Turnover Minimization: A Versatile Shrinkage Portfolio Estimator

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Abstract

We develop a shrinkage model for portfolio choice. It places a layer on a conventional portfolio problem where the optimal portfolio is shrunk towards a reference portfolio. Our model can be easily tailored to accommodate a wide range of portfolio problems with various objectives and constraints while its implementation is simple and straightforward. A data-driven method to determine the shrinkage level is offered. A comprehensive comparative study suggests that our model substantially enhances the performance of its underlying model and outperforms existing shrinkage models as well as the naïve strategy. The naïve strategy serves better as the reference portfolio than the current portfolio.

JEL Classification: G11

Keywords: Turnover minimization; Shrinkage estimator; Equal-weight portfolio; Portfolio optimization

1. Introduction

There has been a long debate on the effectiveness of optimal portfolios and their competitive advantages against the naïve, equal-weight portfolio (*a.k.a.* 1/N rule). In their seminal work, DeMiguel et al. (2009) test fourteen portfolio strategies on seven datasets and find none of them consistently outperform the naïve strategy. They further show that when returns are *i.i.d.* normal, an unrealistically long sample period is required for the Markowitz (1952) mean-variance portfolio to outperform the equal-weight portfolio. Although their results are somewhat exaggerated (*e.g.*, see discussions in Kirby and Ostdiek (2012) and Kan et al. (2016)), the sheer number of citations of their paper reflects the impact it has brought to academia and industry.[1] Undoubtedly, there has been backlash. Kirby and Ostdiek (2012), using similar sets of data, find that a mean-variance strategy constrained to invest only in risky assets outperforms the naïve strategy. Bessler et al. (2014) show that a strategy based on the Black and Litterman (1992) framework outperforms the naïve strategy when applied to a multi-asset dataset.

The race between the naïve and optimal strategies essentially depends upon the predictability of input parameters, *i.e.*, expected returns and covariance matrix. If both input parameters

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[1]Based on Google Scholar search, their paper has been cited 1,539 times at the time of writing (August 2017).

are unknown, it would be reasonable to assume that all the assets have the same expected return and variance. Alternatively, based on asset pricing models, the same return-risk ratio could be assumed for all assets. Both assumptions lead to the equal-weight portfolio as the optimal portfolio.[2] If only the variances are known, we may assume the covariances of all asset pairs are equal and so are the expected returns. This would lead to the volatility timing strategy of Kirby and Ostdiek (2012). If the covariance matrix is known, the expected returns could be assumed the same across assets, in which case, the minimum-variance portfolio would be optimal. If all the input parameters are known, the Markowitz (1952) mean-variance portfolio should be the choice. In reality, input parameters will be estimated with errors and a portfolio strategy that takes estimation errors into account, *e.g.*, a Bayesian method or robust optimization, would be preferred.

From this perspective, good performance of the naïve strategy merely reaffirms the difficulty of reliable input parameter estimation. Even when the input parameters can be predicted to a certain degree, the classical mean-variance strategy could result in a ruinous allocation due to its high parameter sensitivity and error-maximizing property (Michaud, 1989), and it is crucial to address estimation errors for successful utilization of portfolio optimization.

There has been a considerable amount of effort dedicated to address input parameter uncertainty and portfolio sensitivity. One pilar has been formed by the Bayesian approach: *e.g.*, Klein and Bawa (1976), Brown (1976, 1978), Jorion (1986), Black and Litterman (1992), Pástor (2000), Pástor and Stambaugh (2000). For a review of Bayesian models, the reader is referred to Avramov and Zhou (2010). More recently, the robust optimization that finds an optimal portfolio under a worst-case scenario became popular: *e.g.*, Goldfarb and Iyengar (2003), Fabozzi et al. (2007), Ceria and Stubbs (2016). The shrinkage estimator first proposed by Kan and Zhou (2007) optimally combines two or more portfolios so that expected utility loss is minimized. This approach has been adopted later by Tu and Zhou (2011), DeMiguel et al. (2015), and Kan et al. (2016), among others. Other approaches include imposing weight constraints (Jagannathan and Ma, 2003) or using a shrinkage method for parameter estimation (Ledoit and Wolf, 2004).

Although these models have shown some degree of success, they are also subject to limitations. Most importantly, most models assume knowledge on the distributions of estimation errors, whereas their estimation can be as difficult as parameter estimation. The distributions of estimation errors are an important determinant of asset allocation and their misspecification can result in poor portfolio performance. Bayesian approaches normally assume that the covariance matrix is precisely known and focus on the estimation of the expected returns. Whilst the covariance matrix can indeed be estimated with smaller errors, as Kan and Zhou (2007) show, its estimation error can have a nontrivial impact on asset allocation and portfolio performance when combined with the estimation error of the expected returns. Furthermore, Bayesian updates are carried out at the input parameter level, which is not necessarily optimal from the portfolio perspective.

In contrast, shrinkage estimators recognize the uncertainty of both input parameters and

[2]Pflug et al. (2012) also show that the naïve portfolio is optimal when estimation errors are high.

find an optimal combination of multiple portfolios from the portfolio perspective by minimizing expected utility loss (or equivalently, maximizing expected out-of-sample utility). Nevertheless, shrinkage estimators suffer from model parameter uncertainty: the model parameters (coefficients on portfolios) are a function of the unknown input parameters and therefore inherit input parameter uncertainty. This can lead to a nontrivial utility loss. Shrinkage estimators also lack practicality. As they maximize the expected out-of-sample utility, the risk aversion parameter needs to be defined, which is not straightforward especially for institutional investors. It is also difficult to incorporate constraints such as the short-sale constraint into these models.

In this paper, we propose a new portfolio model which is termed **turnover minimization**. The main idea of the turnover minimization is to minimize the distance between an optimal portfolio and a reference portfolio subject to return and/or risk constraints.[3] In contrast to maximizing utility, this mitigates the error maximizing property of the classical mean-variance portfolio. This approach is also consistent with the decision making process of institutional investors: they often prefer to have a stable portfolio that meets their return/risk targets rather than a portfolio that maximizes return or minimizes risk. As detailed in the next section, the turnover minimization has several advantages compared to existing models that account for estimation errors: it does not require an explicit assumption of error distribution; and it can be easily incorporated into conventional portfolio problems with any types of constraints.

The turnover minimization is motivated by the observation that even when the optimal portfolio contains extreme weights, there often exists a near-optimal portfolio with more balanced weights. Consider a two-asset allocation problem with the expected returns and the covariance matrix:

$$\mu = \begin{bmatrix} 0.10 \\ 0.15 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.16 & 0.15 \\ 0.15 & 0.25 \end{bmatrix}.$$

If we maximize the expected return while constraining the variance under 0.2, the optimal portfolio will be $w^* = [0.30\ 0.70]'$ with the expected return of 0.135. Now if we only require 95% of the expected return of the optimal portfolio, *i.e.*, $\mu_p = 0.95 \cdot 0.135 = 0.128$, and make the portfolio as close to the equal-weight portfolio as possible, we obtain $w^* = [0.43\ 0.57]'$. That is, a more balanced asset allocation can be achieved with a little sacrifice of optimality. As it turns out, this property holds in a wide range of portfolio optimization problems.

We test our model through a comprehensive comparative study that involves various portfolio models. In particular, we choose models that incorporate the equal-weight portfolio, as well as standard ones. With the desirable characteristics of the equal-weight portfolio such as low turnover and no-short-sale, and its recent role as a benchmark in portfolio studies, it was no surprise to see the emergence of portfolio models that incorporate it. Tu and Zhou (2011) combine the equal-weight portfolio with an optimal portfolio so that the expected utility loss is minimized: Markowitz (1952) rule, Jorion (1986) rule, Kan and Zhou (2007) rule, and MacKinlay and Pástor (2000) rule are considered for the optimal portfolio. Bessler et al. (2014) incorporate the equal-weight portfolio in the Black-Litterman framework by deriving the equi-

[3] Minimizing the distance from a reference portfolio leads to lower turnover even when the reference portfolio is not the current portfolio, hence the name – turnover minimization.

librium return from it. Beside these, we also test a new model based on the work of Treynor and Black (1973), in which the equal-weight portfolio replaces the market portfolio.

The empirical studies suggest that our model outperforms all the other models in terms of the Sharpe ratio both before and after transaction costs. This result is robust to the datasets, test period, and other variations.

The rest of the paper is organized as follows. Section 2 develops the turnover minimization, where a method to calibrate the model is also proposed. Section 3 carries out empirical analysis: the proposed model is first examined via a simulation study and compared with other models in a comprehensive empirical study involving thirteen datasets. Section 4 concludes the paper. The implementation details of the models used in the empirical analysis can be found in Appendix A, and the full empirical results are provided in the accompanying internet appendix (IA).

2. Turnover Minimization

The turnover minimization aims to minimize the distance from a reference portfolio subject to return/risk constraints. Roughly, the problem can be written in the form:

$$\begin{aligned} \min_{w}\ & (w - w_0)'(w - w_0) \\ \text{subject to}\ & \textit{return/risk constraints} \\ & \textit{and other constraints}, \end{aligned}$$

where w_0 is the reference portfolio which can be any known portfolio at the time of rebalancing. In this paper, the equal-weight portfolio, w_{ew}, and the current portfolio, w_{t-}, are considered for the reference portfolio. As illustrated later in this section, the return and risk constraints are not entirely exogenously given but endogenously determined so as to maximize portfolio performance.

The rationale behind turnover minimization is at least twofold: minimizing turnover mitigates the error maximizing property of the classical portfolio optimization and yields a more robust portfolio; investors are not necessarily return/risk optimizers. They often prefer a more robust portfolio as long as it meets their return/risk targets.

The turnover minimization is formulated as a two-stage optimization problem: classical portfolio optimization and turnover minimization. Consider the following return maximization problem of N assets subject to a variance constraint:

$$\begin{aligned} \max_{w}\ & w'\mu \\ \text{subject to}\ & w'\Sigma w \le \sigma_T^2 \\ & w \in \mathcal{D}, \end{aligned} \tag{1}$$

where $\mu \in \mathbb{R}^N$ and $\Sigma \in \mathbb{R}^{N\times N}$ are the mean and covariance matrix of N asset returns in excess of the risk-free rate, $w \in \mathbb{R}^N$ is the portfolio weights, σ_T^2 is a risk tolerance (target variance), and $\mathcal{D}$ denotes the feasible set of w defined by other constraints such as the budget constraint or short-sale constraint. Denoting the optimal portfolio that solves (1) by w^*, the expected return of w^* is given by $\mu_p^* = w^{*\prime}\mu$. The minimum-turnover portfolio, w_{tm}, is then obtained by solving the second stage problem:

$$\begin{aligned} w_{tm} = \underset{w}{\operatorname{argmin}}\ & (w - w_0)'(w - w_0) \\ \text{subject to } & w'\Sigma w \le \sigma_T^2 \\ & w \in \mathcal{D} \\ & w'\mu \ge (1-\tau)\mu_p^*, \end{aligned} \tag{2}$$

where $\tau > 0$ denotes the proportion of the optimal value the investor is willing to sacrifice in order to obtain a more robust portfolio. When $\tau = 0$, the minimum-turnover portfolio will be the same as the optimal portfolio of the first stage, whereas when $\tau \to \infty$, it will become the reference portfolio unless other constraints are binding.

The turnover minimization is intuitive in that it first finds the optimal portfolio for the underlying portfolio problem in the first stage and then moves it towards the reference portfolio by tolerating sub-optimality while satisfying all the constraints imposed in the first stage. Turnover minimization can be easily incorporated into any portfolio optimization problems such as variance minimization or Sharpe ratio maximization. Below are some examples.

Variance Minimization-Turnover Minimization

$$\begin{aligned} {\sigma_P^2}^* = \min_w\ & w'\Sigma w \\ \text{subject to } & w \in \mathcal{D} \end{aligned} \quad \Rightarrow \quad \begin{aligned} w_{tm} = \underset{w}{\operatorname{argmin}}\ & (w - w_0)'(w - w_0) \\ \text{subject to } & w \in \mathcal{D} \\ & w'\Sigma w \le (1+\tau)^2 {\sigma_P^2}^* \end{aligned} \tag{3}$$

Sharpe Ratio Maximization-Turnover Minimization

$$\begin{aligned} SR^* = \max_w\ & \frac{w'\mu}{\sqrt{w'\Sigma w}} \\ \text{subject to } & w \in \mathcal{D} \end{aligned} \quad \Rightarrow \quad \begin{aligned} w_{tm} = \underset{w}{\operatorname{argmin}}\ & (w - w_0)'(w - w_0) \\ \text{subject to } & w \in \mathcal{D} \\ & \frac{w'\mu}{\sqrt{w'\Sigma w}} \ge (1-\tau) SR^* \end{aligned} \tag{4}$$

Utility Maximization-Turnover Minimization

$$\begin{aligned} U^* = \max_w\ & w'\mu - \frac{\gamma}{2} w'\Sigma w \\ \text{subject to } & w \in \mathcal{D} \end{aligned} \quad \Rightarrow \quad \begin{aligned} w_{tm} = \underset{w}{\operatorname{argmin}}\ & (w - w_0)'(w - w_0) \\ \text{subject to } & w \in \mathcal{D} \\ & w'\mu - \frac{\gamma}{2} w'\Sigma w \ge (1-\tau) U^* \end{aligned} \tag{5}$$

γ : risk aversion coefficient

If the first-stage problem can be formulated as convex programming, the second-stage problem also becomes convex programming and can be efficiently solved using a specialized software package such as CVX, Gurobi, or MOSEK.

2.1. A Closer Look at the Turnover Minimization

Turnover minimization can be viewed as a shrinkage estimator as it shrinks the optimal portfolio towards the reference portfolio. Whilst it is generally impossible to obtain an analytic solution of a turnover minimization problem, the following special case provides some insights of the model and its link to existing models. Consider the utility maximization-turnover minimization problem now with a new distance function, $(w-w_0)'\Sigma(w-w_0)$.[4] The Lagrangian of the problem is give by

$$\mathcal{L} = \frac{1}{2}(w-w_0)'\Sigma(w-w_0) - \lambda\left(w'\mu - \frac{\gamma}{2}w'\Sigma w - (1-\tau)U^*\right). \tag{6}$$

From the first-oder condition, $\frac{\partial\mathcal{L}}{\partial w} = 0$, the minimum-turnover portfolio is given by

$$w_{tm} = \frac{1}{1+\lambda\gamma}w_0 + \frac{\lambda\gamma}{1+\lambda\gamma}w_{mk}, \tag{7}$$

where $w_{mk} = \frac{1}{\gamma}\Sigma^{-1}\mu$ is the optimal portfolio from the first stage, *i.e.*, the utility-maximizing portfolio. The minimum-turnover portfolio is given as a linear combination of the optimal portfolio and the reference portfolio. This is the same as the shrinkage estimator developed by Han (2017) and also the same as the shrinkage estimator of Tu and Zhou (2011) when $w_0 := w_{ew}$. In this regard, the turnover minimization can be considered a more general form of shrinkage estimator that encompasses existing models. The main difference however is that the turnover minimization does not make any particular assumption for the estimation errors and can easily accommodate different types of objective functions and constraints. This flexibility comes at the cost of analytical tractability and τ needs to be calibrated from the data.[5]

When the constraint is binding, *i.e.*, $w'\mu - \frac{\gamma}{2}w'\Sigma w = (1-\tau)U^*$, it can be shown that

$$\lambda = \frac{1}{\gamma}\left(\sqrt{\frac{U^*-U_0}{\tau U^*}} - 1\right), \tag{8}$$

and

$$w_{tm} = \sqrt{\frac{\tau U^*}{U^*-U_0}}w_0 + \left(1 - \sqrt{\frac{\tau U^*}{U^*-U_0}}\right)w_{mk}, \tag{9}$$

where $U_0 = w_0'\mu - \frac{\gamma}{2}w_0'\Sigma w_0$ is the utility of the reference portfolio. Note that the loading on w_0 is 0 when $\tau = 0$ and increases with τ, and w_{tm} eventually becomes equal to w_0 when $\tau = 1 - U_0/U^*$.

[4]This is only for analytical tractability. With Σ, an asset with a larger variance will be penalized more severely for the deviation from w_0, whereas all assets are penalized equally in the original specification. The latter performs slightly better in the empirical analysis.

[5]While a closed form is always preferred, Han (2017) shows that, due to model parameter uncertainty, the closed form solutions offered by Kan and Zhou (2007) and Tu and Zhou (2011) are sub-optimal even when all the assumptions are correct. He argues that the optimal shrinkage level should be higher than that suggested by these models, and cannot be determined analytically.

The distance between w_{tm} and w_0 is given by

$$(w_{tm} - w_0)'(w_{tm} - w_0) = \left(\frac{\lambda\gamma}{1+\lambda\gamma}\right)^2 (w_{mk} - w_0)'(w_{mk} - w_0). \tag{10}$$

If $U_0 = kU^*$ for some constant k, we have

$$\frac{||w_{tm} - w_0||}{||w_{mk} - w_0||} = \frac{\lambda\gamma}{1+\lambda\gamma} = 1 - \sqrt{\frac{\tau}{1-k}}. \tag{11}$$

That is, for a given τ, the normalized distance between w_{tm} and w_0 depends only on the relative size of the utilities, U_0 and U^*, and is independent of input parameters. If U_0 is closer to U^*, the optimal portfolio will shrink towards w_0 more rapidly. Figure 1 shows the relationship between the distance to w_0 and the tolerance τ for different k values. The vertical axis is the normalized distance, $1 - \sqrt{\tau/(1-k)}$. The figure suggests that, even when the utility of the reference portfolio is considerably smaller than that of the Markowitz portfolio, a robust portfolio (*i.e.*, a portfolio close to w_0) can be obtained without any signifiant loss of utility. For instance, when $U_0 = 0.1U^*$, 10% tolerance results in 33% reduction of the distance whereas the reduction increases to 41% when $U_0 = 0.4U^*$. Note that the formula in (11) is derived by minimizing $(w - w_0)'\Sigma(w - w_0)$. When $(w - w_0)'(w - w_0)$ is minimized, the distance is reduced further (see Section 2.3). U^* is a hypothetical maximum utility that can be obtained in the absence of estimation errors. The actual utility of the Markowitz portfolio will be much smaller and so is the utility loss caused by turnover minimization.

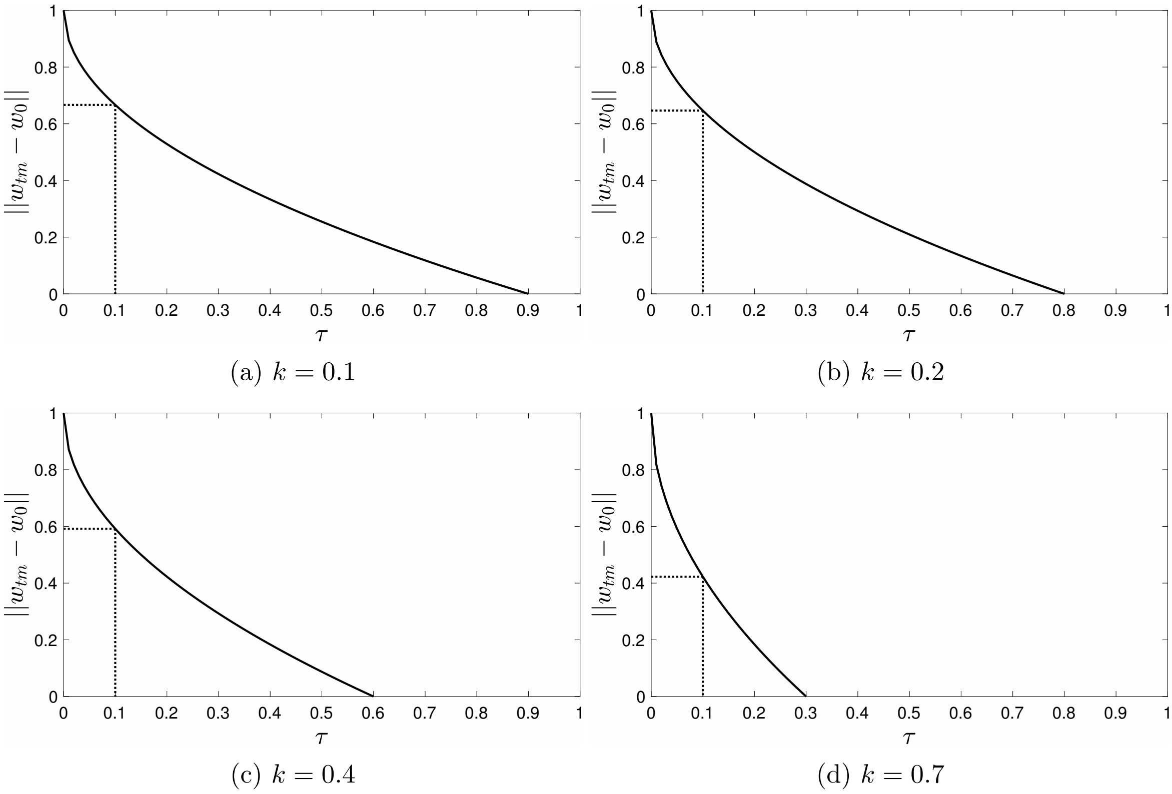
2.2. Calibration of τ

In the turnover minimization, τ determines the degree of shrinkage and the choice of τ is crucial for the performance of portfolio. Except for some special cases with strict assumptions, it is impossible to obtain a closed form for τ.[6] Hence, we adopt a data-driven calibration method described below.

1. For the first ten months of the evaluation period, τ is set to 0.05.
2. When the month $t > 10$, τ is calibrated each month so that the Sharpe ratio during $1, \ldots, t-1$ is maximized. The optimal τ is found via line search spanning the range $[0, 1]$ in its log space.

The same procedure is repeated taking transaction costs into account, *i.e.*, using the Sharpe ratio after transaction costs. The first step is required only for the empirical analysis as there is no data for calibration at the beginning. In practice, one may set a period from the past for initial calibration and recalibrate τ either by rolling the window or by accumulating it as the portfolio evolves.

[6] For example, one could use Equation (7) and maximize the expected out-of-sample utility with respect to τ assuming *i.i.d.* normal returns.



(a) $k = 0.1$ (b) $k = 0.2$

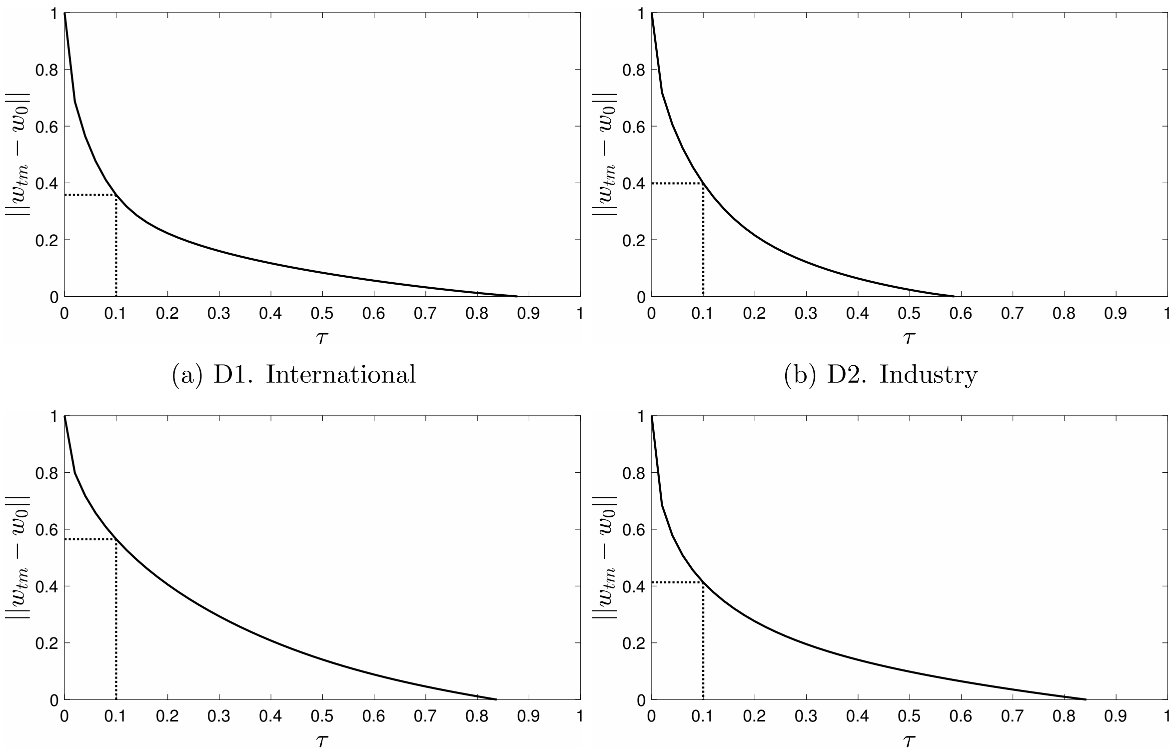
(c) $k = 0.4$ (d) $k = 0.7$

Figure 1: Shrinkage by Turnover Minimization

This figure demonstrates the distance between the minimum-turnover portfolio, w_{tm}, and the reference portfolio, w_0, as a function of τ. The curves are obtained from Equation (11). The distance is normalized by $||w_{mk} - w_0||$.

2.3. A Motivating Example

As illustrated in Section 2.1, shrinking towards a reference portfolio does not necessarily involve a considerable loss of optimality. This is further investigated via simulation. We first estimate μ and Σ from the sample moments of the four datasets used in the simulation study in the next section, and solve the utility maximization-turnover minimization problem in (5) for different values of τ and using the equal-weight portfolio as the reference portfolio. Figure 2 displays the relationship between τ and the distance between the reference portfolio and the minimum-turnover portfolio. As before, the distance is normalized by the distance of w_{mk}. It is remarkable that, for 10% loss of utility ($\tau = 0.1$), the distance to the equal-weight portfolio can normally be reduced by more than half even when its utility is substantially lower than that of the optimal portfolio. This suggests that the turnover minimization could yield a considerably more robust portfolio at the cost of a small fraction of optimality. Again, it is worth emphasizing that the utility loss is being measured against the hypothetical maximum utility and the actual utility of w_{mk} is likely to be much lower due to estimation errors.



(a) D1. International

(b) D2. Industry

(c) D5. Fama-French

(d) D8. Momentum

Figure 2: Minimum-Turnover Portfolios

This figure demonstrates the relationship between the tolerance level τ and the distance from the minimum-turn portfolio to the equal-weight reference portfolio. The distance is normalized by $||w_{mk} - w_0||$, *i.e.*, the distance when $\tau = 0$. The datasets are described in Table 2.

3. Empirical Analysis

In this section, the turnover minimization is evaluated and compared with other portfolio models using real market datasets. Section 3.1 and 3.2 respectively describe the models and the datasets and the remainder of the section discusses simulation and empirical results.

3.1. The Portfolio Models

The portfolio models are listed in Table 1. Their implementation details can be found in Appendix A. W* is the *ex-post* mean-variance optimal portfolio obtained from the sample moments during the evaluation period. It represents the performance of the Markowitz portfolio when no estimation error is present. EW is the equal-weight portfolio. Both W* and EW are rebalanced back to their original allocation every month.

The Markowitz mean-variance portfolio (MK), the global minimum-variance portfolio (MV), and their short-sale-constrained versions (MK+, MV+) are also tested. OC(+) and VT are the optimal constrained portfolio (with the short-sale constraint) and the volatility timing strategy of Kirby and Ostdiek (2012), respectively. TZMK and TZKZ are the shrinkage estimators of Tu and Zhou (2011) which respectively combine the Markowitz rule and the Kan and Zhou rule with the 1/N rule. TB+ is a new model which is an extension of the active portfolio model of Treynor and Black (1973). We use the equal-weight portfolio as the market portfolio and identify active assets by regressing asset returns on the return of the equal-weight portfolio. BL+ is an extension by Bessler et al. (2014) of the Black-Litterman model, in which the prior is derived from the equal-weight portfolio instead of the market portfolio.

Four versions of the turnover minimization are tested: TMKE(+) and TMK0(+) are the turnover minimization associated with Sharpe ratio maximization and TMVE(+) and TMV0(+) are the turnover minimization associated with variance minimization. The last letter indicates the reference portfolio: 'E' for the equal-weight portfolio and '0' for the current portfolio. '+' denotes the short-sale constraint. All turnover minimization models are tested using a calibrated τ as well as a constant $\tau = 0.05$.

3.2. The Data

The portfolio models are tested on the thirteen datasets described in Table 2. These are similar to the datasets used in DeMiguel et al. (2009) and Kirby and Ostdiek (2012) but more comprehensive. In the table, the sample period refers to the evaluation period during which portfolios are rebalanced monthly. The international dataset (D1) has a shorter evaluation period due to data availability.

Input parameters are estimated monthly from a rolling estimation window, $T = 60$, 120, or 240 months. The same evaluation period is used regardless of the estimation window size so that the empirical results can be compared across window sizes. For instance, when $T = 240$, the parameters are estimated from 1931.01 to 1950.12 in the first month and when $T = 120$, they are estimated from 1941.01 to 1950.12.

Table 1: The Portfolio Models

This table lists the portfolio models considered in the empirical analysis. The '+' in the abbreviation denotes a model with the short-sale constraint (applied only to the risky assets). The details of the models are described in Section 2 and Appendix A.

Abbreviation	Description
W*	*Ex-post* mean-variance optimal portfolio
EW	Equal-weight portfolio
Classical models	
MK, MK+	Markowitz (1952) mean-variance portfolio
MV, MV+	Global minimum-variance portfolio
Kirby and Ostdiek (2012)	
OC, OC+	Optimal constrained portfolio: MK(+) without the risk-free asset
VT	Volatility timing strategy
Tu and Zhou (2011)	
TZMK	Combination of MK and EW
TZKZ	Combination of Kan and Zhou (2007) three-fund rule and EW
Incorporating the 1/N rule (in place of the market portfolio)	
TB+	Treynor and Black (1973)
BL+	Black and Litterman (1992)
Turnover Minimization	
TMKE(τ), TMKE(τ)+	Sharpe ratio maximization-Turnover Minimization, $w_0 = w_{ew}$
TMK0(τ), TMK0(τ)+	Sharpe ratio maximization-Turnover Minimization, $w_0 = w_{t-}$
TMVE(τ), TMVE(τ)+	Variance minimization-Turnover Minimization, $w_0 = w_{ew}$
TMV0(τ), TMV0(τ)+	Variance minimization-Turnover Minimization, $w_0 = w_{t-}$
τ: tolerance factor	

3.3. Simulation Studies

The effects of the turnover minimization is first examined via a simulation. We choose datasets D1, D2, D5, and D8 and calculate the sample mean and covariance matrix of each dataset over the evaluation period. These are regarded as true parameters. Under $i.i.d$ normal assumption, the maximum likelihood estimates of the input parameters are distributed as follows:

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\Sigma}{T}\right), \quad \hat{\Sigma} \sim \mathcal{W}_N(T-1, \Sigma)\frac{1}{T}, \tag{12}$$

where T is the estimation window size, and $\mathcal{N}$ and $\mathcal{W}_N$ respectively denote a normal distribution and N-dimensional Wishart distribution.

$\hat{\mu}$ and $\hat{\Sigma}$ are randomly sampled from the above distributions and portfolios are constructed based on these parameters. This is repeated 10,000 times to obtain the expected Sharpe ratio as described below. From the definition of the Sharpe ratio, we have

$$SR = \frac{\mu_p}{\sigma_p}, \tag{13}$$

$$\mu_p = E[w'r] = E[w'\mu], \tag{14}$$

$$\sigma_p^2 = V[w'r] = E[w'\Sigma w] + E[(w'\mu)^2] - E[w'\mu]^2. \tag{15}$$

Following Kan and Wang (2016), we use the unconditional variance instead of $E[w'\Sigma w]$, which

Table 2: The Datasets

This table lists the datasets used in the empirical analysis. The eight international indices in D1 are the gross returns on large/mid cap stocks from eight countries: Canada, France, Germany, Italy, Japan, Switzerland, United Kingdom and USA. All the other datasets consist of the US stocks. The 20 size-sort portfolios (D5, 6, 7, 11, 12, 13) are obtained from the corresponding 25 portfolios by removing the five largest portfolios. All datasets are from K. French website (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`) except D1, which is from the MSCI website (`https://www.msci.com/end-of-day-data-country`).

Dataset	Desciption	N	Sample Period
D1	8 International + World Indices	9	1990.10 - 2015.12
D2	10 Industry Portfolios + Market	11	1951.01 - 2015.12
D3	30 Industry Portfolios + Market	31	1951.01 - 2015.12
D4	3 Fama-French (FF) Factors	3	1951.01 - 2015.12
D5	20 FF Portfolios + Market	21	1951.01 - 2015.12
D6	20 FF Portfolios + FF 3	23	1951.01 - 2015.12
D7	20 FF Portfolios + FF 3 and Momentum	24	1951.01 - 2015.12
D8	10 Momentum Portfolios + Market	11	1951.01 - 2015.12
D9	10 Short-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D10	10 Long-Term Reversal Portfolios + Market	11	1951.01 - 2015.12
D11	20 Size/Momentum Portfolios + Market	21	1951.01 - 2015.12
D12	20 Size/Short-Term Reversal Portfolios + Market	21	1951.01 - 2015.12
D13	20 Size/Long-Term Reversal Portfolios + Market	21	1951.01 - 2015.12

is often used in the literature. This is because using unconditional moments is consistent with the way we evaluate the Sharpe ratio empirically. The expectations are estimated from the simulation as follows.

$$E[w'\mu] = \frac{1}{S}\sum_{s=1}^{S} w^{(s)\prime}\mu, \tag{16}$$

$$E[w'\Sigma w] = \frac{1}{S}\sum_{s=1}^{S} w^{(s)\prime}\Sigma w^{(s)}, \tag{17}$$

$$E[(w'\mu)^2] = \frac{1}{S}\sum_{s=1}^{S} (w^{(s)\prime}\mu)^2, \tag{18}$$

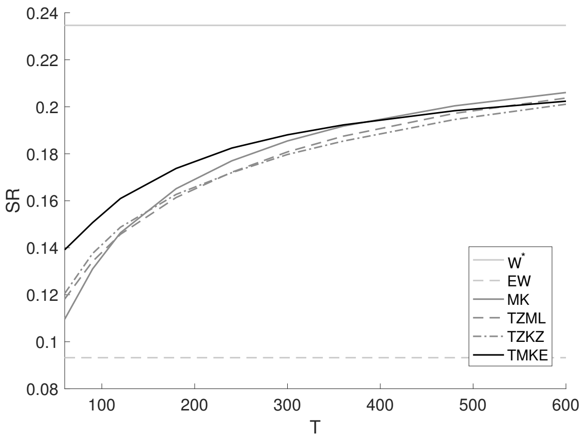
where S is the number of iterations and $w^{(s)}$ is the portfolio obtained in the s-th iteration.

The simulation is repeated using different estimation window sizes. Figure 3 and Table 3 report the results for some selected models. For the turnover minimization, $\tau \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3\}$ are used and Figure 3 presents only the highest Sharpe ratio among them for a given T. This is to examine the potential gain from the turnover minimization when τ is optimally chosen.

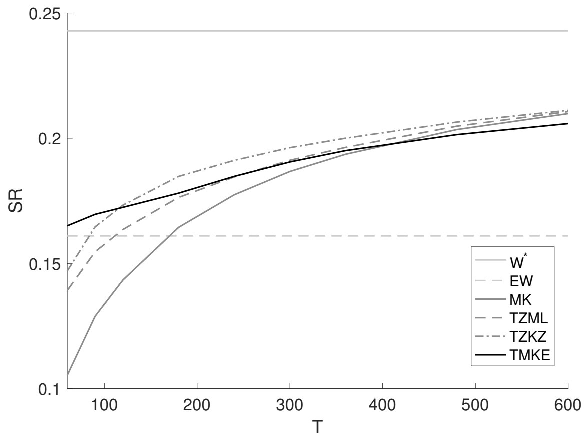
The effectiveness of the turnover minimization is evident. It does not only outperform its underlying model (MK) in all datasets, but also performs superior in comparison to the shrinkage estimators (TZMK and TZKZ) of Tu and Zhou (2011). It is the only model that consistently outperforms EW in all datasets for all T.

When compared with MK, TMKE presents a considerably higher Sharpe ratio especially when T is small, *i.e.*, when the estimation error is large.[7] This is true regardless of the Sharpe

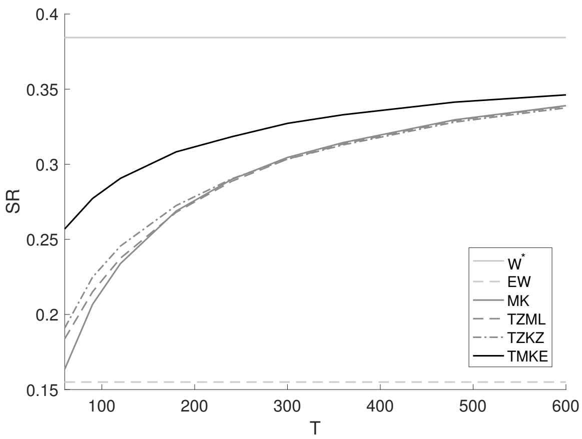
[7]It is unrealistic to assume that returns are *i.i.d.* for an extended period and we are primarily concerned



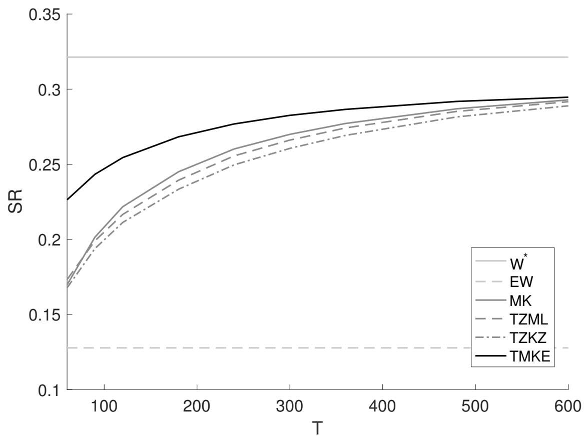
(a) D1. International



(b) D2. Industry



(c) D5. Fama-French



(d) D8. Momentum

Figure 3: Expected Sharpe Ratio

This Figure displays the expected Sharpe ratio of selected portfolio models for different estimation window sizes. The vertical axis represents the Sharpe ratio and the horizontal axis represents the estimation window size. The Sharpe ratio of TMKE is the highest Sharpe ratio obtained from different values of $\tau \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3\}$.

Table 3: Expected Sharpe Ratio

This table reports the expected Sharpe ratio of selected portfolio models for different estimation window sizes. The expected Sharpe ratio is obtained from the simulation described in Section 3.3. The numbers under TMKE(τ) are the values of τ and the bold figures represent the highest Sharpe ratio across τ's.

T	W*	EW	MK	TZMK	TZKZ	TMKE(τ)					
						0.05	0.1	0.15	0.2	0.25	0.3
D1. International											
60	0.235	0.093	0.110	0.118	0.121	0.138	**0.139**	0.139	0.138	0.136	0.133
90	0.235	0.093	0.131	0.134	0.138	0.151	**0.151**	0.149	0.147	0.144	0.140
120	0.235	0.093	0.146	0.146	0.149	**0.161**	0.160	0.158	0.154	0.150	0.146
180	0.235	0.093	0.165	0.161	0.163	**0.174**	0.171	0.167	0.163	0.158	0.152
240	0.235	0.093	0.177	0.172	0.172	**0.182**	0.179	0.174	0.169	0.163	0.156
300	0.235	0.093	0.185	0.181	0.180	**0.188**	0.183	0.178	0.172	0.165	0.158
360	0.235	0.093	0.192	0.187	0.185	**0.192**	0.187	0.181	0.174	0.167	0.159
480	0.235	0.093	0.200	0.197	0.195	**0.198**	0.192	0.185	0.178	0.169	0.161
600	0.235	0.093	0.206	0.204	0.201	**0.202**	0.195	0.188	0.180	0.171	0.162
D2. Industry											
60	0.243	0.161	0.105	0.139	0.147	0.146	0.152	0.157	0.160	0.163	**0.165**
90	0.243	0.161	0.129	0.155	0.165	0.157	0.162	0.165	0.167	0.169	**0.170**
120	0.243	0.161	0.143	0.164	0.173	0.165	0.168	0.170	0.172	**0.172**	0.172
180	0.243	0.161	0.164	0.176	0.185	0.177	0.178	**0.178**	0.178	0.177	0.176
240	0.243	0.161	0.177	0.185	0.191	**0.185**	0.184	0.183	0.182	0.180	0.178
300	0.243	0.161	0.187	0.191	0.196	**0.191**	0.189	0.187	0.184	0.181	0.178
360	0.243	0.161	0.194	0.196	0.200	**0.195**	0.192	0.189	0.186	0.182	0.179
480	0.243	0.161	0.203	0.205	0.206	**0.201**	0.197	0.193	0.188	0.184	0.179
600	0.243	0.161	0.210	0.211	0.211	**0.206**	0.200	0.195	0.190	0.184	0.179
D5. Fama-French											
60	0.384	0.155	0.164	0.184	0.191	0.224	0.239	0.248	0.253	0.256	**0.257**
90	0.384	0.155	0.207	0.215	0.225	0.258	0.270	0.276	**0.277**	0.276	0.273
120	0.384	0.155	0.234	0.237	0.246	0.279	0.288	**0.291**	0.290	0.286	0.280
180	0.384	0.155	0.269	0.268	0.272	0.304	**0.308**	0.306	0.301	0.294	0.284
240	0.384	0.155	0.290	0.289	0.290	0.318	**0.318**	0.313	0.306	0.296	0.284
300	0.384	0.155	0.305	0.303	0.303	**0.327**	0.325	0.318	0.308	0.297	0.284
360	0.384	0.155	0.314	0.313	0.313	**0.333**	0.329	0.320	0.309	0.297	0.283
480	0.384	0.155	0.330	0.329	0.328	**0.341**	0.334	0.323	0.311	0.296	0.281
600	0.384	0.155	0.339	0.339	0.338	**0.346**	0.337	0.325	0.311	0.296	0.280
D8. Momentum											
60	0.321	0.128	0.170	0.173	0.168	0.211	0.219	0.224	**0.226**	0.226	0.224
90	0.321	0.128	0.201	0.199	0.194	0.234	0.241	**0.243**	0.243	0.240	0.235
120	0.321	0.128	0.222	0.217	0.211	0.249	0.254	**0.254**	0.252	0.247	0.240
180	0.321	0.128	0.245	0.239	0.233	0.267	**0.268**	0.266	0.260	0.253	0.243
240	0.321	0.128	0.260	0.256	0.250	**0.277**	0.276	0.271	0.264	0.254	0.243
300	0.321	0.128	0.270	0.266	0.261	**0.283**	0.280	0.273	0.264	0.253	0.241
360	0.321	0.128	0.277	0.274	0.269	**0.287**	0.282	0.274	0.264	0.252	0.239
480	0.321	0.128	0.287	0.285	0.282	**0.292**	0.285	0.276	0.264	0.251	0.237
600	0.321	0.128	0.293	0.292	0.289	**0.295**	0.287	0.276	0.264	0.250	0.236

ratio of the equal-weight portfolio: even when the Sharpe ratio of EW is substantially smaller than that of MK, a higher Sharpe ratio is obtained by shrinking MK towards EW. Logically, the τ associated with the maximum Sharpe ratio decreases with T, *i.e.* as the estimation error diminishes.

The shrinkage estimators of Tu and Zhou (2011), in contrast, only marginally outperform MK even though they also incorporate the equal-weight portfolio and assume the exact knowledge of the error distribution. This can be attributed to three reasons. First, these models maximize the expected out-of-sample utility rather than the Sharpe ratio. Utility maximization depends on the choice of the risk aversion parameter (3, in this paper) and does not necessarily lead to the maximum Sharpe ratio. Model parameter uncertainty is another reason for the unsatisfactory performance. As the coefficients of the portfolios are functions of the unknown true input parameters, they have to be estimated and inevitably inherit the input parameter uncertainty. This results in lower-than-expected performance. Lastly, the portfolio of Tu and Zhou (2011) is restricted to a linear combination of an optimal portfolio and the equal-weight portfolio, whereas the minimum-turnover portfolio does not assume any particular structure.

The only case when TMKE underperforms TZKZ is in D2 when $T \geq 240$. This is, however, because we test only a few discrete values of τ. Table 3 shows that the optimal τ (highlighted in boldface) reaches its minimum rather quickly. With a wider range and more finely divided values of τ, outperformance of the turnover minimization would be more prominent.

Overall, the simulation suggests that, with a carefully chosen τ, the turnover minimization can improve its underlying model substantially and could also outperform other shrinkage estimators that involve the same shrinkage target.

3.4. Empirical Studies

3.4.1. Portfolio Construction and Evaluation

The input parameters are estimated every month during the evaluation period via the maximum likelihood estimator from a rolling estimation window of size T = 60, 120, or 240. Then the portfolios from the models in Table 1 are rebalanced monthly based on these input parameters and monthly portfolio returns are computed. The turnover minimization is tested with three different types of τ: a constant τ set to 0.05 and calibrated τ's, τ_b and τ_a, respectively for before and after transaction costs.

For out-of-sample performance evaluation, the Sharpe ratio (SR) before and after transaction costs as well as turnover (TO) are calculated. They are defined as follows:

$$SR = \frac{\bar{r}_p}{s_p}, \tag{19}$$

$$TO = \frac{1}{KN} \sum_{t=1}^{K} \sum_{i=1}^{N} |w_{i,t} - w_{i,t^-}|, \tag{20}$$

where $\bar{r}_p$ and s_p are respectively the mean and standard deviation of the portfolio returns

about the results from small T's.

over the evaluation period, K and N are the number of months in the evaluation period and the number of assets, and $w_{i,t-}$ and $w_{i,t}$ are the weights of asset i immediately before and after rebalancing at time t. For the Sharpe ratio after transaction costs, transaction costs are set to 50 basis points for both buying and selling risky assets and 0 for the risk-free asset. The actual transaction costs of institutional investors are likely to be lower than this and this assumption adversely affects the performance of optimal strategies which normally carry higher turnover than the naïve strategy. For the statistical inference of the Sharpe ratio, the p-value of the Sharpe ratio difference from the equal-weight portfolio is calculated using the method of Memmel (2003).

Since different portfolio models have different criteria and some are constrained to invest only in the risky assets, comparing models on a level playing field is not straightforward. To mitigate the effects from these discrepancies, we constrain all the models to have the same variance. Variance targeting can be accomplished by adjusting portfolio weights as follows:

$$w := w\frac{\hat{\sigma}_p}{\sigma_T}, \tag{21}$$

where $\hat{\sigma}_p^2 = w'\hat{\Sigma}w$ is the *ex-ante* variance of the optimal portfolio and σ_T^2 is the target variance. For W*, the true covariance matrix is used instead of $\hat{\Sigma}$. σ_T^2 is set to the variance of the equal-weight portfolio over the entire sample period.

In order to ensure that the effectiveness of the turnover minimization is not driven by the variance constraint, models are also tested under the standard utility maximization objective.[8] We use the quadratic expected utility with the risk aversion parameter of 3.

The out-of-sample performances of the portfolio models are evaluated based on the results from the 120-month estimation window. The main findings are robust across different settings; minor differences are discussed in the robustness check in Section 3.4.4. The full empirical results including the results from the turnover minimization with the constant τ and those from different estimation windows can be found in IA.

3.4.2. The Performance of the Turnover Minimization

Table 4 and 5 report the Sharpe ratio before and after transaction costs, and Table 6 and 7 report turnover, respectively under variance targeting and utility maximization. To facilitate comparison, the turnover minimization models are also compared with their underlying models in Figure 4 and 5. The empirical results are consistent with the findings from the simulation and reaffirm the effectiveness of the turnover minimization.

Comparing with their underlying counterparts, the turnover minimization models shrinking towards the equal-weight portfolio present a higher Sharpe ratio and lower turnover. If we first look at the results from variance targeting in Table 4, TMKE and TMKE+, on average, have

[8] Exceptions are EW, MV(+), and VT. These models by construction does not utilize the mean return whereas adjusting a portfolio so as to maximize utility involves the mean return as the adjustment formula is given by: $w := w\frac{1}{\gamma}\frac{\hat{\mu}_p}{\hat{\sigma}_p^2}$, where $\hat{\mu}_p = w'\hat{\mu}$, $\hat{\sigma}_p^2 = w'\hat{\Sigma}w$, and γ is the risk aversion parameter. Therefore, maximizing utility using these models would spoil their key characteristic.

the Sharpe ratio 0.265 and 0.191 before transaction costs and 0.187 and 0.176 after transaction costs, whereas the corresponding values of MK and MK+ are 0.231 and 0.184 before transaction costs and 0.063 and 0.166 after transaction costs. Similarly, TMVE(+) has the Sharpe ratio 0.198 (0.178) and 0.148 (0.168) respectively before and after transaction costs, whereas the corresponding values of MV(+) are 0.156 (0.166) and 0.051 (0.156). A similar observation can be made under utility maximization. The improvement is consistent across datasets and more prominent after transaction costs, owing to the significantly lower turnover of the proposed model. Few exceptions occur in D6, D7, and D12 under utility maximization, where MK has a slightly higher Sharpe ratio than TMKE before transaction costs. However, this relationship is reversed after taking transaction costs into account.

When compared with EW, TMKE, TMKE+ and TMVE+ outperform EW in all datasets and TMVE in twelve datasets before transaction costs under variance targeting. If we count only statistically significant cases at 10%, they are respectively 7, 9, 11 and 10. Among other models, only TZKZ outperforms EW in all datasets (8 times statistically significant). The superior performance of the turnover minimization is largely maintained even after the conservatively set transaction costs. In particular, the short-sale constrained models (TMKE+ and TMVE+) outperform EW in all datasets after transaction costs (7 and 9 times statistically significant, respectively). No other models show the same level of performance.

The turnover minimization models continue to perform superior under utility maximization, but the performances of TMKE and TMKE+ are less pronounced. In contrast, TMVE and TMVE+ maintain a similar level of performance and TMVE+, in particular, outperforms EW in all datasets both before and after transaction costs (11 and 9 times statistically significant, respectively). In fact, TMVE+ is the only strategy that outperforms EW in all datasets under utility maximization. The overall difference between variance targeting and utility maximization can be attributed to the fact that variance targeting is less susceptible to the estimation error of the mean as it forces the portfolio to the target variance.

When short-sale is not allowed, the underlying portfolios are closer to EW and the benefit from shrinking towards EW becomes rather limited. Nevertheless, incorporating turnover minimization improves the performance of the underlying models consistently across datasets and optimization criteria. This makes TMKE+ and TMVE+ the best performing long-only models respectively under variance targeting and utility maximization.

As evidenced from the superior performance of the short-sale constrained turnover minimization models, it is often beneficial to constrain portfolio weights to reduce turnover and leverage. Besides, many financial institutions do not allow short positions in their portfolio. The turnover minimization admits the flexibility of adding these constraints while accounting for parameter uncertainty.

Comparing reference portfolios, shrinking towards the current portfolio turns out to be less effective. TMK0(+) and TMV0(+) perform considerably weaker than their equal-weight counterparts, TMKE(+) and TMVE(+). These models usually underperform their underlying models before transaction costs and perform comparably only after transaction costs, owing to their lower turnover. A similar observation has been made by Han (2017), who compares expected turnovers to show that the equal-weight portfolio is a more effective shrinkage target. This contradicts the widely-accepted belief that accounting for transaction costs yields a more robust portfolio and enhance performance.

The calibration of τ proves to be effective. In the presence of transaction costs, high turnover is harmful and a higher level of shrinkage would be desired. The calibration results reported in Table 8 and 9 are in accordance with this conjecture: τ_a calibrated under transaction costs is greater than τ_b calibrated without transaction costs. Table 6 and 7 show that turnover is indeed substantially lower when τ_a is employed. Consequently, the turnover minimization models with τ_b performs better before transaction costs whereas those with τ_a perform better after transaction costs. Both versions outperform the constant-τ version.[9]

3.4.3. The Performance of Other Models

Among the other models, Tu and Zhou (2011) shrinkage estimators, TZMK and TZKZ, perform comparably to TMKE before transaction costs: their average Sharpe ratios before transaction costs are 0.248 and 0.255 under variance targeting and 0.242 and 0.247 under utility maximization, whereas the corresponding values of TMKE are 0.265 and 0.260. Between the two, TZKZ appears to perform better than TZMK. Nevertheless, their performance is significantly deteriorated once transaction costs are taken into account due to their high turnover. This is common for most optimal strategies that allow short-sale such as MK and OC. While TZMK and TZKZ enhance successfully the Sharpe ratio and reduce turnover in comparison to the Markowitz model, they are still characterized by costly portfolio rebalancing and underperform EW after transaction costs in most datasets. This reaffirms the need for the ability to incorporate constraints in shrinkage models.

Among the strategies that incorporate the 1/N rule, the variation of the Black-Litterman model (BL+) performs best when transaction costs are taken into account. Nevertheless, it outperforms EW statistically significantly only in three datasets under utility maximization and is generally outperformed by the turnover minimization models.

Another model that is worth noting is the volatility timing (VT) of Kirby and Ostdiek (2012). Although it outperforms EW only marginally (the average Sharpe ratios before (after) transaction costs are respectively 0.151 (0.148) and 0.144 (0.142) under variance targeting, and 0.155 (0.153) and 0.150 (0.148) under utility maximization), the difference is statistically significant in eleven datasets even after transaction costs. This is because the portfolio weights of VT are entirely determined by the relative size of the variances of asset returns, which are very stable over time. This leads to a stable VT portfolio and consequently a low standard deviation of the return difference between VT and EW. With the superior performance of VT, it could be that using VT as the shrinkage target could enhance the performance of the turnover minimization even further. This topic, however, is not further pursued in this paper.

All in all, the turnover minimization with the equal-weight portfolio as the reference portfolio performs superior in comparison to the other strategies. The unconstrained models (TMKE and TMVE) perform best before transaction costs while the short-sale constrained counterparts

[9] The full results using different τ's can be found in IA.

Table 4: Sharpe Ratio under Variance Targeting

This table reports the Sharpe ratios of the portfolio models in Table 1 under variance targeting. Input parameters are estimated from a rolling window of size $T = 120$ and transaction costs are assumed to be 50 basis points for risky assets and 0 for the risk-free asset. The columns represent the datasets described in Table 2. In the turnover minimization models, τ_b (τ_a) denotes the τ calibrated without (with) transaction costs. The Sharpe ratios statistically significantly higher at 10% than that of EW are marked by *.

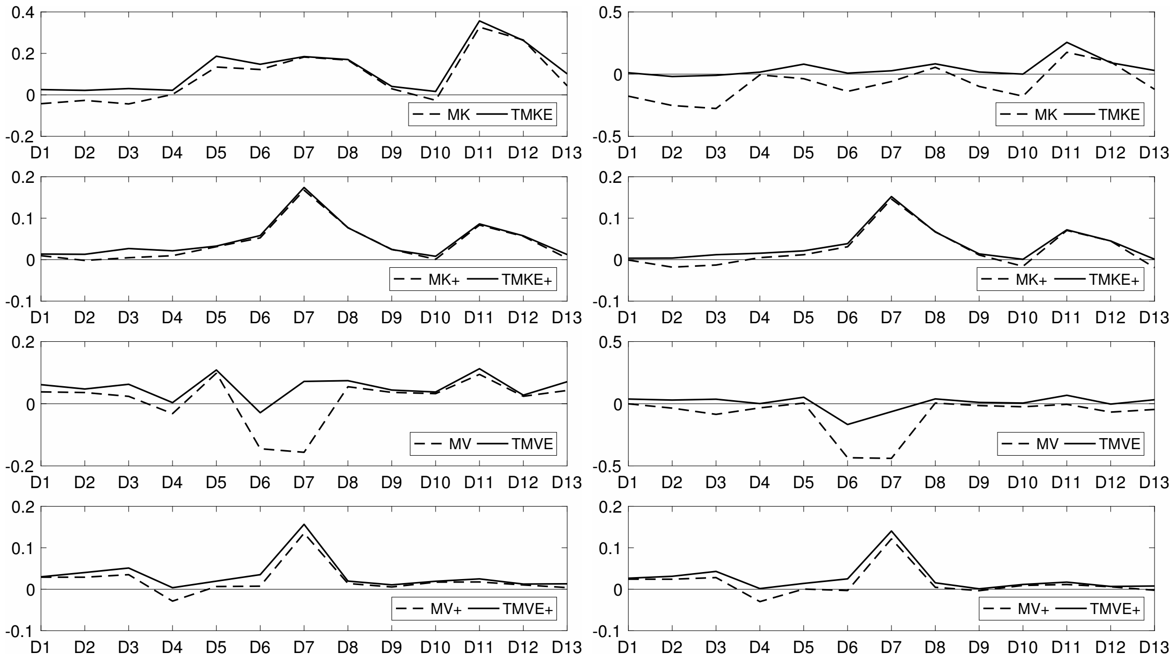
	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	Mean
Before Transaction Cost														
W*	0.234	0.243	0.289	0.216	0.384	0.389	0.444	0.321	0.244	0.208	0.461	0.466	0.332	0.325
EW	0.083	0.156	0.146	0.185	0.150	0.152	0.161	0.121	0.133	0.148	0.142	0.139	0.158	0.144
MK	0.041	0.129	0.102	0.187	0.284*	0.275*	0.343*	0.289*	0.162	0.122	0.469*	0.404*	0.202	0.231
MK+	0.093	0.154	0.151	0.195	0.181*	0.205*	0.328*	0.199*	0.157*	0.149	0.226*	0.195*	0.162	0.184
MV	0.121	0.192	0.170	0.153	0.248*	0.008	0.005	0.176*	0.169*	0.181*	0.237*	0.163	0.201	0.156
MV+	0.112*	0.185*	0.181*	0.156	0.157	0.160	0.296*	0.135*	0.138	0.165*	0.160*	0.150	0.162	0.166
OC	0.102	0.153	0.127	0.165	0.301*	0.274*	0.345*	0.286*	0.182*	0.144	0.464*	0.400*	0.208	0.242
OC+	0.099	0.147	0.148	0.170	0.163	0.167	0.173	0.193*	0.150*	0.153	0.202*	0.185*	0.160	0.162
VT	0.093*	0.171*	0.160*	0.155	0.159*	0.164*	0.186*	0.131*	0.135*	0.151*	0.151*	0.141	0.162*	0.151
TB+	0.082	0.160	0.173	0.146	0.176*	0.195*	0.283*	0.189*	0.146*	0.151	0.225*	0.198*	0.160	0.176
BL+	0.091	0.157	0.160	0.189	0.178*	0.200*	0.307*	0.196*	0.160*	0.154	0.219*	0.185*	0.168	0.182
TZMK	0.043	0.160	0.145	0.188	0.308*	0.302*	0.352*	0.295*	0.171*	0.154	0.468*	0.406*	0.237*	0.248
TZKZ	0.064	0.184	0.167	0.202	0.308*	0.286*	0.342*	0.291*	0.184*	0.167	0.465*	0.404*	0.250*	0.255
TMKE(τ_b)	0.109	0.177	0.176	0.207	0.337*	0.300*	0.345*	0.292*	0.173	0.165	0.499*	0.401*	0.260*	0.265
TMKE+(τ_b)	0.097	0.168	0.173*	0.206	0.183*	0.211*	0.335*	0.199*	0.157*	0.156	0.229*	0.197*	0.171*	0.191
TMK0(τ_b)	0.008	0.074	0.041	0.138	0.274*	0.204	0.331*	0.246*	0.165	0.065	0.460*	0.392*	0.176	0.198
TMK0+(τ_b)	0.069	0.165	0.147	0.182	0.180*	0.185	0.320*	0.197*	0.156*	0.139	0.225*	0.194*	0.163	0.179
TMVE(τ_b)	0.145*	0.203*	0.208*	0.188	0.259*	0.123	0.232*	0.196*	0.177*	0.186*	0.255*	0.167	0.229*	0.198
TMVE+(τ_b)	0.113*	0.196*	0.197*	0.189	0.170*	0.188*	0.317*	0.141*	0.143*	0.168*	0.167*	0.152	0.171*	0.178
TMV0(τ_b)	0.114	0.181	0.184	0.159	0.250*	0.005	0.001	0.163*	0.155	0.182*	0.233*	0.157	0.189	0.152
TMV0+(τ_b)	0.109*	0.178	0.190*	0.159	0.150	0.164	0.293*	0.124	0.132	0.157	0.163*	0.145	0.148	0.162
After Transaction Cost														
W*	0.209	0.215	0.248	0.210	0.336	0.325	0.386	0.294	0.224	0.190	0.422	0.414	0.299	0.290
EW	0.081	0.153	0.143	0.179	0.148	0.150	0.158	0.119	0.131	0.146	0.140	0.137	0.156	0.142
MK	-0.097	-0.100	-0.134	0.173	0.111	0.011	0.099	0.174	0.032	-0.031	0.316*	0.236*	0.034	0.063
MK+	0.080	0.135	0.130	0.184	0.160	0.181	0.305*	0.187*	0.142	0.130	0.211*	0.182*	0.137	0.166
MV	0.080	0.117	0.057	0.145	0.153	-0.284	-0.281	0.125	0.116	0.122	0.135	0.070	0.110	0.051
MV+	0.105*	0.177	0.171	0.149	0.149	0.147	0.280*	0.124	0.127	0.155	0.152	0.143	0.154	0.156
OC	0.011	-0.020	-0.078	0.150	0.140	0.085	0.169	0.195*	0.083	0.015	0.320*	0.254*	0.047	0.105
OC+	0.083	0.128	0.128	0.157	0.141	0.140	0.144	0.181*	0.131	0.135	0.186*	0.175*	0.136	0.143
VT	0.090*	0.167*	0.156*	0.148	0.157*	0.161*	0.182*	0.128*	0.133*	0.149*	0.149*	0.139	0.160*	0.148
TB+	0.073	0.145	0.152	0.128	0.152	0.158	0.250*	0.176*	0.124	0.139	0.213*	0.185*	0.131	0.156
BL+	0.079	0.145	0.146	0.179	0.165*	0.182	0.287*	0.186*	0.150*	0.141	0.209*	0.172*	0.153	0.169
TZMK	-0.057	-0.063	-0.072	0.175	0.173	0.090	0.140	0.192*	0.068	0.047	0.326*	0.253*	0.119	0.107
TZKZ	-0.022	-0.020	-0.042	0.190	0.181	-0.010	0.062	0.192*	0.094	0.075	0.326*	0.252*	0.144	0.110
TMKE(τ_a)	0.092	0.134	0.133	0.195	0.228*	0.158	0.185*	0.202*	0.148	0.146	0.396*	0.229*	0.186	0.187
TMKE+(τ_a)	0.084	0.157	0.155	0.194	0.170*	0.189*	0.310*	0.186*	0.145*	0.147	0.212*	0.182*	0.157	0.176
TMK0(τ_a)	-0.058	-0.033	-0.035	0.130	0.216*	0.180	0.324*	0.251*	0.152	0.043	0.365*	0.323*	0.148	0.154
TMK0+(τ_a)	0.064	0.143	0.134	0.149	0.157	0.174	0.302*	0.187*	0.145*	0.140	0.219*	0.189*	0.161	0.167
TMVE(τ_a)	0.118*	0.182*	0.179*	0.180	0.200*	-0.017	0.094	0.158*	0.141	0.152	0.208*	0.134	0.188*	0.148
TMVE+(τ_a)	0.107*	0.184*	0.186*	0.180	0.162*	0.175*	0.299*	0.135*	0.132	0.158*	0.158*	0.144	0.164	0.168
TMV0(τ_a)	0.100	0.185*	0.165	0.152	0.191*	-0.139	-0.137	0.126	0.140	0.117	0.182	0.066	0.119	0.097
TMV0+(τ_a)	0.101	0.173	0.182*	0.153	0.131	0.152	0.283*	0.112	0.137	0.156	0.153	0.145	0.145	0.156

Table 5: Sharpe Ratio under Utility Maximization

This table reports the Sharpe ratios of the portfolio models in Table 1 under utility maximization. Input parameters are estimated from a rolling window of size $T = 120$ and transaction costs are assumed to be 50 basis points for risky assets and 0 for the risk-free asset. The columns represent the datasets described in Table 2. In the turnover minimization models, τ_b (τ_a) denotes the τ calibrated without (with) transaction costs. The Sharpe ratios statistically significantly higher at 10% than that of EW are marked by *.

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	Mean
Before Transaction Cost														
W*	0.234	0.243	0.289	0.216	0.384	0.389	0.444	0.321	0.244	0.208	0.461	0.466	0.332	0.325
EW	0.093	0.161	0.149	0.192	0.155	0.157	0.165	0.128	0.138	0.156	0.148	0.143	0.164	0.150
MK	0.029	0.134	0.106	0.177	0.298*	0.287*	0.342*	0.282*	0.171	0.137	0.450*	0.411*	0.201	0.233
MK+	0.056	0.158	0.150	0.184	0.169	0.201*	0.317*	0.182*	0.153	0.141	0.208*	0.182*	0.146	0.173
MV	0.122	0.181	0.157	0.153	0.252*	0.005	-0.001	0.178*	0.174*	0.175	0.239*	0.175	0.203	0.155
MV+	0.122*	0.174	0.170	0.157	0.163	0.160	0.282*	0.143*	0.146	0.170*	0.165*	0.155	0.166	0.167
OC	0.041	0.067	0.061	0.114	0.235*	0.289*	0.346*	0.264*	0.129	0.078	0.409*	0.395*	0.153	0.199
OC+	0.073	0.130	0.130	0.163	0.179*	0.174	0.208*	0.200*	0.152	0.133	0.214*	0.205*	0.159	0.163
VT	0.102*	0.170*	0.160*	0.160	0.165*	0.166*	0.184*	0.138*	0.142*	0.159	0.157*	0.146*	0.169*	0.155
TB+	0.029	0.148	0.157	0.130	0.163	0.188	0.280*	0.172*	0.143	0.147	0.205*	0.183*	0.142	0.161
BL+	0.062	0.167	0.160	0.189	0.161	0.194*	0.302*	0.184*	0.149	0.144	0.199*	0.168	0.151	0.172
TZMK	0.048	0.164	0.142	0.171	0.303*	0.296*	0.335*	0.274*	0.185*	0.170	0.434*	0.410*	0.219*	0.242
TZKZ	0.056	0.180	0.164	0.189	0.306*	0.282*	0.320*	0.267*	0.195*	0.183	0.429*	0.408*	0.236*	0.247
TMKE(τ_b)	0.083	0.169	0.170	0.201	0.335*	0.281*	0.332*	0.294*	0.211*	0.159	0.484*	0.401*	0.256*	0.260
TMKE+(τ_b)	0.061	0.168	0.170	0.199	0.171	0.207*	0.320*	0.184*	0.152	0.146	0.212*	0.182*	0.165	0.180
TMK0(τ_b)	-0.002	0.095	0.021	0.152	0.277*	0.238*	0.302*	0.230*	0.185	0.112	0.438*	0.389*	0.173	0.201
TMK0+(τ_b)	0.019	0.161	0.134	0.145	0.168	0.190	0.304*	0.178*	0.152	0.128	0.207*	0.179*	0.146	0.162
TMVE(τ_b)	0.152*	0.200*	0.199*	0.195	0.263*	0.128	0.219*	0.200*	0.182*	0.183	0.264*	0.193*	0.234*	0.201
TMVE+(τ_b)	0.122*	0.174	0.186*	0.195	0.176*	0.192*	0.316*	0.148*	0.149*	0.172*	0.171*	0.158*	0.175*	0.180
TMV0(τ_b)	0.117	0.175	0.175	0.165	0.251*	-0.004	-0.003	0.168*	0.158	0.151	0.233*	0.174	0.187	0.150
TMV0+(τ_b)	0.118*	0.165	0.171	0.165	0.157	0.161	0.289*	0.139	0.137	0.165	0.169*	0.148	0.146	0.164
After Transaction Cost														
W*	0.198	0.195	0.217	0.203	0.289	0.250	0.291	0.270	0.214	0.182	0.369	0.351	0.269	0.254
EW	0.091	0.158	0.146	0.187	0.153	0.156	0.163	0.126	0.136	0.155	0.146	0.142	0.163	0.148
MK	-0.158	-0.507	-0.857	0.143	-0.052	-0.346	-0.312	0.081	-0.015	-0.119	0.073	-0.010	-0.091	-0.167
MK+	0.040	0.135	0.125	0.155	0.147	0.167	0.269*	0.167*	0.135	0.120	0.190*	0.166	0.119	0.149
MV	0.083	0.111	0.051	0.146	0.162	-0.167	-0.173	0.128	0.123	0.119	0.145	0.085	0.118	0.072
MV+	0.114*	0.167	0.160	0.150	0.155	0.152	0.272*	0.132	0.135	0.160	0.158	0.150	0.158	0.159
OC	-0.150	-0.519	-0.886	0.084	-0.102	-0.101	-0.061	0.068	-0.066	-0.177	0.051	0.006	-0.138	-0.153
OC+	0.057	0.114	0.113	0.150	0.159	0.153	0.183	0.191*	0.135	0.114	0.201*	0.196*	0.136	0.146
VT	0.100*	0.167*	0.157*	0.154	0.163*	0.163*	0.180*	0.136*	0.140*	0.157	0.156*	0.144*	0.167*	0.153
TB+	0.016	0.126	0.133	0.088	0.139	0.145	0.232*	0.153	0.119	0.128	0.190*	0.167	0.109	0.134
BL+	0.049	0.152	0.142	0.173	0.147	0.171	0.274*	0.173*	0.138	0.131	0.187*	0.155	0.136	0.156
TZMK	-0.072	-0.214	-0.209	0.141	0.128	0.026	0.043	0.131	0.061	0.030	0.217*	0.180	0.075	0.041
TZKZ	-0.032	-0.112	-0.126	0.161	0.156	-0.017	0.006	0.142	0.095	0.075	0.234*	0.197	0.108	0.068
TMKE(τ_a)	0.024	0.031	0.011	0.171	0.210*	0.139	0.163	0.208*	0.148	0.135	0.322*	0.187*	0.157	0.147
TMKE+(τ_a)	0.049	0.165	0.144	0.169	0.153	0.167	0.273*	0.167*	0.138	0.139	0.191*	0.165*	0.140	0.159
TMK0(τ_a)	-0.127	-0.230	-0.208	0.124	0.117	0.112	0.215	0.204*	0.127	0.046	0.243*	0.253*	0.088	0.074
TMK0+(τ_a)	0.004	0.154	0.103	0.139	0.141	0.156	0.277*	0.162*	0.134	0.118	0.196*	0.168*	0.137	0.145
TMVE(τ_a)	0.121*	0.169	0.177*	0.188	0.206*	0.039	0.132	0.164*	0.153*	0.152	0.221*	0.142	0.195*	0.159
TMVE+(τ_a)	0.116*	0.167	0.175*	0.188	0.167*	0.183*	0.301*	0.144*	0.142*	0.163*	0.163*	0.150	0.169	0.171
TMV0(τ_a)	0.103	0.170	0.154	0.160	0.176	-0.131	-0.088	0.132	0.148	0.103	0.197*	0.105	0.139	0.105
TMV0+(τ_a)	0.110	0.162	0.170	0.160	0.139	0.156	0.281*	0.123	0.144	0.162	0.161	0.150	0.149	0.159

(TMKE+ and TMVE+) perform better when subject to transaction costs. The turnover minimization allows us to enjoy the benefits of the shrinkage estimator without losing modelling flexibility.



(a) Before transaction cost (b) After transaction cost

Figure 4: Sharpe Ratio under Variance Targeting

This figure compares the turnover minimization models with their underlying models under variance targeting when $T = 120$. The vertical axis represents the Sharpe ratio difference from EW and the horizontal axis represents the datasets.

3.4.4. Robustness Check

Comprehensive robustness tests further extend the empirical study and verify its findings. We first repeat the same analysis using different estimation window sizes, $T = 60$ and 240. The same analysis is also applied to ten additional datasets. These datasets are the same as those in Table 2 but exclude the market and factor portfolios. We finally analyze the performance in sub-periods. The complete set of results can be found in IA.

The results on the new datasets are qualitatively similar to those presented in Section 3.4.2 with some minor differences. The ranking of the portfolios are largely unchanged and the turnover minimization continues to perform superior. Under both variance targeting and utility maximization, TMKE and TMVE perform superior on average before and after transaction costs. In general, portfolios tend to perform slightly worse before transaction costs and better after transaction costs in the new datasets. This is perhaps because excluding the factor portfolios reduces the size of the feasible set while simultaneously mitigating leverage and turnover as the factor portfolios cannot be sold short to buy other assets.

The turnover minimization remains superior across different estimation windows. With the smaller estimation window ($T = 60$), optimal strategies tend to perform poorer especially after

Table 6: Turnover under Variance Targeting

This table reports the turnover of the portfolio models in Table 1 under variance targeting when $T = 120$. Turnover is defined by the formula in (20). The columns represent the datasets described in Table 2. In the turnover minimization models, τ_b (τ_a) denotes the τ calibrated without (with) transaction costs.

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	Mean
W*	0.026	0.021	0.012	0.007	0.023	0.026	0.022	0.021	0.017	0.014	0.019	0.025	0.016	0.019
EW	0.003	0.002	0.001	0.008	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.002
MK	0.128	0.180	0.098	0.019	0.109	0.144	0.127	0.112	0.125	0.145	0.107	0.111	0.104	0.116
MK+	0.014	0.016	0.007	0.014	0.011	0.010	0.010	0.010	0.013	0.016	0.008	0.007	0.013	0.011
MV	0.048	0.070	0.052	0.011	0.063	0.189	0.173	0.051	0.051	0.058	0.069	0.064	0.059	0.074
MV+	0.008	0.007	0.004	0.010	0.004	0.006	0.007	0.010	0.009	0.009	0.004	0.003	0.004	0.007
OC	0.092	0.138	0.085	0.019	0.101	0.102	0.091	0.088	0.097	0.124	0.101	0.098	0.100	0.095
OC+	0.017	0.016	0.006	0.016	0.010	0.010	0.010	0.010	0.016	0.015	0.008	0.005	0.012	0.012
VT	0.003	0.003	0.001	0.009	0.001	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.002
TB+	0.010	0.013	0.007	0.023	0.012	0.016	0.014	0.012	0.018	0.010	0.007	0.007	0.015	0.012
BL+	0.013	0.010	0.005	0.012	0.007	0.008	0.008	0.008	0.008	0.010	0.005	0.007	0.007	0.008
TZMK	0.099	0.169	0.080	0.017	0.081	0.109	0.105	0.099	0.096	0.094	0.099	0.101	0.069	0.094
TZKZ	0.088	0.161	0.081	0.015	0.078	0.156	0.141	0.096	0.083	0.084	0.096	0.101	0.064	0.096
TMKE(τ_b)	0.077	0.154	0.078	0.019	0.064	0.109	0.117	0.090	0.109	0.128	0.084	0.098	0.056	0.091
TMKE+(τ_b)	0.013	0.016	0.006	0.014	0.011	0.011	0.010	0.008	0.013	0.012	0.008	0.006	0.007	0.010
TMK0(τ_b)	0.100	0.164	0.067	0.013	0.039	0.092	0.035	0.051	0.058	0.096	0.081	0.087	0.039	0.071
TMK0+(τ_b)	0.005	0.011	0.005	0.013	0.010	0.009	0.006	0.007	0.013	0.004	0.004	0.003	0.003	0.007
TMVE(τ_b)	0.051	0.019	0.011	0.010	0.043	0.098	0.074	0.052	0.046	0.057	0.057	0.065	0.034	0.047
TMVE+(τ_b)	0.010	0.008	0.004	0.010	0.005	0.005	0.007	0.004	0.010	0.010	0.005	0.004	0.005	0.007
TMV0(τ_b)	0.026	0.023	0.007	0.008	0.055	0.142	0.073	0.043	0.018	0.040	0.037	0.050	0.053	0.044
TMV0+(τ_b)	0.006	0.005	0.003	0.008	0.003	0.005	0.005	0.005	0.003	0.005	0.003	0.002	0.002	0.004
TMKE(τ_a)	0.028	0.016	0.008	0.020	0.031	0.011	0.013	0.053	0.022	0.021	0.058	0.092	0.035	0.031
TMKE+(τ_a)	0.012	0.009	0.005	0.014	0.009	0.011	0.010	0.008	0.012	0.008	0.007	0.006	0.004	0.009
TMK0(τ_a)	0.043	0.121	0.031	0.010	0.029	0.020	0.016	0.025	0.022	0.037	0.045	0.034	0.026	0.035
TMK0+(τ_a)	0.005	0.004	0.002	0.011	0.002	0.004	0.005	0.004	0.003	0.002	0.002	0.002	0.001	0.004
TMVE(τ_a)	0.026	0.008	0.005	0.010	0.016	0.071	0.068	0.024	0.015	0.039	0.027	0.005	0.019	0.026
TMVE+(τ_a)	0.012	0.008	0.004	0.010	0.004	0.005	0.007	0.005	0.003	0.007	0.004	0.006	0.004	0.006
TMV0(τ_a)	0.008	0.006	0.006	0.008	0.019	0.070	0.062	0.024	0.013	0.022	0.032	0.021	0.027	0.024
TMV0+(τ_a)	0.005	0.004	0.002	0.008	0.002	0.005	0.005	0.002	0.002	0.003	0.002	0.001	0.002	0.003

Table 7: Turnover under Utility Maximization

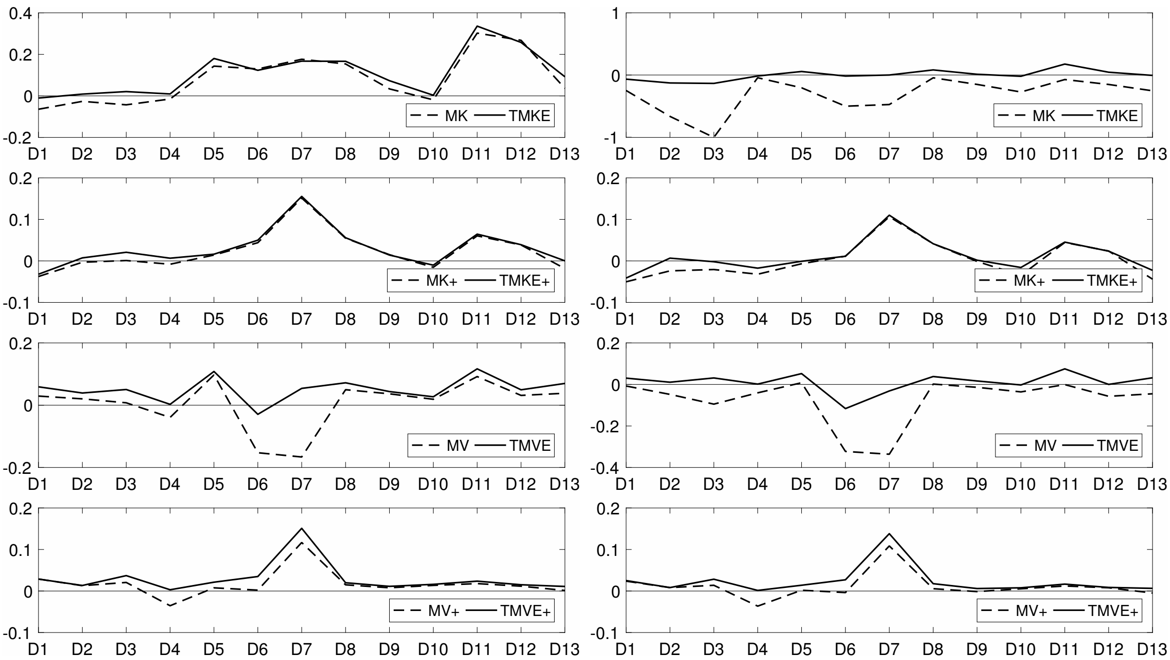
This table reports the turnover of the portfolio models in Table 1 under utility maximization when $T = 120$. Turnover is defined by the formula in (20). The columns represent the datasets described in Table 2. In the turnover minimization models, τ_b (τ_a) denotes the τ calibrated without (with) transaction costs.

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	Mean
W*	0.065	0.071	0.045	0.060	0.116	0.156	0.187	0.098	0.046	0.033	0.131	0.165	0.066	0.095
EW	0.002	0.002	0.001	0.007	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002
MK	0.434	4.160	4.359	0.244	1.084	2.173	2.380	0.811	0.625	0.828	1.613	1.621	0.776	1.624
MK+	0.026	0.047	0.021	0.199	0.021	0.034	0.078	0.026	0.028	0.034	0.018	0.014	0.024	0.044
MV	0.034	0.043	0.024	0.007	0.032	0.005	0.005	0.035	0.037	0.039	0.032	0.033	0.028	0.027
MV+	0.007	0.004	0.002	0.007	0.003	0.001	0.001	0.008	0.008	0.007	0.003	0.002	0.003	0.004
OC	0.403	3.886	4.271	0.152	0.856	1.128	1.273	0.745	0.539	0.622	1.414	1.460	0.703	1.342
OC+	0.018	0.015	0.006	0.026	0.009	0.008	0.008	0.009	0.015	0.017	0.007	0.005	0.012	0.012
VT	0.002	0.002	0.001	0.007	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002
TB+	0.019	0.040	0.021	0.247	0.022	0.044	0.069	0.032	0.036	0.026	0.016	0.014	0.027	0.047
BL+	0.015	0.022	0.011	0.053	0.010	0.016	0.024	0.013	0.014	0.017	0.009	0.009	0.011	0.017
TZMK	0.135	0.650	0.242	0.141	0.244	0.330	0.381	0.320	0.198	0.210	0.477	0.441	0.136	0.300
TZKZ	0.092	0.432	0.146	0.130	0.183	0.286	0.318	0.230	0.143	0.158	0.358	0.327	0.099	0.223
TMKE(τ_b)	0.141	1.610	0.524	0.223	0.499	1.018	2.153	0.451	0.285	0.648	1.091	1.287	0.347	0.791
TMKE+(τ_b)	0.016	0.043	0.019	0.196	0.021	0.036	0.079	0.024	0.029	0.025	0.019	0.012	0.016	0.041
TMK0(τ_b)	0.207	2.290	1.562	0.168	0.656	1.087	0.913	0.447	0.309	0.545	1.342	1.214	0.344	0.853
TMK0+(τ_b)	0.014	0.035	0.015	0.165	0.020	0.032	0.067	0.021	0.028	0.017	0.012	0.008	0.009	0.034
TMVE(τ_b)	0.042	0.019	0.006	0.008	0.022	0.005	0.004	0.032	0.038	0.036	0.026	0.030	0.017	0.022
TMVE+(τ_b)	0.009	0.006	0.002	0.008	0.004	0.001	0.002	0.003	0.008	0.009	0.004	0.003	0.003	0.005
TMV0(τ_b)	0.017	0.012	0.004	0.006	0.027	0.004	0.003	0.026	0.012	0.024	0.015	0.024	0.024	0.015
TMV0+(τ_b)	0.005	0.002	0.001	0.006	0.002	0.001	0.001	0.005	0.003	0.003	0.002	0.001	0.001	0.002
TMKE(τ_a)	0.080	0.164	0.068	0.210	0.112	0.052	0.059	0.204	0.109	0.062	0.307	0.242	0.115	0.137
TMKE+(τ_a)	0.012	0.037	0.016	0.191	0.019	0.032	0.080	0.023	0.027	0.023	0.016	0.013	0.010	0.038
TMK0(τ_a)	0.187	0.984	0.316	0.168	0.187	0.124	0.166	0.189	0.133	0.155	0.377	0.320	0.156	0.266
TMK0+(τ_a)	0.014	0.026	0.012	0.164	0.011	0.023	0.052	0.016	0.018	0.014	0.010	0.007	0.006	0.029
TMVE(τ_a)	0.020	0.006	0.004	0.008	0.010	0.004	0.004	0.015	0.015	0.033	0.014	0.003	0.011	0.011
TMVE+(τ_a)	0.012	0.006	0.002	0.008	0.003	0.002	0.001	0.004	0.005	0.005	0.003	0.003	0.003	0.004
TMV0(τ_a)	0.006	0.004	0.003	0.006	0.010	0.003	0.003	0.014	0.009	0.015	0.013	0.010	0.012	0.008
TMV0+(τ_a)	0.004	0.002	0.001	0.006	0.001	0.001	0.001	0.001	0.001	0.002	0.001	0.000	0.001	0.002

Table 8: Calibrated τ under Variance Targeting

This table reports the mean of the calibrated τ's for each turnover minimization model under variance targeting when $T = 120$. $\bar{\tau}_b$ ($\bar{\tau}_a$) denotes the mean over the sample period of the τ calibrated without (with) transaction costs. Transaction costs are assumed to be 50 basis points for risky assets and 0 for the risk-free asset.

		D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13
TMKE	$\bar{\tau}_b$	0.456	0.289	0.266	0.030	0.128	0.099	0.033	0.032	0.035	0.343	0.050	0.047	0.142
	$\bar{\tau}_a$	0.649	0.865	0.882	0.040	0.370	0.538	0.327	0.139	0.663	0.834	0.151	0.077	0.349
TMKE+	$\bar{\tau}_b$	0.205	0.122	0.064	0.025	0.005	0.011	0.008	0.009	0.001	0.095	0.008	0.018	0.083
	$\bar{\tau}_a$	0.272	0.639	0.252	0.032	0.048	0.041	0.014	0.013	0.019	0.269	0.016	0.026	0.181
TMK0	$\bar{\tau}_b$	0.524	0.494	0.722	0.245	0.221	0.227	0.224	0.189	0.229	0.347	0.070	0.092	0.319
	$\bar{\tau}_a$	0.687	0.857	0.917	0.271	0.345	0.453	0.291	0.226	0.306	0.571	0.239	0.236	0.417
TMK0+	$\bar{\tau}_b$	0.199	0.268	0.236	0.091	0.006	0.044	0.062	0.012	0.003	0.264	0.014	0.034	0.190
	$\bar{\tau}_a$	0.214	0.565	0.558	0.162	0.086	0.130	0.079	0.024	0.060	0.322	0.026	0.040	0.356
TMVE	$\bar{\tau}_b$	0.087	0.213	0.292	0.294	0.103	0.944	0.972	0.044	0.110	0.004	0.031	0.149	0.079
	$\bar{\tau}_a$	0.134	0.251	0.420	0.337	0.267	0.988	0.988	0.110	0.295	0.107	0.145	0.808	0.211
TMVE+	$\bar{\tau}_b$	0.009	0.071	0.045	0.293	0.077	0.404	0.122	0.050	0.042	0.009	0.041	0.148	0.060
	$\bar{\tau}_a$	0.013	0.080	0.052	0.327	0.090	0.463	0.130	0.070	0.108	0.019	0.060	0.179	0.078
TMV0	$\bar{\tau}_b$	0.080	0.303	0.445	0.276	0.091	0.483	0.947	0.063	0.115	0.030	0.035	0.141	0.074
	$\bar{\tau}_a$	0.112	0.380	0.582	0.281	0.420	0.988	0.988	0.222	0.170	0.206	0.096	0.695	0.344
TMV0+	$\bar{\tau}_b$	0.010	0.083	0.061	0.275	0.179	0.350	0.216	0.083	0.056	0.032	0.125	0.114	0.181
	$\bar{\tau}_a$	0.014	0.122	0.119	0.280	0.265	0.409	0.225	0.120	0.064	0.039	0.147	0.121	0.194



(a) Before transaction cost (b) After transaction cost

Figure 5: Sharpe Ratio under Utility Maximization

This figure compares the turnover minimization models with their underlying models under utility maximizatoin when $T = 120$. The vertical axis represents the Sharpe ratio difference from EW and the horizontal axis represents the datasets.

transaction costs. This is because input parameter estimates change more abruptly resulting in higher turnover. Notwithstanding, the turnover minimization successfully calibrates τ and maintains superior performance distancing itself further from the underlying model. The comparison of the performances from $T = 120$ and $T = 240$ before transaction costs suggests that increasing the estimation window size does not necessarily reduce estimation errors: optimal strategies indeed perform poorer when $T = 240$. This signifies that using a parametric approach for estimation error can potentially be dangerous. For example, TZMK and TZKZ assume that the estimation errors will be smaller when T is larger and accordingly increase the weight on the optimal portfolio while reducing the weight on the equal-weight portfolio. The turnover minimization remains as the best performing strategy when $T = 240$.

Figure 6 and 7 portrays the sub-period performance of the turnover minimization models respectively under variance targeting and utility maximization. Each sub-period is ten-year long and five-year apart from each other except for the last sub-period which is shorter due to the size of the whole sample period. The charts on the left show the average Sharpe ratio across the sub-periods and the charts on the right show the percentage of the sub-periods in which a strategy outperforms EW (outperformance ratio). The solid black line represents a turnover minimization model and the gray dotted line represents its underlying model. The sub-period analysis results of all models can be found in IA.

In line with the findings in Section 3.4.2, the turnover minimization models generally outperform EW as well as their underlying models. They outperform EW more frequently than their underlying models in most datasets. All turnover minimization models have a high outper-

Table 9: Calibrated τ under Utility Maximization

This table reports the mean of the calibrated τ's for each turnover minimization model under utility maximization when $T = 120$. $\bar{\tau}_b$ ($\bar{\tau}_a$) denotes the mean over the sample period of the τ calibrated without (with) transaction costs. Transaction costs are assumed to be 50 basis points for risky assets and 0 for the risk-free asset.

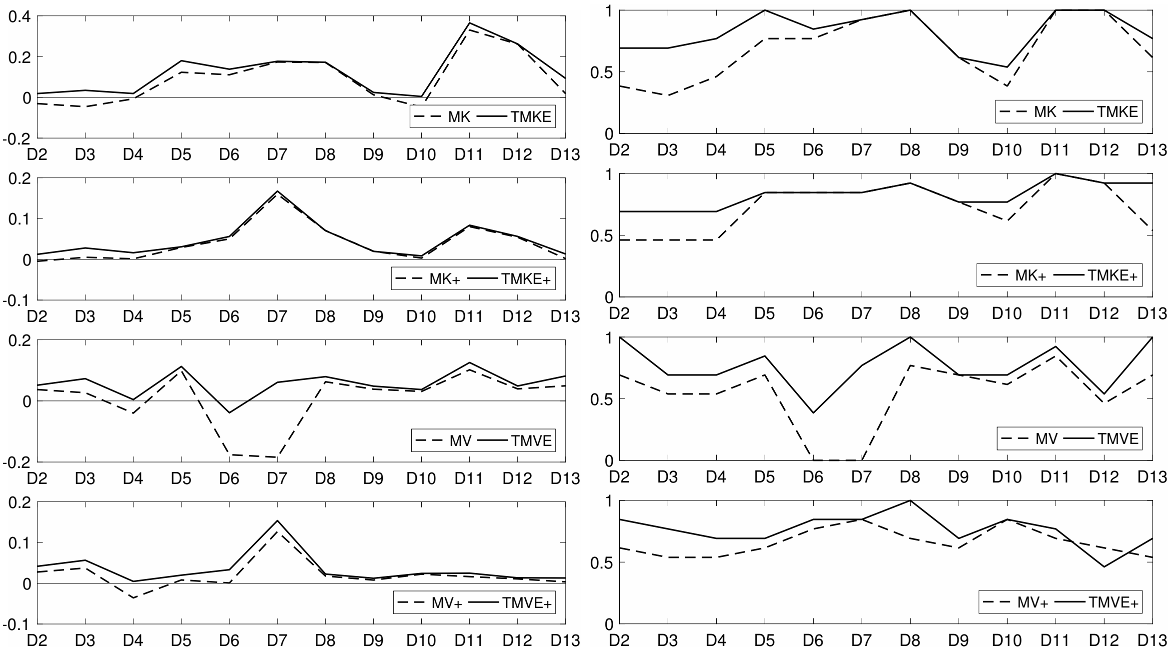
		D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13
TMKE	$\bar{\tau}_b$	0.277	0.150	0.284	0.039	0.129	0.139	0.034	0.058	0.100	0.124	0.058	0.048	0.139
	$\bar{\tau}_a$	0.404	0.825	0.887	0.059	0.372	0.421	0.360	0.187	0.240	0.734	0.284	0.314	0.383
TMKE+	$\bar{\tau}_b$	0.339	0.150	0.104	0.030	0.004	0.011	0.006	0.009	0.001	0.091	0.006	0.019	0.075
	$\bar{\tau}_a$	0.352	0.508	0.298	0.035	0.033	0.054	0.010	0.012	0.016	0.211	0.014	0.027	0.202
TMK0	$\bar{\tau}_b$	0.349	0.706	0.801	0.201	0.202	0.214	0.181	0.162	0.232	0.319	0.091	0.107	0.287
	$\bar{\tau}_a$	0.513	0.859	0.915	0.207	0.472	0.501	0.358	0.277	0.359	0.702	0.376	0.359	0.523
TMK0+	$\bar{\tau}_b$	0.277	0.390	0.331	0.147	0.006	0.043	0.062	0.010	0.001	0.425	0.014	0.040	0.270
	$\bar{\tau}_a$	0.307	0.601	0.480	0.140	0.063	0.143	0.085	0.024	0.045	0.525	0.022	0.046	0.330
TMVE	$\bar{\tau}_b$	0.095	0.339	0.266	0.253	0.103	0.940	0.970	0.052	0.075	0.007	0.036	0.158	0.084
	$\bar{\tau}_a$	0.162	0.476	0.390	0.285	0.263	0.986	0.988	0.133	0.260	0.173	0.143	0.789	0.218
TMVE+	$\bar{\tau}_b$	0.009	0.192	0.043	0.251	0.075	0.417	0.137	0.044	0.039	0.009	0.055	0.118	0.083
	$\bar{\tau}_a$	0.016	0.214	0.049	0.283	0.087	0.440	0.143	0.062	0.086	0.022	0.082	0.132	0.095
TMV0	$\bar{\tau}_b$	0.092	0.575	0.563	0.262	0.075	0.494	0.946	0.095	0.097	0.080	0.043	0.112	0.084
	$\bar{\tau}_a$	0.124	0.676	0.712	0.262	0.373	0.768	0.988	0.209	0.141	0.235	0.090	0.541	0.303
TMV0+	$\bar{\tau}_b$	0.010	0.344	0.254	0.261	0.163	0.402	0.225	0.071	0.045	0.054	0.112	0.111	0.176
	$\bar{\tau}_a$	0.014	0.348	0.272	0.261	0.232	0.413	0.228	0.115	0.054	0.064	0.138	0.110	0.177

formance ratio before transaction costs under variance targeting. Although the outperformance ratio deteriorates after transaction costs, the short-sale constrained models perform robustly. Under utility maximization, the performance of TMKE and TMKE+ is less pronounced while TMVE and TMVE+ consistently perform superior. In particular, TMVE+ performs robustly in all circumstances. Overall, the sub-period analysis confirms the robustness of the findings in Section 3.4.2.

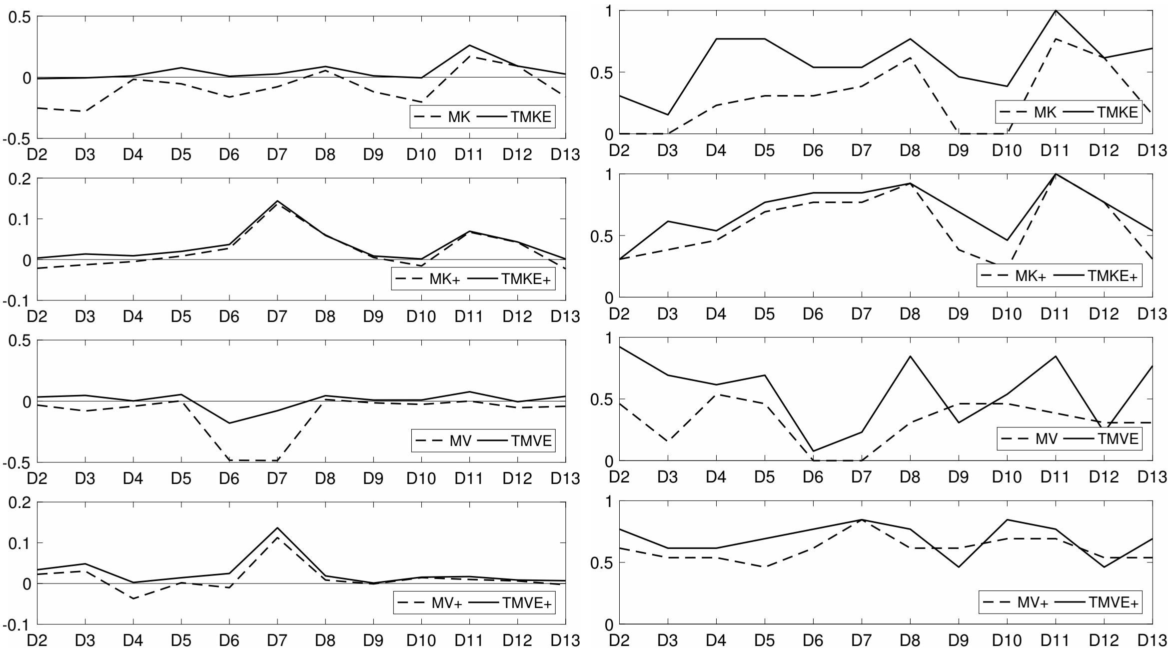
4. Concluding Remarks

In this paper, we develop a versatile shrinkage portfolio estimator, turnover minimization. It places an additional layer on a conventional portfolio problem in which the optimal portfolio found in the original problem is shrunk towards a reference portfolio. Unlike existing shrinkage models, our model does not assume the distribution of estimation errors but determines the optimal shrinkage level from the data. This nonparametric approach makes our model particularly suitable when the distribution of the input parameter estimates is unknown. Another advantage of our model is that it can be easily tailored to accommodate a wide range of portfolio problems with various objectives and constraints. This flexibility is particularly beneficial to practitioners who often encounter various constraints imposed by internal policy or regulation. The implementation is straightforward whilst the gain from the added layer is substantial.

We evaluate our model against various portfolio models including well-known classical models to existing shrinkage models via a comprehensive comparative analysis. The simulation and empirical studies reveal that the turnover minimization enhances the performance of its underlying model and outperforms the other models. While unconstrained turnover minimization



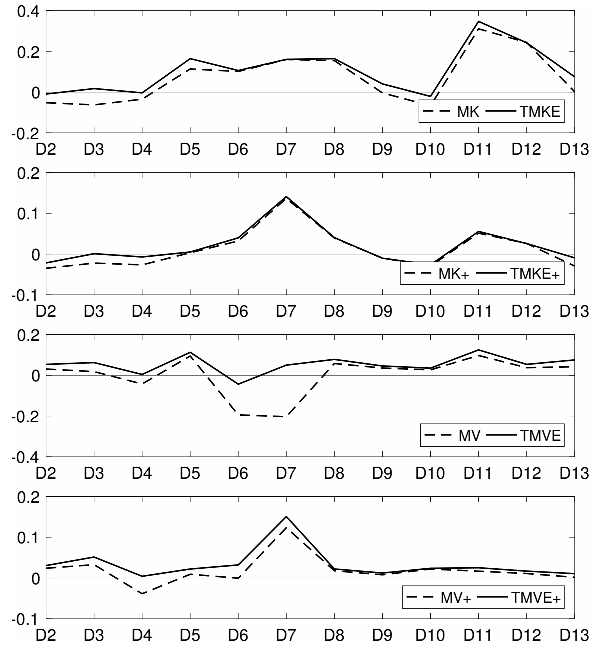
(a) Sharpe ratio before transaction costs (b) Outperformance



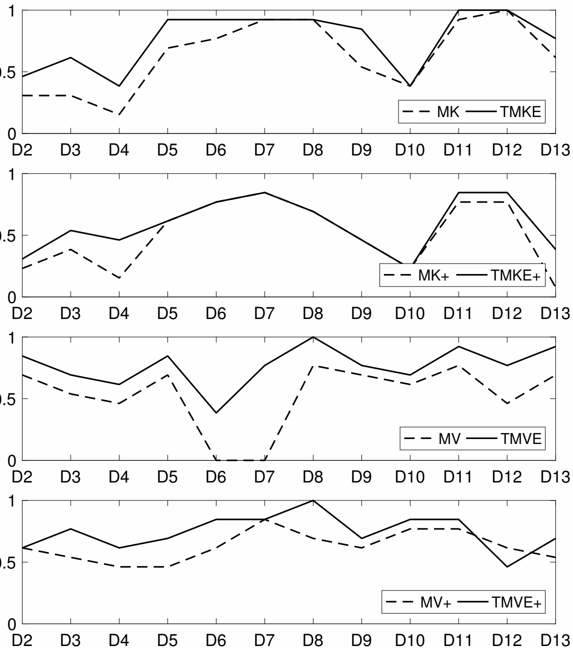
(c) Sharpe ratio after transaction costs (d) Outperformance

Figure 6: Sub-Period Performance under Variance Targeting.eps

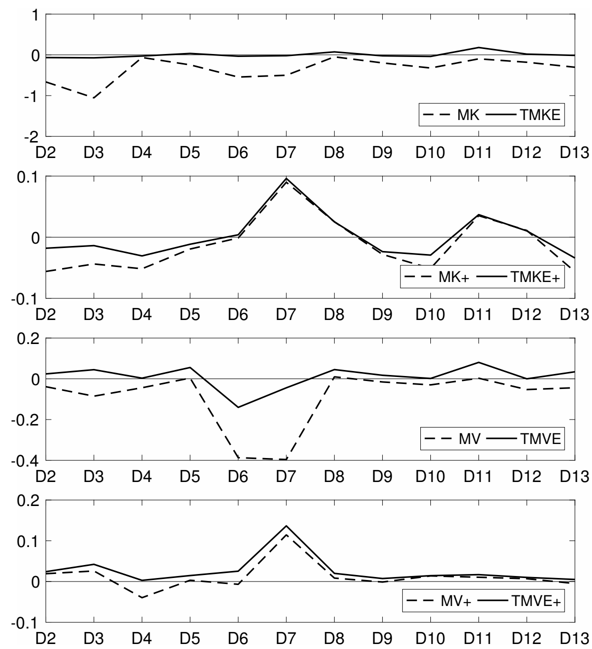
This Figure demonstrates the sub-period performance of the turnover minimization models under variance targeting when $T = 120$. The charts on the left show the mean Sharpe ratio difference from EW across sub-periods and the charts on the right show the percentage of the sub-periods in which a strategy outperforms EW. The solid black line represents a turnover minimization model and the gray dotted line represents its underlying model. Each sub-period is ten-year long and five-year apart from each other except for the last sub-period which is shorter due to the size of the whole sample period.



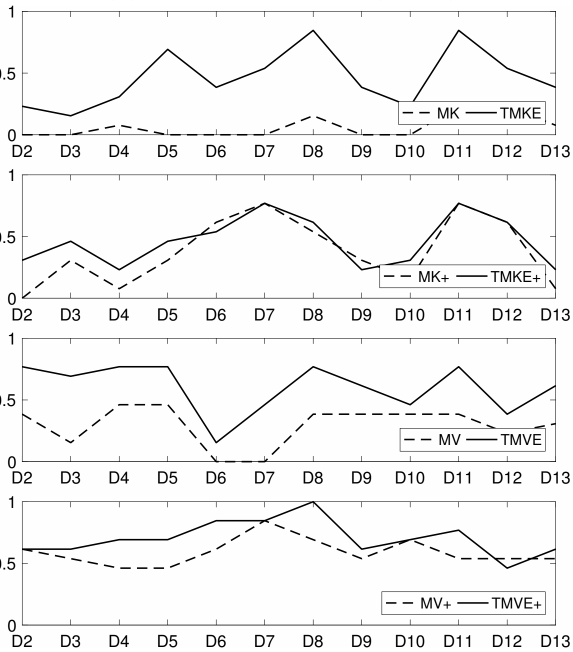
(a) Sharpe ratio before transaction cost



(b) Outperformance, P(>EW)



(c) Sharpe ratio after transaction cost



(d) Outperformance, P(>EW)

Figure 7: Sub-Period Performance under Utility Maximization

This Figure demonstrates the sub-period performance of the turnover minimization models under utility maximization when $T = 120$. The charts on the left show the mean Sharpe ratio difference from EW across sub-periods and the charts on the right show the percentage of the sub-periods in which a strategy outperforms EW. The solid black line represents a turnover minimization model and the gray dotted line represents its underlying model. Each sub-period is ten-year long and five-year apart from each other except for the last sub-period which is shorter due to the size of the whole sample period.

models perform superior without transaction costs, they are outperformed by their short-sale constrained counterparts when subject to large transaction costs. This highlights the advantage of our model being capable of accommodating constraints. We also find that the equal-weight portfolio serves better as the shrinkage target compared to the current portfolio. The effectiveness of the turnover minimization is reaffirmed through a comprehensive robustness check.

We propose a simple calibration method to determine the shrinkage level and test only two types of shrinkage targets. Different shrinkage targets and more sophisticated calibration methods could enhance the performance of the turnover minimization further.

A. Implementation of the Models

This section describes the implementation details of the models in Table 1. For the full details of each model, the reader is referred to the original papers.

Under variance targeting, a portfolio, w, is adjusted as follows to meet the variance target, σ_T^2:

$$w := w\frac{\sigma_T}{\sqrt{w'\hat{\Sigma}w}}. \tag{A.1}$$

A.1. *Ex-post* Optimal Portfolio, W*

The *ex-post* optimal portfolio maximizes the utility using the true μ and Σ:

$$w^* = \frac{1}{\gamma}\Sigma^{-1}\mu. \tag{A.2}$$

When w^* is adjusted to meet the variance target, the true covariance matrix Σ is used instead of its estimate, $\hat{\Sigma}$. W* is rebalanced back to the optimal portfolio every month.

A.2. Equal-Weight Portfolio, EW

The equal-weight portfolio allocates the wealth equally to the risky assets:

$$w_{ew} = \frac{1}{N}1_N, \tag{A.3}$$

where 1_N is an N-dimensional vector of ones. EW is rebalanced monthly.

A.3. Markowitz (1952) Mean-Variance Portfolio, MK(+)

The mean-variance portfolio can be obtained from the formulas below.

- Unconstrained Utility Maximization (MK)

$$w_{mk} = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}. \tag{A.4}$$

- Short-sale Constrained Utility Maximization (MK+)

$$
\begin{aligned}
w_{mk+} = \underset{w}{\operatorname{argmax}}\ & w'\hat{\mu} - \frac{\gamma}{2} w'\hat{\Sigma} w \\
\text{subject to } & w_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{A.5}
$$

A.4. Global Minimum-Variance Portfolio, MV(+)

The global minimum-variance portfolio can be obtained from the formulas below.

- Unconstrained Variance Minimization (MV)

$$
w_{mv} = \frac{\hat{\Sigma}^{-1} 1_N}{1_N' \hat{\Sigma}^{-1} 1_N}. \tag{A.6}
$$

- Short-sale Constrained Variance Minimization (MV+)

$$
\begin{aligned}
w_{mv+} = \underset{w}{\operatorname{argmin}}\ & w'\hat{\Sigma} w \\
\text{subject to } & w' 1_N = 1 \\
& w_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{A.7}
$$

A.5. Optimal Constrained Portfolio (Kirby and Ostdiek, 2012), OC(+)

Kirby and Ostdiek (2012) show that a mean-variance portfolio constrained to invest only in the risky assets and have the same expected return as the naïve portfolio outperforms the naïve portfolio. The same strategy is considered here, but to be consistent with other strategies, variance is constrained to meet the target rather than the expected return.

The OC portfolio under variance targeting can be obtained from

$$
\begin{aligned}
w_{oc} = \underset{w}{\operatorname{argmax}}\ & w'\hat{\mu} \\
\text{subject to } & w' 1_N = 1 \\
& w'\hat{\Sigma} w \leq \sigma_T^2.
\end{aligned}
\tag{A.8}
$$

As OC invests only in the risky assets, the variance constraint is imposed during optimization. Other OC optimization problems are similarly defined.

A.6. Volatility Timing (Kirby and Ostdiek, 2012), VT

Kirby and Ostdiek (2012) also introduce portfolio strategies based on volatility. One of their volatility-based strategies that utilizes only the variance is considered. The portfolio from the volatility timing strategy is determined by the formula

$$
w_{vt,i} = \frac{\left(1/\hat{\sigma}_i^2\right)^m}{\sum_{i=1}^{N} \left(1/\hat{\sigma}_i^2\right)^m}, \quad i = 1, \ldots, N, \tag{A.9}
$$

where m is a tuning parameter which determines the aggressiveness of the weight adjustment in response to changes in the volatility of the asset and $\hat{\sigma}_i^2$ is the sample variance of the i-th asset return. In the empirical analysis, m is set to 1.

A.7. Black and Litterman (1992) Model, BL+

Black and Litterman (1992) introduce a Bayesian asset allocation model where subjective investor views can be incorporated in the market portfolio. Their framework is adopted by Bessler et al. (2014) who use the naïve portfolio as a proxy for the market portfolio and sample mean as investor views. The implementation procedure is as follows.

The equilibrium return implied by the equal-weight portfolio is given by

$$\bar{\mu} = \gamma \Sigma \mu_{ew}, \tag{A.10}$$

where γ is the risk aversion parameter. The equilibrium return is assumed to be an unbiased estimate of the true mean:

$$\bar{\mu} = \mu + \eta, \quad \eta \sim N(0, \bar{\Sigma}), \tag{A.11}$$

where $\bar{\Sigma} = \kappa \hat{\Sigma}$ for some constant κ. The investor view is defined as the sample mean, $\hat{\mu}$, and is assumed to be an unbiased estimator of μ:

$$\hat{\mu} = \mu + \epsilon, \quad \epsilon \sim N(0, \Omega). \tag{A.12}$$

Ω represents the uncertainty of the view and is assumed to be of the form $\kappa \text{diag}(\hat{\Sigma})$, where $\text{diag}(\hat{\Sigma})$ is a diagonal matrix derived from $\hat{\Sigma}$. From (A.11) and (A.12), the mean and covariance matrix of the asset returns can be estimated via the generalized least squares and are given by:

$$\tilde{\mu} = \bar{\mu} + \bar{\Sigma}(\bar{\Sigma} + \Omega)^{-1}(\hat{\mu} - \bar{\mu}), \tag{A.13}$$

$$\tilde{\Sigma} = \hat{\Sigma} + \bar{\Sigma} - \bar{\Sigma}(\bar{\Sigma} + \Omega)^{-1}\bar{\Sigma}. \tag{A.14}$$

γ and κ are respectively set to 3 and 0.1. The Black-Litterman optimal portfolio is obtained by solving the constrained mean-variance problem in (A.5) with the mean and covariance estimates defined above.

A.8. Treynor and Black (1973) Model, TB+

Treynor and Black (1973) develop an active portfolio strategy for an optimal allocation between active assets (assets with abnormal excess returns) and the market portfolio. We adopt their model and use the naïve portfolio as a proxy for the market portfolio.

The "active assets" are first identified by regressing asset returns on the naïve portfolio returns:

$$r_{it} = \alpha_i + \beta_i r_{ew,t} + e_{it}, \quad t = 1, \ldots, T, \tag{A.15}$$

where r_{it} and $r_{ew,t}$ are respectively the returns of asset i and the naïve portfolio at time t in excess of the risk free rate. The assets with a significant α_i at 5% are identified as active assets. The regression is carried out and active assets are identified every month.

The optimal portfolio from the active assets and the equal-weight portfolio can be obtained by solving the usual mean-variance problem with the equal-weight portfolio added in the asset pool. Treynor and Black (1973) provide a closed form solution for an unconstrained problem but it needs to be solved numerically when subject to the short-sale constraint:

$$\begin{aligned} w_{tb+} &= \underset{w}{\text{argmax}}\ w'\hat{\mu} - \frac{\gamma}{2} w'\hat{\Sigma} w \\ &\text{subject to } w_i + \frac{1}{M} w_{M+1} \geq 0, \quad i = 1, \dots, M, \end{aligned} \tag{A.16}$$

where M denotes the number of active assets, and $\hat{\mu} \in \mathbb{R}^{M+1}$ and $\hat{\Sigma} \in \mathbb{R}^{(M+1)\times(M+1)}$ are the input parameter estimates of the M active assets and the equal-weight portfolio ($(M+1)$-th asset). Note that since the $(M+1)$-th asset is the equal-weight portfolio, the short-sale constraint has a different form.

A.9. Tu and Zhou (2011) Models, TZMK and TZKZ

Tu and Zhou (2011) develop a shrinkage portfolio model that combines an optimal portfolio with the naïve portfolio. They specifically consider an optimal mix of the naïve portfolio with the Markowitz (1952) rule, Jorion (1986) rule, Kan and Zhou (2007) rule, and MacKinlay and Pástor (2000) rule. In this paper, the Markowitz (1952) rule and Kan and Zhou (2007) rule are considered.

- MK+EW (TZMK)

  The Tu and Zhou (2011) portfolio that combines the equal-weight portfolio with the Markowitz portfolio is given by

$$w_{tzml} = \hat{a} w_{mk} + (1 - \hat{a}) w_{ew}, \tag{A.17}$$

  where

$$\hat{a} = \frac{\pi_2}{\pi_1 + \pi_2}, \tag{A.18}$$

$$\pi_1 = w_{ew}'\hat{\Sigma} w_{ew} - \frac{2}{\gamma} w_{ew}'\hat{\mu} + \frac{1}{\gamma^2}\tilde{\theta}^2, \tag{A.19}$$

$$\pi_2 = \frac{1}{\gamma^2}(c_1 - 1)\tilde{\theta}^2 + \frac{c_1}{\gamma^2}\frac{N}{T} \tag{A.20}$$

$$c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}, \tag{A.21}$$

  where $\tilde{\theta}^2$ is an estimate of $\theta = \mu\Sigma^{-1}\mu$. Tu and Zhou (2011) suggest to use the estimator of Kan and Zhou (2007):

$$\tilde{\theta}^2 = \frac{(T-N-2)\hat{\theta}^2 - N}{T} + \frac{2(\hat{\theta}^2)^{N/2}(1+\hat{\theta}^2)^{-(T-2)/2}}{T B_{\hat{\theta}^2/(1+\hat{\theta}^2)}(N/2, (T-N)/2)}, \tag{A.22}$$

where $\hat{\theta}^2 = \hat{\mu}'\hat{\Sigma}^-1\hat{\mu}$, and $B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy$ is an incomplete beta function.

- KZ+EW (TZKZ)

  The Tu and Zhou (2011) portfolio that combines the equal-weight portfolio with the Kan and Zhou (2007) three-fund rule is given by

$$w_{tzkz} = \hat{a}w_{kz} + (1-\hat{a})w_{ew}, \tag{A.23}$$

  where w_{kz} is the Kan-Zhou rule defined below, and

$$\hat{a} = \frac{\pi_1 - \pi_{13}}{\pi_1 - 2\pi_{13} + \pi_3}, \tag{A.24}$$

$$\pi_{13} = \frac{1}{\gamma^2}\tilde{\theta}^2 - \frac{1}{\gamma}w'_{ew}\hat{\mu} + \frac{1}{\gamma c_1}\left(\hat{\eta}w'_{ew}\hat{\mu} + (1-\hat{\eta})\hat{\mu}_g w'_{ew}1_N\right)$$
$$- \frac{1}{\gamma^2 c_1}\left(\hat{\eta}\hat{\mu}'\tilde{\Sigma}^{-1}\hat{\mu} + (1-\hat{\eta})\hat{\mu}_g\hat{\mu}'\tilde{\Sigma}^{-1}1_N\right), \tag{A.25}$$

$$\pi_3 = \frac{1}{\gamma^2}\tilde{\theta}^2 - \frac{1}{\gamma^2 c_1}\left(\tilde{\theta}^2 - \frac{N}{T}\hat{\eta}\right). \tag{A.26}$$

  π_1 is as defined above and $\hat{\eta}$ and $\hat{\mu}_g$ are as given below.

- Kan and Zhou (2007) Three-Fund Rule

  The three-fund rule of Kan and Zhou (2007) is given by

$$w_{kz} = \hat{a}w_{mk} + \hat{b}\tilde{w}_{mv}, \tag{A.27}$$

  where $\tilde{w}_{mv} = \frac{1}{\gamma}\hat{\Sigma}^{-1}1_N$, and

$$\hat{a} = \frac{1}{c_1}\hat{\eta}, \quad \hat{b} = \frac{1}{c_1}(1-\hat{\eta}), \quad \hat{\eta} = \frac{\tilde{\phi}^2}{\tilde{\phi}^2 + N/T}. \tag{A.28}$$

  $\tilde{\phi}^2$ is given by

$$\tilde{\phi}^2 = \frac{(T-N-1)\hat{\phi}^2 - (N-1)}{T} + \frac{2(\tilde{\phi}^2)^{(N-1)/2}(1+\tilde{\phi}^2)^{-(T-2)/2}}{TB_{\tilde{\phi}^2/(1+\tilde{\phi}^2)}((N-1)/2,(T-N+1)/2)}, \tag{A.29}$$

  where

$$\hat{\phi}^2 = (\hat{\mu} - \hat{\mu}_g 1_N)'\,\hat{\Sigma}^{-1}\,(\hat{\mu} - \hat{\mu}_g 1_N), \quad \hat{\mu}_g = \frac{\hat{\mu}'\hat{\Sigma}^{-1}1_N}{1'_N\hat{\Sigma}^{-1}1_N}. \tag{A.30}$$

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