factors, as the log likelihood increases almost 10,000 points in both the Gaussian and t-factor copula's. The Gaussian multi-factor models also perform relatively better than the DECO and cDCC model, while the multi-factor t-copula models outperform the Block DECO model as well.

Table 5 shows two additional interesting results beyond our main result. First, the fit of t-factor copulas is considerably better than the Gaussian factor copula models with differences around 2000 points in the maximized log-likelihood. Second, there is a strong persistence in the time-varying factor loadings, as the value of B is around 0.97 for most of the estimated (t-) factor copula models. These two findings confirm the empirical analysis of OP2017b on the log-differences of U.S. CDS spreads.

[insert Figure 2]

Figure 2 shows the different factor loadings corresponding with the FC-Mf-Full-t model, which has the richest factor structure and the best statistical fit. The top sub-figure shows the equi-factor loading, hence the systemtic part that hits the dependence structure between all 100 assets. The equi factor loading increases in particular during crises periods, such as 2003, 2008-2009 and in the end of 2011. The middle sub-figure shows the differential effects of three industries with respect to the equi factor loading: the Capital Goods (blue line), Financials (red line) and the Health industry (yellow line). Notably, the loadings of the Financials with respect to the latent factor exceed the loadings of the Health and Capital Goods industries during the period 2001-2009 and after 2013. Huge upward spikes are visible as well for the Financials during the heat of the global financial crisis (2007-2009). The lower sub-figure depicts the evaluation of the industry-specific factor loadings. The main take-away is that the three depicted industry factors behave quite differently: the Capital Goods factor loadings are considerably lower than the Health and Financial factor loadings, while the last two industries seems negatively related to each-other. Hence allowing for different industry factors with their own GAS loadings shows distinctive patterns, which has a enormous effect on the statistical fit of the model, as shown before.

[insert Figure 3]

Figure 3 plots the implied within correlation of Financials and Capital Goods industry factor loadings according to the Mf-Full-t model and a benchmark (i.e. the Block-DECO model). The sub-figures shows a similar pattern of the factor loadings, although the correlation within Financials is around 0.8 or higher in 2003, 2008 and 2011 (crisis periods) for the multi-factor copula, while according to the Block-DECO model these within correlations are around 0.7. In addition, the differences between the within correlations of both industries are much smaller in the period January 2007-June 2008 implied by the Multi-factor t-copula model than implied by the benchmark model. We would like to emphasize that these findings does not imply that one pattern is automatically better, as we do not know the true correlation pattern. We therefore conduct an out-of-sample density forecast exercise in the next sub-section to discriminate between the forecasting power of factor copulas and the cDCC/DECO models.

The effect of including realized equi-correlation into the model specification of the factor loadings is listed in Table 6, where parameter estimates are shown of the Gaussian and t-factor copula models including the high-frequency based correlation (see (20)).[4] Comparing this table with Table 5 clearly suggests that including the realized correlation improves the fit of the model, especially for one- and two-factor copula models. For example, the log-likelihood of the FC-1f-Group N model increases with more than 700 points from 66,030 to 66,766 by including only one exogenous variable to the specification of the factor loadings. The effect seems somewhat stronger for the Gaussian factor copulas than for the t-factor copulas. Moreover, the effect declines rapidly when the factor structure enriches: in case

[4]We have also experienced with heterogeneous realized group correlations, where we average each pairwise realized correlation within an industry. The results did not improve upon including the simple realized equi-correlation into the specification of the factor loadings.

of the MF-Full-t model, the log-likelihood increase equals just 14 points (from 83,262 to 83,278). Finally, note that the inclusion of the realized equi-correlation negatively affects the persistence parameter B. Hence the importance of the High-Frequency based innovation increases relative to the score-based innovation. This result has also been found in (multivariate) volatility models (Noureldin *et al.*, 2012, see for example). The HF-based innovation is more important than the score-based innovation based on daily returns which is not surprising as the former contains more information.

Figure 4 shows the impact of including the realized correlation on the fit of the correlations according to the FC-1f-Group-t model. The upper panel depicts the correlation within Capital Goods company returns (blue line) and Financial company returns (red line) without the realized correlation, the lower panel shows the same correlations, but now with inclusion of the realized equi-correlation. The figure shows that including this variable into the factor model produces more dynamic correlation patterns, especially in crises periods. Hence High-Frequency based information adjust changes in dependence much faster than information based on daily returns.

To summarize the in-sample findings, the statistical fit of multi-factor models is considerably higher than the fit of one-factor models. In addition, the multi-factor models outperform recent benchmarks as the cDCC and (block) DECO models. Including realized equi-correlation has a positive effect on the fit, although this effect declines when the factor structure of the copula enriches.

4.4 Multivariate Density Forecasts

A natural way to assess the out-of-sample (OOS) forecasting performance of our various factor copula models is to consider multivariate density forecasts, as we have closed-form copula densities. We follow Salvatierra and Patton (2015) as we do not only compute density forecasts over the entire support, but also on the left joint tail, using the approach of Diks *et al.* (2014). Using the log scoring rule (see Mitchell and Hall (2005), Amisano and Giacomini (2007), the multivariate one-step ahead density forecasts boils down to the OOS copula log-likelihood. Note that the marginal distributions drop out from the multivariate density forecasts as all considered (factor-) copulas and benchmarks has similar marginal specifications.

Using a moving-window with an in-sample period of 1000 observations (which corresponds roughly to four calender years), leaves $P = 2521$ observations for the out-of-sample period, starting at 28 December 2004. Hence the OOS period includes the Great Financial Crisis. We re-estimate each model after roughly two calender months (i.e. 50 observations).

Define the difference in the log score between two copula density forecasts M_1 and M_2 as

$$d_{ls,t} = S_{ls,t}(\boldsymbol{u}_t, M_1) - S_{ls,t}(\boldsymbol{u}_t, M_2) \tag{29}$$

for $t = 1001, 1002, \ldots, T-1$ with $S_{ls,t}(\boldsymbol{u}_t, M_j)$ $(j = 1, 2)$ the log score at time t of the density forecast corresponding to model M_j,

$$S_{ls,t}(\boldsymbol{u}_t, M_j) = \log c_t(\boldsymbol{u}_t | R_t, \mathcal{F}_{t-1}, M_j) \tag{30}$$

where $c_t(\cdot)$ is the Gaussian or t-copula density. The null- and alternative hypotheses of equal predictive ability are now given by

$$H_0 : \mathbb{E}[d_{ls}] \quad = 0 \tag{31}$$

$$H_A : \mathbb{E}[d_{ls}] \quad \neq 0 \tag{32}$$

for all P OOS forecasts. This hypothesis can be tested by means of a Diebold and Mariano (1995) (DM) test statistic

$$DM_{ls} = \frac{\bar{d}}{\sqrt{\hat{\sigma}^2/P}} \tag{33}$$

with $\bar{d}$ the out-of-sample average of the log score differences and $\hat{\sigma}^2$ a HAC-consistent variance estimator of the true variance σ^2 of d_{ls}. This test-statistic is asymptotically N(0,1) distributed under the assumptions of the framework of Giacomini and White (2006). A significantly positive value of DM_{ls} means that model M_1 has superior forecast performance over model M_2.

Diks *et al.* (2014) propose to use the conditional likelihood (cl) score function to compare density forecasts instead of the log score, which is in our case given by

$$S_{cl,t}(\boldsymbol{u}_t, M_j) = [\log c_t(\boldsymbol{u}_t|R_t, \mathcal{F}_{t-1}, M_j) - \log C_t(\mathbf{q})]\, I[\boldsymbol{u}_t < \mathbf{q}] \tag{34}$$

where $\mathbf{q}$ is a $N \times 1$ vector and $C_t(\cdot)$ the conditional Copula function. Hence (34) is the log-likelihood of model M_j conditional on the fact that $\boldsymbol{u}_t < \mathbf{q}$. For any q between 0 and 1, this boils down to the joint lower region $\prod_{i=1}^{N}[0, q]$. Obviously, when $q = 1$ we are back to the log score. The above test-framework can now be used again, where H_0 and H_A change into $\mathbb{E}[d_{cl}] = 0$ vs. $\mathbb{E}[d_{cl}] \neq 0$ and

$$d_{cl,t} = S_{cl,t}(\boldsymbol{u}_t, M_1) - S_{cl,t}(\boldsymbol{u}_t, M_2). \tag{35}$$

We consider $q = [0.01, 0.05, 0.10, 0.25]$ such that we compare the copula density forecasts in the joint lower 1, 5, 10 and 25% tail.

Since we deal with a lot of models due to various factor structures and hence many different density forecasts, we also consider the Model Confidence Set (MCS) of Hansen *et al.* (2011) with a significance level op 5%, applied to the (minus) log score values and conditional likelihoods, to correct for the interdependence between all models.

[insert Table 7]

Table 7 shows the results of comparing copula density forecasts over the whole region (Panel A) and over the joint lower tails (Panels B.1-B.4), based on the factor copula's and the benchmarks. We show the mean of the log score (conditional likelihood) as well as the pair-wise DM test statistics of the Mf-Full-t model against all other models. Finally, we show the p-values of the Model-Confidence-Set approach. The table shows three interesting results. First, Panel A shows that the Mf-Full-t model has superior predictive ability of the whole support of the copula density compared to all one- and two-factor copulas and the benchmarks, as indicated by both the the pairwise DM test-statistics and the MCS results. Hence allowing for different industry-factors and an additional factor with industry-specific loadings leads to superior density forecasts of the copula density. Second, when considering the 5, 10 and 25% joint left tail of the copula density, again the Multi-factor t-copula model significantly outperforms our competitors. In case of the lower 1% joint tail, this model plays at par with the cDCC model. The superiority of the multi-factor copula models in density forecasts is an important result, as the left joint tail is of particular interest with respect to risk-management purposes. Third, similar to the in-sample results, the most gain is obtained by allowing for industry factors. For example, adding one factor with time-varying loadings to the 1f-Group-t model increases the average log-score by 1.4 points (from 20.96 to 22.31), however allowing for different industry factor loadings implies an additional increase in 2.4 points as the average log-score of the Mf-t model equals 24.76.

[insert Table 8]

Table 8 confirms our in-sample results on including the realized correlation into the factor loading specification. The table shows pair-wise DM test statistics on equal predictive ability on the whole support and the lower 1,5, 10 and 25% joint lower tail of the copula density of each type of factor copula, with and without including the realized correlation. For the most one-factor models, the difference in the log score or conditional likelihood is statistically significant. Hence including the realized correlation improves the density forecasts. This result confirms Salvatierra and Patton (2015), who find a similar result in bivariate copula's. Notably, the impact is higher in the class of Gaussian factor copulas than t-factor copulas. When the factor structure enriches, this effect becomes insignificant or even becomes negatively significant, hence including the realized equi-correlation worsens the predictive ability. Only in case of the lower joint 1% tail, the effect seems positive for all considered factor copulas, but it is not statistically significant for the two- and multi-t-factor copulas.

[insert Table 9]

Finally, Table 9 shows the MCS results applied to the set all of factor copula models, hence with and without including the realized correlation, and in including the benchmark models. We present only the models that stay within the confidence set plus the associated p-value. We find again that the Mf-Full-t copula model is superior in the full support, as well as in the 5, 10 and 25% joint lower tail. Further, it predicts at-par with the cDCC model in the 1% lower tail. Moreover, including realized measures into the loading specification has some positive influence as the MF-Full-t (RM) model also belongs to the confidence set in case of the 1 and 5% joint lower tail.

In sum, we conclude that the one-step ahead copula density forecast of the whole support as well as the left 5, 10 and 25% joint lower tail of multi-(t)-factor models are superior against one-factor models and the DCC/DECO class of models. Including the realized equi-correlation improves the accuracy of the density forecasts of one- and two-factor copula models, but the effect vanishes for multi-factor t-copula models.

5 Conclusions

We have introduced various factor structures within the class of closed-form factor copula models for high dimensions, building on recent work of Oh and Patton (2017b) and Creal and Tsay (2015). The factor structures are based on group-specific characteristics. We introduce new score-driven dynamics for the time-varying factor loadings. The resulting factor copula specification is computationally tractable and in closed form such that parameters can straightforwardly be estimated by Maximum Likelihood. In addition, an important feature of our model is that it easily allows for inclusion of covariates into the model, while avoiding any positive definite restrictions.

We model the dependency across 100 equity returns listed at the S&P 500 index over the period 2001-2014 and show that the multi-factor copula model based on industry characteristics has a better fit than one-factor models and benchmarks such as the cDCC and (Block-)DECO models. Out of sample, one-step ahead density forecasts are superior to or competes with our benchmarks when considering the whole support, as well as the 1, 5, 10 or 25% joint lower tail of the copula density. Finally, the inclusion of realized equi-correlation into the factor copula specification improves density forecasts, although the effect diminishes when the factor specification enriches.

References

Aielli, G.P. (2013), Dynamic conditional correlation: on properties and estimation, *Journal of Business & Economic Statistics* **31**, 282–299.

Amisano, G. and R. Giacomini (2007), Comparing density forecasts via weighted likelihood ratio tests, *Journal of Business and Economic Statistics* **25**, 177–190.

Andersen, T., T. Bollerslev, F.X. Diebold and P. Labys (2003), Modeling and forecasting realized volatility, *Econometrica* **71**, 529–626.

Azzalini, A. and A. Capitanio (2003), Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **65**(2), 367–389.

Barndorff-Nielsen, O.E., P.R. Hansen, A. Lunde and N. Shephard (2009), Realized kernels in practice: trades and quotes, *Econometrics Journal* **12**, 1–32.

Blasques, C., S.J. Koopman and A. Lucas (2015), Information theoretic optimality of observation driven time series models for continuous responses, *Biometrika* **102**(2), 325–343.

Brownlees, C.T. and G.M. Gallo (2006), Financial econometric analysis at ultra-high frequency: Data handling concerns, *Computational Statistics and Data Analysis* **51**, 2232–2245.

Cherubini, U., S. Mulinacci, F. Gobbi and S. Romagnoli (2011), *Dynamic Copula methods in finance*, Vol. 625, John Wiley & Sons.

Christoffersen, P., V. Errunza, K. Jacobs and H. Langlois (2012), Is the potential for international diversification disappearing? A dynamic copula approach, *Review of Financial Studies* **25**, 3711–3751.

Christoffersen, P., V. Errunza, K. Jacobs and X. Jin (2014), Correlation dynamics and international diversification benefits, *International Journal of Forecasting* **30**(3), 807–824.

Creal, D., B. Schwaab, S.J. Koopman and A. Lucas (2014), Observation driven mixed-measurement dynamic factor models with an application to credit risk, *Review of Economics and Statistics* **96**(5), 898–915.

Creal, D. and R.S. Tsay (2015), High dimensional dynamic stochastic copula models, *Journal of Econometrics* **189**, 335–345.

Creal, D., S.J. Koopman and A. Lucas (2013), Generalized autoregressive score models with applications, *Journal of Applied Econometrics* **28**, 777–795.

Diebold, F.X. and R.S. Mariano (1995), Comparing predictive accuracy, *Journal of Business and Economic Statistics* **13**, 253–263.

Diks, C., V. Panchenko, O. Sokolinskiy and D. van Dijk (2014), Comparing the accuracy of multivariate density forecasts in selected regions of the copula support, *Journal of Economic Dynamics and Control* **48**, 79–94.

Embrechts, P., R. Frey and A. McNeil (2005), *Quantitative Risk Management*, Vol. 10, Princeton Series in Finance Princeton.

Engle, R.F. (2002), Dynamic conditional correlation: a simple class of multivariate generalized autoregressive conditional heteroskedasticity models, *Journal of Business and Economic Statistics* **20**, 339–350.

Engle, R.F. and B. Kelly (2012), Dynamic equicorrelation, *Journal of Business Economic Statistics* **30**, 212–228.

Engle, R.F, N. Shephard and K. Sheppard (2008), Fitting and testing vast dimensional time-varying covariance models, Working Paper.

Engle, R.F., V.K. Ng and M. Rothschild (1990), Asset pricing with a factor-ARCH covariance structure: Empirical estimates for treasury bills, *Journal of Econometrics* **45**(1-2), 213–237.

Fama, E.F. and K.R. French (1993), Common risk factors in the returns on stocks and bonds, *Journal of financial economics* **33**(1), 3–56.

Fama, E.F. and K.R. French (2016), Dissecting anomalies with a five-factor model, *The Review of Financial Studies* **29**(1), 69–103.

Fan, J., Y. Fan and J. Lv (2008), High dimensional covariance matrix estimation using a factor model, *Journal of Econometrics* **147**(1), 186–197.

Fan, J., Y. Liao and M. Mincheva (2011), High dimensional covariance matrix estimation in approximate factor models, *Annals of statistics* **39**(6), 3320.

Fan, Y. and A.J. Patton (2014), Copulas in econometrics, *Annual Review of Economics* **6**, 179–200.

Galeano, P., M. Ausín and H. Nguyen (2017), Parallel Bayesian Inference for High Dimensional Dynamic Factor Copulas, Working Paper.

Giacomini, R. and H. White (2006), Tests of conditional predictive ability, *Econometrica* **74**(6), 1545–1578.

Hafner, C.M. and H. Manner (2012), Dynamic stochastic copula models: Estimation, inference and applications, *Journal of Applied Econometrics* **27**, 269–295.

Hansen, P. and A. Lunde (2011), Forecasting volatility using high frequency data, *The Oxford Handbook of Economic Forecasting, Oxford: Blackwell* pp. 525–556.

Hansen, P.R., A. Lunde and J.M. Nason (2011), The model confidence set, *Econometrica* **79**(2), 453–497.

Harvey, A.C. (2013), *Dynamic Models for Volatility and Heavy Tails: with Applications to Financial and Economic Time Series*, Cambridge University Press.

Krupskii, P. and H. Joe (2013), Factor copula models for multivariate data, *Journal of Multivariate Analysis* **120**, 85–101.

Ledoit, O. and M. Wolf (2003), Improved estimation of the covariance matrix of stock returns with an application to portfolio selection, *Journal of Empirical Finance* **10**, 603–621.

Lucas, A., B. Schwaab and X. Zhang (2014), Conditional euro area sovereign default risk, *Journal of Business and Economic Statistics* **32**(2), 271–284.

Lucas, A., B. Schwaab and X. Zhang (2017), Modeling financial sector joint tail risk in the euro area, *Journal of Applied Econometrics* **32**(1), 171–191.

McNeil, A.J., R. Frey and P. Embrechts (2015), *Quantitative risk management: Concepts, techniques and tools*, Princeton university press.

Mitchell, J. and S.G. Hall (2005), Evaluating, comparing and combining density forecasts using the KLIC with an application to the Bank of England and NIESR fan-charts of inflation, *Oxford Bulletin of Economics and Statistics* **67**(s1), 995–1033.

Noureldin, D., N. Shephard and K. Sheppard (2012), Multivariate high-frequency-based volatility (HEAVY) models, *Journal of Applied Econometrics* **27**, 907–933.

Oh, D.H. and A.J. Patton (2017a), Modeling dependence in high dimensions with factor copulas, *Journal of Business & Economic Statistics* **35**(1), 139–154.

Oh, D.H. and A.J. Patton (2017b), Time-varying systemic risk: Evidence from a dynamic copula model of cds spreads, *Journal of Business & Economic Statistics* pp. 1–15.

Opschoor, A., P. Janus, A. Lucas and D. van Dijk (2017), New HEAVY models for fat-tailed realized covariances and returns, *Journal of Business and Economic Statistics* pp. 1–15.

Patton, A.J. (2006), Modelling asymmetric exchange rate dependence, *International Economic Review* **47**, 527–556.

Patton, A.J. (2009), Copula-based models for financial time series, *Handbook of financial time series*, Springer, pp. 767–785.

Patton, A.J. et al. (2012), Copula methods for forecasting multivariate time series, *Handbook of economic forecasting* **2**, 899–960.

Salvatierra, I.D.L. and A.J. Patton (2015), Dynamic copula models and high frequency data, *Journal of Empirical Finance* **30**, 120–135.

Table 1: Various factor structures and their properties

This table summarizes the various factor structures that are proposed given that there are N assets allocated to G different groups. We show the number of factors, the number of different scaled factor loadings, the dimension of the scaled factor loading matrix and the existence of an equi factor, group-specific factor and/or group-specific loadings.

Name	# factors	# $\lambda's$	eq-factor	gr-factor	gr-loading	dim $\tilde{\boldsymbol{L}}_t^\top$
FC-1f-Equi	1	1	yes	no	no	$N \times 1$
FC-1f-Group	1	G	yes	no	yes	$N \times 1$
FC-2f-Equi-Group	2	G+1	yes	no	yes	$N \times 2$
FC-Mf-Equi-Group	G+1	G+1	yes	yes	no	$N \times (G+1)$
FC-Mf-Full	G+2	2G+1	yes	yes	yes	$N \times (G+2)$

Table 2: Parameter estimates of Multi-Factor-Copula DGP

This table provides Monte Carlo results of parameter estimates from simulated multi-factor Gaussian and t-copula processes, as given in (23)-(26). $B(N)$ and $B(t)$ denote the value of B in case of the Guassian (N) and t-factor copulas respectively. The table reports the mean and standard deviation in parentheses based on 1000 replications.

Coef.	True	FC-N	FC-t	FC-N	FC-t
		T = 500		T = 1000	
Panel A: FC-Mf-Equi-Group model					
ω_{eq}	0.010	0.012 (0.005)	0.011 (0.003)	0.011 (0.003)	0.011 (0.002)
ω_1	-0.050	-0.058 (0.014)	-0.057 (0.011)	-0.054 (0.009)	-0.055 (0.009)
ω_2	-0.043	-0.051 (0.012)	-0.049 (0.010)	-0.047 (0.008)	-0.047 (0.007)
ω_3	-0.037	-0.043 (0.010)	-0.042 (0.009)	-0.039 (0.007)	-0.040 (0.006)
ω_4	-0.030	-0.035 (0.009)	-0.034 (0.007)	-0.032 (0.006)	-0.033 (0.005)
ω_5	-0.023	-0.027 (0.007)	-0.027 (0.006)	-0.025 (0.005)	-0.026 (0.004)
ω_6	-0.017	-0.019 (0.006)	-0.019 (0.005)	-0.018 (0.004)	-0.018 (0.003)
ω_7	-0.010	-0.012 (0.005)	-0.011 (0.003)	-0.011 (0.003)	-0.011 (0.003)
ω_8	-0.003	-0.004 (0.005)	-0.004 (0.003)	-0.004 (0.003)	-0.004 (0.002)
ω_9	0.003	0.004 (0.005)	0.004 (0.003)	0.004 (0.003)	0.004 (0.002)
ω_{10}	0.010	0.012 (0.005)	0.011 (0.003)	0.011 (0.003)	0.011 (0.002)
A_{eq}	0.010	0.010 (0.001)	0.010 (0.002)	0.010 (0.001)	0.010 (0.001)
$A_{gr.f}$	0.020	0.019 (0.004)	0.018 (0.004)	0.019 (0.003)	0.019 (0.003)
$B(N)$	0.920	0.907 (0.020)		0.915 (0.013)	
$B(t)$	0.970		0.966 (0.007)		0.967 (0.005)
ν_C	30		30.37 (2.33)		30.27 (1.50)
Panel B: FC-Mf-Full model					
ω_{eq}	0.010	0.010 (0.009)	0.010 (0.003)	0.010 (0.005)	0.010 (0.002)
ω_1	-0.050	-0.057 (0.016)	-0.053 (0.011)	-0.053 (0.009)	-0.053 (0.011)
ω_2	-0.043	-0.049 (0.014)	-0.046 (0.009)	-0.046 (0.008)	-0.046 (0.008)
ω_3	-0.037	-0.042 (0.016)	-0.039 (0.006)	-0.039 (0.007)	-0.038 (0.005)
ω_4	-0.030	-0.034 (0.013)	-0.031 (0.005)	-0.032 (0.006)	-0.031 (0.004)
ω_5	-0.023	-0.026 (0.008)	-0.025 (0.004)	-0.025 (0.005)	-0.024 (0.003)
ω_6	-0.017	-0.019 (0.008)	-0.017 (0.003)	-0.018 (0.004)	-0.017 (0.002)
ω_7	-0.010	-0.011 (0.005)	-0.010 (0.002)	-0.010 (0.003)	-0.010 (0.001)
ω_8	-0.003	-0.004 (0.004)	-0.003 (0.002)	-0.004 (0.003)	-0.004 (0.001)
ω_9	0.003	0.004 (0.004)	0.004 (0.002)	0.004 (0.003)	0.003 (0.001)
ω_{10}	0.010	0.011 (0.005)	0.010 (0.002)	0.010 (0.003)	0.010 (0.002)
$\omega_{gr.l}$	0.010	0.011 (0.009)	0.010 (0.002)	0.010 (0.004)	0.010 (0.001)
A_{eq}	0.020	0.021 (0.005)	0.019 (0.005)	0.020 (0.003)	0.020 (0.003)
$A_{gr.f}$	0.015	0.014 (0.005)	0.013 (0.003)	0.014 (0.003)	0.014 (0.002)
$A_{gr.l}$	0.010	0.011 (0.008)	0.010 (0.002)	0.010 (0.003)	0.010 (0.001)
$B(N)$	0.920	0.909 (0.029)		0.915 (0.014)	
$B(t)$	0.980		0.979 (0.003)		0.980 (0.002)
ν_C	30		30.42 (2.32)		30.26 (1.61)

Table 3: S&P 500 constituents

This table lists ticker symbols of 100 companies listed at the S&P 500 index during the period January 2, 2001 until December 31, 2014. All Tickers are grouped per industry.

Ind Nr.	Industry	# Comp.	Tickers
1	Capital Goods	10	AA,BA,CAT,HON,F,NOC,UTX,A,IR,GD
2	Finance	19	AXP,JPM,AIG,BAC,C,KEY,MTB,COF,USB,BBT,STI,WFC,GS,MS,MMC,HIG,PNC,XL,MCO
3	Energy	12	GE,XOM,BHI,MUR,SLB,CVX,HAL,OXY,APC,SU,CNX,PXD
4	Consumer Services	14	HD,MCD,WMT,TGT,BXP,DIS,JCP,NLY,ANF,EQR,WY,RCL,WSM,TV
5	Consumer Non-Durables	9	KO,MO,SYY,PEP,CL,AVP,GIS,CPB,EL
6	Health Care	11	PFE,ABT,BAX,JNJ,LLY,THC,MMM,MRK,BMY,MDT,CI
7	Public Utilities	7	AEP,AEE,DUK,SO,WMB,VZ,EXC
8	Technology	5	IBM,DOV,HPQ,TSM,CSC
9	Basic Industries	9	PG,DD,FLR,DOW,AES,IP,ATI,LPX,POT
10	Transportation	4	LUV,UPS,NSC,FDX

Table 4: Marginal distribution parameter estimates

This table reports summaries of the maximum likelihood parameter estimates of the HEAVY GAS tF models of (27) on 100 different daily time series of equity returns and realized variances. The columns present the mean and quantiles from the cross-sectional distribution of the parameters listed in the rows. Data are observed over the period January 2, 2001 until December 31, 2014 ($T = 3521$ trading days).

	Mean	5%	25%	Med	75%	95%
ϕ_0	0.029	-0.031	0.012	0.027	0.046	0.092
ϕ_1	-0.009	-0.053	-0.026	-0.009	0.009	0.027
ϕ_2	-0.013	-0.048	-0.029	-0.011	0.000	0.019
ω	0.055	0.027	0.034	0.044	0.068	0.126
α	0.831	0.621	0.746	0.826	0.910	1.016
β	0.982	0.969	0.978	0.982	0.989	0.996
ν_0	8.47	4.84	6.32	8.17	9.95	13.28
ν_1	22.87	16.87	20.35	22.30	25.00	30.71
ν_2	14.68	10.67	12.92	14.86	16.39	19.07

	$\sharp$ of rejections
KS test for Student's t dist of std. residuals	10

Table 5: Factor Copula parameter estimates and benchmarks

This table reports maximum likelihood parameter estimates of various factor copula models, the (Block) DECO model of Engle and Kelly (2012) and the cDCC model of Engle (2002), applied to daily equity returns of 100 assets listed at the S&P 500 index. We consider five different factor copula models, see Table 1 for the definition of their abbreviations. Panel A.1 denote the factor models with a Gaussian copula density, Panel A.2 list the parameter estimates corresponding with the t-factor copula. Further, Panel B denotes the estimates of our benchmark models. In case of the cDCC model, the table shows parameters estimates obtained by the Composite Likelihood (CL) method. Standard errors are provided in parenthesis and constructed from the numerical second derivatives of the log-likelihood function. We report the copula log-likelihood for all models. The sample is January 2, 2001 until December 31, 2014 (3521 observations).

Model	ω_{eq}	A_{eq}	A_{ind}	A_{gr}	B	ν_C	LogL
Panel A.1: Gaussian factor copula's							
FC-1f-Equi	-0.036	0.007			0.886		63,888
	(0.009)	(0.001)			(0.026)		
FC-1f-Group				0.011	0.979		66,030
				(0.001)	(0.005)		
FC-2f-Equi-Group	-0.021	0.008		0.025	0.942		70,979
	(0.007)	(0.001)		(0.002)	(0.016)		
FC-Mf-Equi-Group	-0.019	0.008	0.030		0.928		80,121
	(0.005)	(0.001)	(0.003)		(0.014)		
FC-Mf-Full	-0.004	0.025	0.015	0.025	0.985		80,426
	(0.005)	(0.002)	(0.002)	(0.002)	(0.003)		
Panel A.2: t-factor copula's							
FC-1f-Equi	-0.020	0.020			0.934	38.49	67,256
	(0.005)	(0.002)			(0.013)	(1.65)	
FC-1f-Group				0.007	0.988	35.63	69,604
				(0.001)	(0.003)	(1.12)	
FC-2f-Equi-Group	-0.012	0.019		0.032	0.955	41.98	72,796
	(0.007)	(0.002)		(0.004)	(0.019)	(1.74)	
FC-Mf-Equi-Group	-0.006	0.020	0.026		0.973	48.02	82,445
	(0.003)	(0.002)	(0.003)		(0.008)	(0.43)	
FC-Mf-Full	0.000	0.062	0.013	0.018	0.992	46.47	83,262
	(0.002)	(0.009)	(0.001)	(0.002)	(0.001)	(1.61)	
Panel B: Benchmarks							
cDCC(CL)		0.019			0.964		75,604
		(0.001)			(0.001)		
DECO N		0.067			0.903		63,374
		(0.002)			(0.004)		
DECO t		0.065			0.908	33.66	67,049
		(0.004)			(0.006)	(0.88)	
Block-DECO		0.046			0.932		80,886
		(0.000)			(0.001)		

Table 6: Inserting the realized equi-correlation into Factor Copulas

This table reports maximum likelihood parameter estimates of various factor copula models applied to daily equity returns of 100 assets and their realized equi-correlations. We consider five different factor copula models, see Table 1 for the definition of their abbreviations. Panel A.1 denote the factor models with a Gaussian copula density, Panel A.2 list the parameter estimates corresponding with the t-factor copula. Standard errors are provided in parenthesis and constructed from the numerical second derivatives of the log-likelihood function. We report the copula log-likelihood for all models. The sample is January 2, 2001 until December 31, 2014 (3521 observations).

Model	ω_{eq}	A_{eq}	A_{ind}	A_{gr}	γ_{RC}	B	ν_C	LogL
Panel A.1: Gaussian factor copula's								
FC-1f-Equi	0.113	0.006			0.261	0.730		64,373
	(0.016)	(0.001)			(0.032)	(0.035)		
FC-1f-Group				0.013	0.155	0.862		66,766
				(0.001)	(0.009)	(0.011)		
FC-2f-Equi-Group	0.016	0.008		0.016	0.015	0.996		71,513
	(0.006)	(0.001)		(0.001)	(0.008)	(0.001)		
FC-Mf-Equi-Group	0.108	0.006	0.042		0.213	0.808		80,455
	(0.015)	(0.001)	(0.003)		(0.026)	(0.022)		
FC-Mf-Full	0.138	0.011	0.030	0.052	0.313	0.878		80,561
	(0.016)	(0.001)	(0.002)	(0.005)	(0.040)	(0.021)		
Panel A.2: t-factor copula's								
FC-1f-Equi	0.116	0.014			0.249	0.761	40.57	67,398
	(0.013)	(0.002)			(0.025)	(0.023)	(1.42)	
FC-1f-Group				0.010	0.080	0.930	38.54	69,926
				(0.001)	(0.020)	(0.017)	(0.74)	
FC-2f-Equi-Group	0.074	0.015		0.045	0.157	0.843	44.24	72,908
	(0.011)	(0.002)		(0.005)	(0.022)	(0.024)	(1.97)	
FC-Mf-Equi-Group	0.065	0.015	0.040		0.123	0.894	50.66	82,500
	(0.016)	(0.002)	(0.003)		(0.028)	(0.020)	(2.30)	
FC-Mf-Full	0.037	0.060	0.013	0.018	0.046	0.992	46.79	83,278
	(0.016)	(0.008)	(0.001)	(0.002)	(0.022)	(0.002)	(1.68)	

Table 7: One-step ahead copula density forecasts

This table provides the accuracy of one-step ahead copula density forecasts of 100 daily returns of the S&P500 index, obtained by various factor copula models, assuming a Gaussian or Student's t distribution (denoted by N or t). The columns represent two types of one-factor copulas (one equi-factor or one factor with group-specific loadings, denoted by 1f-eq and 1f-gr), one two-factor model (an equi-factor plus a factor with group-specific loadings denoted by 2f) and two types of multi-factor copula models (one equi-factor plus G group-specific factors (i.e. the Mf model) and the Mf model plus an additional factor with group-specific loadings, denoted as Mf-F respectively). In addition, the last four columns show results of the benchmarks: the cDCC model of Engle (2002) and the (Block) DECO model of Engle and Kelly (2012). Panel A denotes the accuracy of the full support of the density, whereas Panel B.1 - Panel B.4 list results of the joint lower 1, 5, 10 and 25% tail of the copula density. In each panel, we present first the mean of the log score or conditional likelihood (S_{ls} and S_{cl}). The number in bold represents the model that has the highest log score or conditional likelihood across all models. Second, we list the DM-statistic of a pairwise test on predictive accuracy of the MF-F-t model against all other models. A positive DM-statistic implies that the MF-F-t model has superior density forecasts against the alternative models. Third, we present the p-value associated with the Model Confidence Set of Hansen *et al.* (2011), based on a significance level of 5%. Bold numbers in this row represent those models which stay within the model confidence set. The out-of-sample period goes from January 2005 until December 2014 and contains 2519 observations.

	1f-eq-N	1f-eq-t	1f-gr-N	1f-gr-t	2f-N	2f-t	Mf-N	Mf-t	Mf-F-N	Mf-F-t	cDCC	DECO-N	DECO-t	BL-DECO
Panel A: full support														
S_{ls}	19.36	20.31	19.96	20.96	20.44	22.31	24.14	24.76	24.23	**24.90**	21.59	19.20	20.24	24.46
DM_{ls}	35.79	37.05	34.31	35.07	10.83	30.63	13.68	8.55	12.65		12.84	36.93	38.09	6.39
MCS p-val	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)	(0.00)	(0.00)	(0.00)
Panel B.1: 1% joint left tail														
S_{cl}	1.479	1.522	1.476	1.521	1.503	1.543	1.533	1.564	1.535	1.568	**1.577**	1.469	1.519	1.527
DM_{cl}	4.77	3.45	4.87	3.80	4.55	3.36	4.39	2.15	4.27		-0.61	4.89	3.68	3.79
MCS p-val	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.14)	(0.00)	**(0.53)**	**(1.00)**	(0.00)	(0.00)	(0.00)
Panel B.2 5% joint left tail														
S_{cl}	4.211	4.335	4.248	4.379	4.283	4.463	4.494	4.590	4.501	**4.606**	4.579	4.186	4.328	4.505
DM_{ls}	9.33	8.13	8.97	7.52	7.83	7.13	7.56	4.43	7.30		0.90	9.80	8.39	5.66
MCS p-val	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	**(1.00)**	(0.01)	(0.00)	(0.00)	(0.00)
Panel B.3 10% joint left tail														
S_{cl}	6.171	6.396	6.266	6.499	6.313	6.626	6.730	6.898	6.748	**6.922**	6.806	6.126	6.384	6.751
DM_{cl}	12.84	11.46	11.84	10.08	11.06	10.24	9.37	4.63	9.26		2.67	13.27	11.70	7.14
MCS p-val	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)	(0.00)	(0.00)	(0.00)
Panel B.4 25% joint left tail														
S_{cl}	9.22	9.62	9.44	9.86	9.23	10.18	10.60	10.89	10.63	**10.94**	10.53	9.11	9.60	10.66
DM_{cl}	19.54	17.85	17.87	15.75	5.11	15.37	11.86	6.27	11.52		5.10	20.16	18.13	7.83
MCS p-val	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)	(0.00)	(0.00)	(0.00)

Table 8: Realized equi-correlations and one-step ahead copula density forecasts

This table provides the accuracy of one-step ahead copula density forecasts of 100 daily returns of the S&P 500 index, obtained by various factor copula models with and without inclusion of the realized equi-correlation, assuming a Gaussian or Student's t distribution (denoted by N or t). We list the mean of the log score or conditional likelihood (S_{ls} and S_{cl}) of the factor copula model with and without inclusion of the realized equi-correlation and the DM-statistic of a pairwise test on predictive accuracy between the two models. The test is done over the whole support, as well as over the joint lower 1,5,10 and 25% tail of the copula density. A positive DM-statistic implies that including the realized correlation implies superior density forecasts against the base model. The superscripts ***, ** and * indicate significance at the 1%, 5% and 10% level respectively. The out-of-sample period goes from January 2005 until December 2014 and contains 2519 observations.

	Full support			1% joint left tail			5% joint left tail			10% joint left tail			25% joint left tail		
Model	S_{ls}	S_{ls}^{RM}	DM_{ls}	S_{ls}	S_{ls}^{RM}	DM_{ls}	S_{ls}	S_{ls}^{RM}	DM_{ls}	S_{ls}	S_{ls}^{RM}	DM_{ls}	S_{ls}	S_{ls}^{RM}	DM_{ls}
1f-eq-N	19.36	19.49	4.26^{***}	1.479	1.494	3.80^{***}	4.211	4.249	4.65^{***}	6.171	6.227	5.40^{***}	9.22	9.28	3.75^{***}
1f-eq-t	20.31	20.35	4.43^{***}	1.560	1.563	3.14^{***}	4.389	4.398	3.88^{***}	6.508	6.519	3.78^{***}	9.62	9.63	2.82^{***}
1f-gr-N	19.96	20.08	2.44^{**}	1.476	1.498	3.99^{***}	4.248	4.300	4.44^{***}	6.266	6.338	4.85^{***}	9.44	9.51	2.88^{***}
1f-gr-t	20.96	20.99	1.25	1.559	1.569	3.11^{***}	4.436	4.457	2.91^{***}	6.612	6.636	2.69^{***}	9.86	9.88	1.47
2f-N	20.44	21.43	2.51^{**}	1.503	1.516	1.73^{*}	4.283	4.370	3.18^{***}	6.313	6.447	3.84^{***}	9.23	9.85	1.90^{*}
2f-t	22.31	22.31	0.09	1.581	1.583	0.28	4.518	4.502	−1.16	6.722	6.715	−0.34	10.18	10.20	0.65
Mf-N	24.14	24.18	1.31	1.533	1.544	3.90^{***}	4.494	4.520	3.89^{***}	6.730	6.766	4.05^{***}	10.60	10.63	2.18^{**}
Mf-t	24.76	24.36	-8.30^{***}	1.604	1.606	0.92	4.651	4.642	−1.24	6.982	6.935	-3.96^{***}	10.89	10.77	-5.88^{***}
Mf-F-N	24.23	24.25	0.64	1.535	1.546	3.48^{***}	4.501	4.527	3.94^{***}	6.748	6.778	3.37^{***}	10.63	10.67	2.80^{***}
Mf-F-t	24.90	24.58	-6.67^{***}	1.609	1.610	0.60	4.667	4.660	−1.01	7.007	6.972	-2.87^{***}	10.94	10.85	-4.51^{***}

Table 9: Model Confidence Set results on copula density forecasts of all models

This table reports test results on equal predictive accuracy of one-step ahead copula density forecasts by means of the Model Confidence Set (MCS) procedure of Hansen *et al.* (2011). The copula density forecast are obtained from 24 different models, applied to daily equity returns of 100 assets listed at the S&P 500 index. We have five different types of Factor Copula models: two types of one-factor copulas (one equi-factor or one factor with group-specific loadings), one two-factor model (an equi-factor plus a factor with group-specific loadings) and two types of multi-factor copula models (the Mf model that contains one equi-factor and G group specific factors and the Mf-Full model that adds another factor with group-specific loadings to the Mf model). Each type of model is further discriminated across distribution (Gaussian vs. a Student's t) and inclusion of the realized equi-correlation into the factor copula model specification (RM). This results in 20 different factor copulas. In addition, we have the cDCC model of Engle (2002) and the (Block) DECO model of Engle and Kelly (2012) as benchmarks. Panel A denotes the accuracy of the full support of the density, whereas Panel B.1 - Panel B.4 list results of the joint lower 1,5,10 and 25% tail of the copula density. The table reports the models that stay within the MCS, as well as the associated p-value. The out-of-sample period goes from January 2005 until December 2014 and contains 2519 observations.

Models in MCS	MCS p-value
Panel A: full support	
Mf-Full-t	(1,00)
Panel B.1: 1% joint lower tail	
Mf-t	(0.12)
Mf-t (RM)	(0.35)
Mf-Full-t	(0.57)
Mf-Full-t(RM)	(0.62)
cDCC	(1.00)
Panel B.2: 5% joint lower tail	
Mf-Full-t (RM)	(0.39)
Mf-Full-t	(1.00)
Panel B.3: 10% joint lower tail	
Mf-Full-t	(1.00)
Panel B.4: 25% joint lower tail	
Mf-Full-t	(1.00)

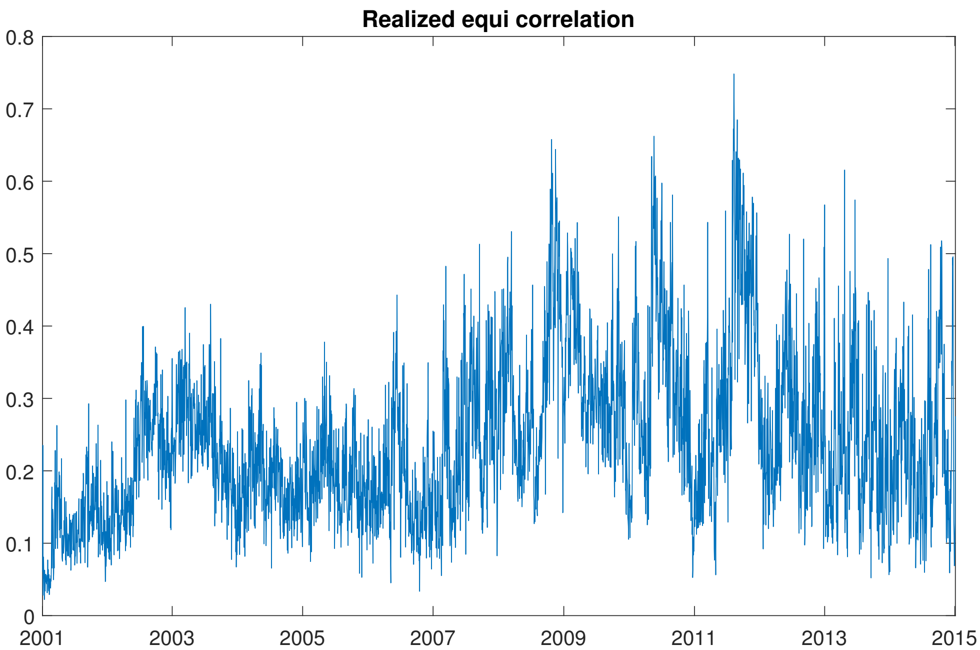

Figure 1: Realized equi-correlations

This figure shows the daily averaged pairwise realized correlations between 100 stocks listed at the S&P 500 index. Pairwise realized correlations are backed out from the 5-minute realized covariance and realized variance respectively. The sample spans the period from January 2, 2001 until December 31, 2014 ($T = 3521$ days).

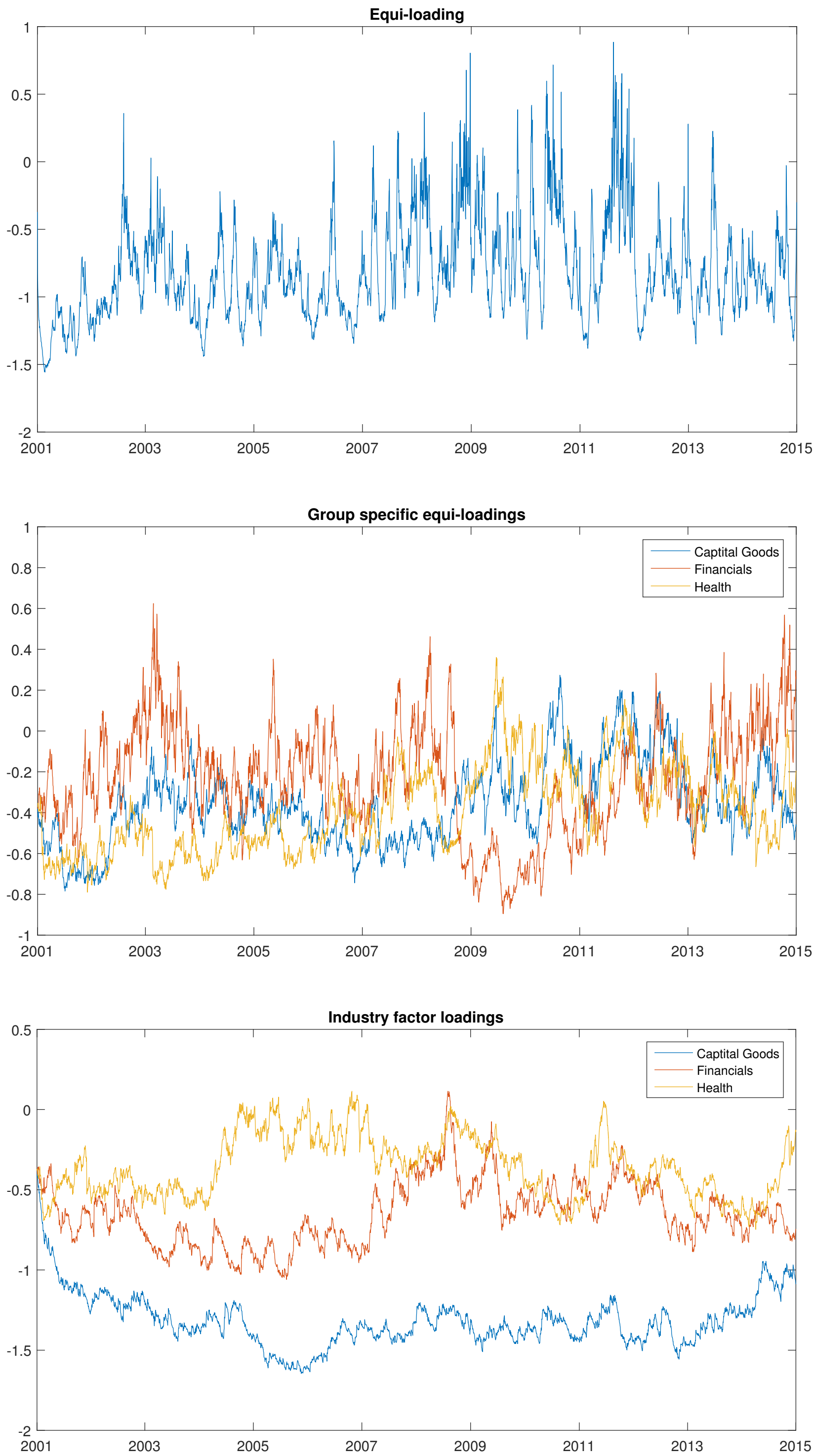

Figure 2: Factor loadings of the Multi-factor-Full t-copula

This figure shows within dependencies through time of stock returns of three types of industries: Financials, Capital Goods and Health companies, according to the Multi-factor-Full t copula model. This model consists of one equi factor (upper panel), one factor with industry-specific loadings (middle panel) and industry-specific factors (lower panel). The sample spans the period from January 2, 2001 until December 31, 2014 ($T = 3521$ days).

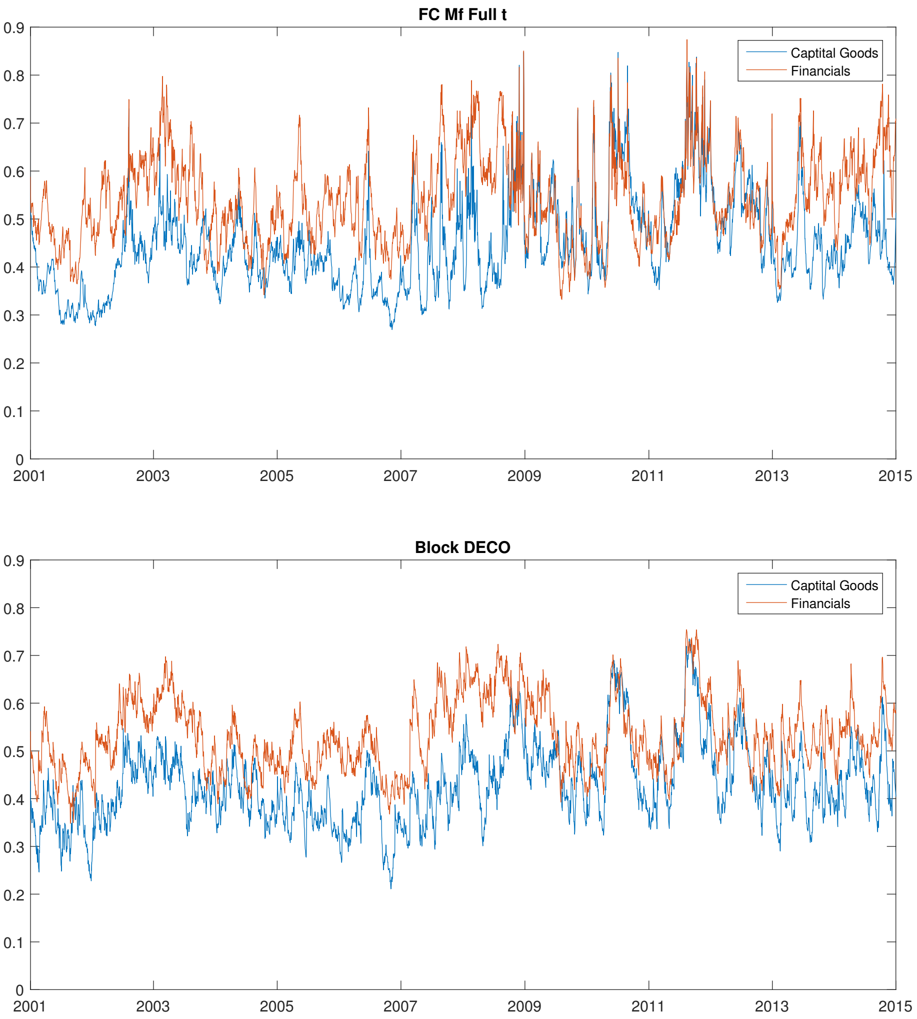

Figure 3: Within group dependencies

This figure shows the within dependence between stock returns of 19 Financial companies (red line) and 11 Capital Goods companies (blue line) through time, according to the Multi-Factor-Full t-copula model (upper panel) and the Block-DECO model (lower panel). The sample spans the period from January 2, 2001 until December 31, 2014 ($T = 3521$ days).

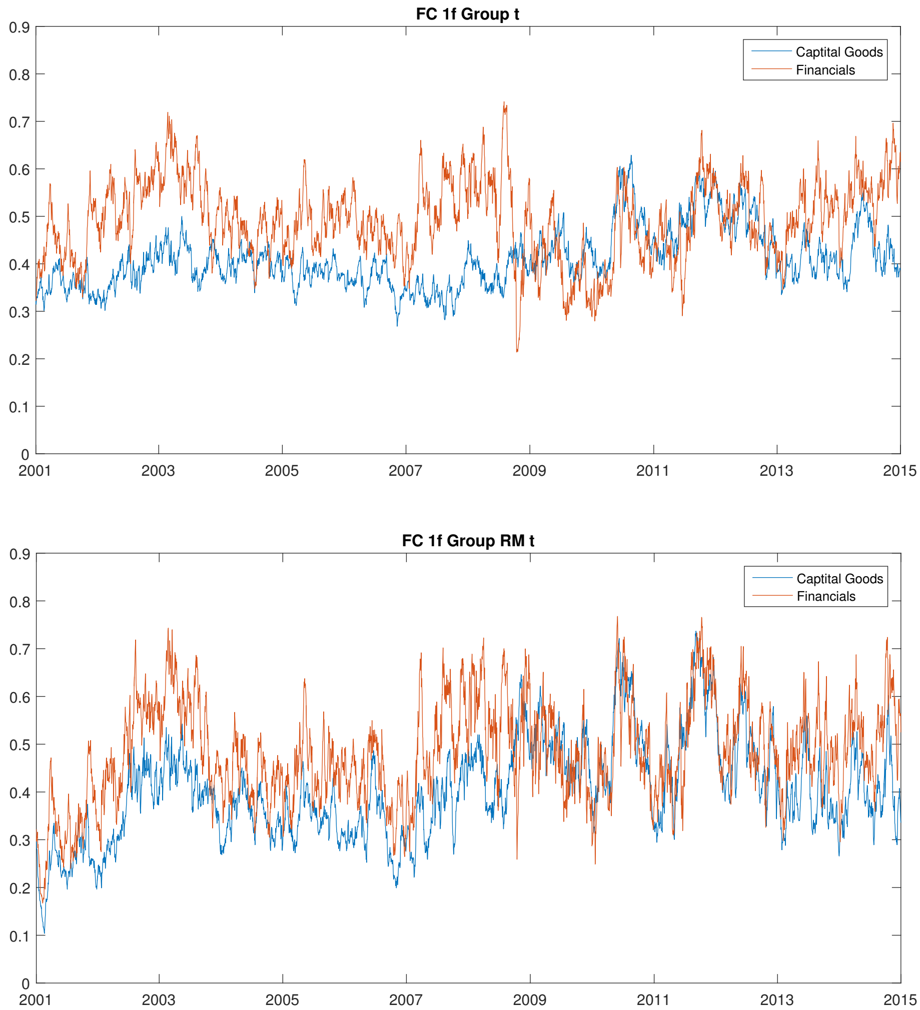

Figure 4: Within group correlations and the effect of realized equi-correlations
This figure shows the within dependence between stock returns of 19 Financial companies (red line) and 11 Capital Goods companies (blue line) through time, according to the factor copula model with one factor with group-specific loadings (upper panel) and the same model with including the realized equi-correlation into the model specification (lower panel). The sample spans the period from January 2, 2001 until December 31, 2014 ($T = 3521$ days).