Bonferroni-Type Tests for Return Predictability with Possibly Trending Predictors*

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Abstract

The Bonferroni Q test of Campbell and Yogo (2006) is routinely used in empirical studies investigating predictability in asset returns because of its near-optimal power properties for strongly persistent and endogenous predictors. Its formulation, however, only allows for a constant mean in the predictor, seemingly at odds with many of the predictors used in practice. We establish the asymptotic size and local power properties of the Q test, and the corresponding Bonferroni t-test of Cavanagh, Elliott and Stock (1995), under a local-to-zero specification for a linear trend in the predictor, revealing that size and power depends on the magnitude of the trend for both. To rectify this we develop with-trend variants of the Q and t tests. We also develop hybrid tests, designed to have good size and power properties when uncertainty exists as to whether or not a linear trend is present in the predictor. These use union-of-rejections and switching mechanisms to capitalise on the relative power advantages of the constant-only tests when a trend is absent and the with-trend tests otherwise. A further extension allows use of a conventional t-test where the predictor appears to be weakly persistent. We show that, overall, our recommended hybrid test offers excellent size and power properties regardless of the presence of a linear trend in the predictor, or the predictor's degrees of persistence and endogeneity. An empirical application to an updated Welch and Goyal (2008) dataset illustrates the practical relevance of our new approach.

Keywords: predictive regression; linear trend; unknown regressor persistence; Bonferroni tests; hybrid tests; union of rejections.

JEL Classifications: C22; C12; G14.

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1 Introduction

The predictability of asset returns using publicly available data has received a great deal of attention in both the economics and finance literature, leading to a large number of published studies examining whether various financial and macroeconomic variables have predictive power for returns. Candidate financial predictor variables considered have included valuation ratios such as the dividend-price or earnings-price ratio, the dividend yield and a variety of interest rate measures. Macroeconomic variables such as inflation and industrial production have also been considered; for early contributions see *inter alia* Fama (1981), Keim and Stambaugh (1986), Campbell (1987), Campbell and Shiller (1988a,b), Fama and French (1988,1989) and Fama (1990).

A common feature of many predictor variables used in empirical studies is that they are highly persistent, with a strong negative correlation found between the innovations to the returns and predictor; see, eg, Campbell and Yogo (2006) [CY] and Welch and Goyal (2008). For these highly persistent and endogenous predictors it can be shown that basing inference on conventional tests can be misleading. For instance, for strongly persistent and endogenous predictors CY show that using the conventional regression t-statistic to test for predictability leads to right-tail tests that are asymptotically oversized, with this oversize more severe the stronger is the persistence or endogeneity of the predictor, other things equal.

As a consequence, numerous tests for predictability have been developed that are designed to deliver robust inference in the presence of strongly persistent and endogenous regressors. These include likelihood-based tests developed by Cavanagh, Elliott and Stock (1995) [CES], Lewellen (2004), CY and Jansson and Moreira (2006), with these approaches explicitly modelling the predictor as an autoregressive process with the dominant root given by $\rho = 1 + cT^{-1}$ where c is a finite constant and T is the sample size. Of these tests, the Bonferroni Q test in particular has been widely adopted in the empirical literature. Other test procedures based on instrumental variable estimation have also recently been proposed, including contributions by Kostakis *et al.* (2015) and Breitung and Demetrescu (2015). Regardless of the approach taken, a common feature of all of these papers is that their primary (or even exclusive) focus is on the constant-only tests, and the properties of these tests in the presence of a (neglected) trend in the predictor are not established.

Assuming that the predictor only contains a deterministic constant would seem justified for some of the candidate predictors considered in previous empirical studies. Few, for example, would argue that for developed countries macroeconomic variables such as inflation or interest rates would likely contain deterministic elements other than a constant.

However, the same is not true for other variables that have been considered in the literature, and in many instances one cannot discount the possibility that a predictor variable may contain a deterministic linear trend. Indeed, in Section 7 of this paper we find there to be statistically significant evidence of a linear trend in many of the financial variables commonly employed as predictors in the Welch and Goyal (2008) datatset. Given the evidence of a potential trend in these predictors it is of great interest to examine the impact of an omitted trend on extant tests for predictability, and also to consider predictability tests that explicitly account for the potential presence of a trend.

In this paper we will consider constant-only and with-trend variants of the Bonferroni Q test of CY and the Bonferroni t test of CES. The Bonferroni approach underpinning these tests is based on constructing an initial confidence interval for the dominant autoregressive root, ρ, in the predictor, by inverting a unit root test, then basing a confidence interval for the predictive regression coefficient on this initial confidence interval for ρ. It is well known in the unit root literature that an omitted deterministic trend impacts the asymptotic distribution of mean-only unit root tests, see eg Harvey *et al.* (2009), and we will show in Section 4 that in the constant-only case, omitting the trend in the predictive regression test stage also impacts the limit distribution of the constant-only predictive regression test statistics of both CY and CES. When the correlation between the innovations to the predictor and returns is negative both of these effects combined will be shown to lead the constant-only Bonferroni Q and t tests to exhibit substantial asymptotic undersize when testing in the right tail, with the tests displaying a subsequent loss of power, and substantial asymptotic oversize when testing in the left tail in the presence of an omitted trend.

The with-trend variants of CY and CES that we consider are based on an initial confidence interval for ρ that uses a trend-augmented unit root test statistic, and a secondary confidence interval for the coefficient on the predictor from a trend-augmented predictive regression. These tests are exact invariant to a linear trend in the predictor. In the unit root testing context it is known that while the inclusion of a trend in the unit root test regression renders inference invariant to the presence of a trend, the power of the resulting trend-augmented unit root tests lags behind their constant-only counterparts when no trend is present, again see Harvey *et al.* (2009). We observe a similar phenomenon in the predictive regression testing context, with the with-trend Bonferroni Q and t-tests displaying inferior power to the constant-only Bonferroni Q and t-tests when no trend is present in the predictor. However, this ranking can be reversed when a trend of reasonable magnitude is present.

In view of the different tests' power rankings across no trend and trend environments, we first propose a union-of-rejections strategy for the right (left) tailed testing context when the correlation is negative (positive), that combines inference from both the constant-only and with-trend Bonferroni Q test. This procedure will be shown to capitalise on the relative power advantages of the constant-only and with-trend tests in the no trend and trend scenarios, respectively, delivering attractive levels of power regardless of whether a trend is or is not present in the predictor. In the case of left (right) tailed testing when the correlation is negative (positive), as the constant-only tests are over-sized, our initial hybrid test is simplified to the with-trend Bonferroni Q test. Such preliminary approaches, however, will be shown to be of most benefit when the predictor is strongly persistent with $\rho = 1 + c/T$ and c close to zero. We then further develop our recommended hybrid test that switches into the with-trend Bonferroni t-test when there is evidence that c is not close to zero, or into the conventional t-test when there is sufficient evidence that the predictor is weakly persistent.

The remainder of this paper is organised as follows. Section 2 introduces the predictive regression model we consider and the assumptions we place on the data generating process (DGP). In Section 3 we give a description of both the constant-only Bonferroni Q and t-tests of CY and CES, respectively, as well as modifications of these tests that account for the presence of a trend in the predictor. In Section 4 we report the limiting distributions of the predictive regression and unit root test statistics used in the test procedures outlined in this paper, and examine the relative local asymptotic power of the constant-only and with-trend Bonferroni type tests. Our proposed hybrid test procedures are outlined in Section 5, and the local asymptotic power of these tests, as well as recommendations on which test to use in practice, are provided in Section 6. The results of an empirical exercise applying our new tests to an updated version of the Welch and Goyal (2008) dataset is provided in Section 7. Section 8 concludes. Additional Monte Carlo simulation results exploring both the local asymptotic and finite sample size and power properties of the tests, together with proofs of the main theorems presented in the paper, are provided in the supplementary appendix.

2 The Predictive Regression DGP

We consider the following predictive regression DGP

$$r_t = \alpha + \beta(x_{t-1} - \gamma(t-1)) + u_t, \qquad t = 2, ..., T \tag{1}$$

where r_t denotes the return on an asset in period t, and x_{t-1} denotes a putative predictor observed at time $t-1$. We assume the process for x_t is given by

$$x_t = \mu + \gamma t + w_t, \qquad t = 1, ..., T \tag{2}$$

$$w_t = \rho w_{t-1} + v_t, \qquad t = 2, ..., T \tag{3}$$

where w_1 is assumed to be an $O_p(1)$ random variable and where u_t and v_t are disturbances, formal assumptions on which are made below.

Remark 2.1. While we permit the potential presence of a linear trend in the predictor, x_t, note that (1) implies that only the detrended component of the predictor enters the DGP for returns, r_t. This assumption is made to rule out the possibility of a linear trend in r_t when $\beta \neq 0$ which is not empirically reasonable. $\diamondsuit$

We make the following assumptions concerning the disturbances u_t and v_t.

Assumption D. *We assume that* $\psi(L)v_t = e_t$ *where* $\psi(L) := \sum_{i=0}^{p-1} \psi_i L^i$ *with* $\psi_0 = 1$ *and* $\psi(1) \neq 0$, *with the roots of* $\psi(L)$ *assumed to be less than one in absolute value. We assume that* $z_t := (u_t, e_t)'$ *is a bivariate martingale difference sequence with respect to the natural filtration* $\mathcal{F}_t := \sigma\{z_s, s \leq t\}$ *satisfying the following conditions: (i)* $E[z_t z_t'] = \begin{bmatrix} \sigma_u^2 & \sigma_{ue} \\ \sigma_{ue} & \sigma_e^2 \end{bmatrix}$, *(ii)* $\sup_t E[u_t^4] < \infty$, *and (iii)* $\sup_t E[e_t^4] < \infty$. *For future reference, we define* $\omega_v^2 := \lim_{T\to\infty} T^{-1} E(\sum_{t=2}^{T} v_t)^2 = \sigma_e^2/\psi(1)^2$ *to be the long run variance of the error process* $\{v_t\}$, *and* $\delta := \sigma_{ue}/\sigma_u\sigma_e$ *as the correlation between the innovations* $\{u_t\}$ *and* $\{e_t\}$.

Remark 2.2. The conditions in Assumption D coincide with the most general set of assumptions considered by CY (see pages 56-57 of CY). The assumptions placed on z_t permit conditional heteroskedasticity in the innovations, but impose unconditional homoskedasticity. The MDS aspect of Assumption D implies the standard assumption made in this literature that the unpredictable component of returns, u_t, is serially uncorrelated. Assumption D allows the dynamics of the predictor variable to be captured by an $AR(p)$, with the degree of persistence of the predictor (strong or weak) controlled by the parameter ρ in (3), as will be formalised in Assumptions S and W below. $\diamondsuit$

As discussed in Section 1, the focus of this paper is on testing the null hypothesis that $(r_t - \alpha)$ is a MDS and, hence, that r_t is not predictable by x_{t-1}; that is, $H_0 : \beta = \beta_0 = 0$ in (1). We focus on developing tests that offer reliable levels of size and power regardless of whether a linear trend is present in the predictor variable x_t under different assumptions regarding the degree of persistence in the predictor. We therefore allow the predictor process $\{x_t\}$ in (2) to satisfy one of the following two assumptions.

Assumption S. *The predictor* $\{x_t\}$ *is strongly persistent, with the autoregressive parameter* ρ *in (3) given by* $\rho = \rho_T = 1 + cT^{-1}$ *with* c *a finite non-zero constant.*

Assumption W. *The predictor* $\{x_t\}$ *is weakly persistent, with the autoregressive parameter* ρ *in (3) fixed and bounded away from unity,* $|\rho| < 1$.

Remark 2.3. Under Assumption S the predictor x_t is a strongly persistent local-to-unity process with the degree of persistence controlled by the parameter c. For $c = 0$, x_t is a pure unit root process, while for $c < 0$, x_t is a stationary but near-integrated process. Finally, for $c > 0$, x_t is a (locally) explosive process. ♢

In order to facilitate an asymptotic power analysis in the strongly persistent case, we consider the following local-to-zero alternative hypothesis for β:

Assumption B. *When the predictor $\{x_t\}$ is strongly persistent, the local alternative hypothesis is given by $H_b : \beta = \beta_T = b(\sigma_u/\omega_v)T^{-1}$, where b is a finite constant.*

Our analysis will consider predictability tests that are invariant to the presence of a trend in the predictor x_t and also tests that depend on the trend parameter γ. In order to enable analysis of the asymptotic behaviour of the latter tests when the predictor x_t is strongly persistent (i.e. when Assumption S holds), at points below we will make use of an additional assumption for γ:

Assumption T. *The trend coefficient γ in (1) and (2) is given by $\gamma = \gamma_T = \kappa\omega_v T^{-1/2}$, where κ is a finite constant.*

Remark 2.4. Under Assumptions B and T, the scalings by T^{-1} and $T^{-1/2}$ in β_T and γ_T, respectively, provide the appropriate Pitman drifts when x_t is strongly persistent, while the scalings by σ_u/ω_v and ω_v are simply convenience measures to ensure that these nuisance parameters do not appear in the subsequent expressions for the limit distributions. Note that Assumption T is not required for test statistics that are invariant to γ. ♢

3 Predictability Tests under Strong Persistence

In this section we outline the candidate Bonferroni predictability tests that we consider for testing the null of no predictability under Assumption S, i.e. that the predictor series is strongly persistent. In each case, we present extant tests which are valid only when assuming that no trend is present in the predictor series x_t, that is assuming $\gamma = 0$. We also consider modified variants that allow for the more general case where it is possible that $\gamma \neq 0$, such that a trend might be present in x_t.

3.1 Bonferroni Q Tests

The first test we consider is the Bonferroni Q test of CY which makes use of an initial confidence interval for $\rho = 1 + cT^{-1}$, where this confidence interval is obtained by inverting a unit root test. CY consider only the possibility of a constant appearing in the predictor series; that is, they impose that $\gamma = 0$ in (2). For a given value of ρ, CY propose a test for the null hypothesis $\beta = \beta_0$ based on the following (infeasible) test statistic

$$\begin{aligned} Q_\mu(\beta_0, \rho) &:= \frac{\sum_{t=2}^T x_{\mu,t-1}\left[r_t - \beta_0 x_{t-1} - \frac{\sigma_{ue}}{\sigma_e \omega_v}(x_t - \rho x_{t-1})\right] + \frac{T}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2)}{\sqrt{\sigma_u^2(1-\delta^2)\sum_{t=2}^T x_{\mu,t-1}^2}} \\ &= Q_\mu(0, \rho) - \beta_0/(s_\mu\sqrt{1-\delta^2}) \end{aligned}$$

where $s_\mu^2 := \sigma_u^2/\sum_{t=2}^T x_{\mu,t-1}^2$, σ_v^2 denotes the short run variance of v_t, and $x_{\mu,t-1}$, $t = 2, ..., T$, are the residuals from regressing x_{t-1} on a constant. A confidence interval for β can then be derived based on the quantity $Q_\mu(\beta, \rho)$. As we will quantify later, the behaviour of this statistic will depend on the trend coefficient γ when $\gamma \neq 0$. In view of this, we now consider a variant of the Bonferroni Q test which is invariant to γ. To obtain such a variant, we replace $x_{\mu,t-1}$ in the CY statistic with $x_{\tau,t-1}$, where $x_{\tau,t-1}$, $t = 2, ..., T$, denotes the residuals from a regression of x_{t-1} on a constant and linear trend. The modified statistic is then

$$\begin{aligned} Q_\tau(\beta_0, \rho) &:= \frac{\sum_{t=2}^T x_{\tau,t-1}\left[r_t - \beta_0 x_{t-1} - \frac{\sigma_{ue}}{\sigma_e \omega_v}(x_t - \rho x_{t-1})\right] + \frac{T}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2)}{\sqrt{\sigma_u^2(1-\delta^2)\sum_{t=2}^T x_{\tau,t-1}^2}} \\ &= Q_\tau(0, \rho) - \beta_0/(s_\tau\sqrt{1-\delta^2}) \end{aligned}$$

where $s_\tau^2 := \sigma_u^2/\sum_{t=2}^T x_{\tau,t-1}^2$.

Remark 3.1. The statistic $Q_\tau(\beta_0, \rho)$ is exact invariant to the value of γ. Moreover, although our DGP excludes the possibility of a linear trend in r_t, $Q_\tau(\beta_0, \rho)$ would also be invariant to such a trend, should one be present; that is, if equation (1) of the DGP was instead $r_t = \alpha + \lambda t + \beta x_{t-1} + u_t$, then $Q_\tau(\beta_0, \rho)$ would be (exact) invariant to both λ and γ. $\diamondsuit$

Under Assumptions D and S, $Q_\mu(\beta_0, \rho)$ admits a standard normal limiting null distribution provided $\gamma = 0$ and, as we will subsequently show in Section 4, $Q_\tau(\beta_0, \rho)$ also admits a standard normal limiting null distribution under these assumptions, regardless of the value of γ. Therefore, when the predictor x_t is strongly persistent a $(1-\alpha)$ confidence interval for β, $[\underline{\beta}_d^Q(\rho, \alpha), \overline{\beta}_d^Q(\rho, \alpha)]$ with $d = \mu$ denoting the constant-only case and $d = \tau$ the with-trend case, can be constructed as:

$$\underline{\beta}_d^Q(\rho, \alpha) = \{Q_d(0, \rho) + z_{\alpha/2}\}s_d\sqrt{1-\delta^2}, \quad \overline{\beta}_d^Q(\rho, \alpha) = \{Q_d(0, \rho) - z_{\alpha/2}\}s_d\sqrt{1-\delta^2} \tag{4}$$

with $z_{\alpha/2}$ denoting the $\alpha/2$ quantile of the standard normal distribution.

The confidence interval in (4), however, implicitly relies on knowledge of the value of $\rho = 1 + cT^{-1}$, where the parameter c cannot be consistently estimated. In the constant only case, $d = \mu$, CY propose obtaining a valid confidence interval for ρ by inverting the constant-only ADF-GLS test[1] of Elliott *et al.* (1996), denoted $DF\text{-}GLS_{\mu}^{\bar{c}}$ henceforth, applied to the predictor x_t using pre-computed (asymptotic) confidence belts for the $DF\text{-}GLS_{\mu}^{\bar{c}}$ test statistic. Denoting this confidence interval for ρ constructed at the α_1 level as $[\underline{\rho}_{\mu}(\alpha_1), \overline{\rho}_{\mu}(\alpha_1)]$ CY show that the confidence interval $[\underline{\beta}_{\mu}^{Q}(\overline{\rho}_{\mu}(\alpha_1), \alpha_2), \overline{\beta}_{\mu}^{Q}(\underline{\rho}_{\mu}(\alpha_1), \alpha_2)]$ has (asymptotic) coverage of at least $(1 - \alpha)$ where $\alpha = \alpha_1 + \alpha_2$.

In the case where $d = \tau$, such that a linear trend is permitted in the predictor, we use the obvious with-trend parallel of the approach taken in CY. Specifically, we obtain a confidence interval for ρ by inverting the with-trend ADF-GLS test of Elliott *et al.* (1996), henceforth denoted $DF\text{-}GLS_{\tau}^{\bar{c}}$, applied to the predictor x_t using pre-computed (asymptotic) confidence belts for the $DF\text{-}GLS_{\tau}^{\bar{c}}$ test statistic. Denoting this confidence interval for ρ constructed at the α_1 level as $[\underline{\rho}_{\tau}(\alpha_1), \overline{\rho}_{\tau}(\alpha_1)]$, the confidence interval $[\underline{\beta}_{\tau}^{Q}(\overline{\rho}_{\tau}(\alpha_1), \alpha_2), \overline{\beta}_{\tau}^{Q}(\underline{\rho}_{\tau}(\alpha_1), \alpha_2)]$ will have (asymptotic) coverage of at least $(1 - \alpha)$ where, again, $\alpha = \alpha_1 + \alpha_2$.

CY show that the confidence interval $[\underline{\beta}_{\mu}^{Q}(\overline{\rho}_{\mu}(\alpha_1), \alpha_2), \overline{\beta}_{\mu}^{Q}(\underline{\rho}_{\mu}(\alpha_1), \alpha_2)]$ suffers from over-coverage, with the asymptotic size of tests based on this confidence interval often well below $(\alpha/2)$, and we found the same for the confidence interval $[\underline{\beta}_{\tau}^{Q}(\overline{\rho}_{\tau}(\alpha_1), \alpha_2), \overline{\beta}_{\tau}^{Q}(\underline{\rho}_{\tau}(\alpha_1), \alpha_2)]$. Therefore, we follow CY and use a refinement where the significance level used to obtain the initial confidence interval for ρ is adapted to the upper and lower bounds separately, and also to the value of δ. Values of this significance level are chosen numerically to minimise over-coverage associated with the confidence interval for β, while ensuring that the asymptotic size of the overall Bonferroni test does not exceed a chosen level across a specified range of c. Denoting the chosen significance levels for the lower and upper confidence bounds for ρ by $\underline{\alpha}_{1,d}^{Q}$ and $\overline{\alpha}_{1,d}^{Q}$, respectively, the confidence interval for ρ can be written as $[\underline{\rho}_{d}(\underline{\alpha}_{1,d}^{Q}), \overline{\rho}_{d}(\overline{\alpha}_{1,d}^{Q})]$, and the resulting $(1 - \alpha_2)$ level confidence interval for β is obtained as $[\underline{\beta}_{d}^{Q}(\overline{\rho}_{d}(\overline{\alpha}_{1,d}^{Q}), \alpha_2), \overline{\beta}_{d}^{Q}(\underline{\rho}_{d}(\underline{\alpha}_{1,d}^{Q}), \alpha_2)]$ where

$$\begin{aligned}
\underline{\beta}_{d}^{Q}(\overline{\rho}_{d}(\overline{\alpha}_{1,d}^{Q}), \alpha_2) &= \{Q_d(0, \overline{\rho}_{d}(\overline{\alpha}_{1,d}^{Q})) + z_{\alpha_2/2}\} s_d \sqrt{1 - \delta^2}, \\
\overline{\beta}_{d}^{Q}(\underline{\rho}_{d}(\underline{\alpha}_{1,d}^{Q}), \alpha_2) &= \{Q_d(0, \underline{\rho}_{d}(\underline{\alpha}_{1,d}^{Q})) - z_{\alpha_2/2}\} s_d \sqrt{1 - \delta^2}.
\end{aligned}$$

For a given value of δ, CY propose selecting $\underline{\alpha}_{1,d}^{Q}$ and $\overline{\alpha}_{1,d}^{Q}$ such that one-sided tests for predictability constructed in this manner have an asymptotic size of exactly $\alpha_2/2$ for some

[1] In the context of both $DF\text{-}GLS_{\mu}^{\bar{c}}$ and the $DF\text{-}GLS_{\tau}^{\bar{c}}$ statistic defined below, $\bar{c}$ denotes the parameter used for quasi-difference demeaning/detrending the data. We follow Elliott *et al.* (1996) and set $\bar{c} = -7$ for $DF\text{-}GLS_{\mu}^{\bar{c}}$ and $\bar{c} = -13.5$ for $DF\text{-}GLS_{\tau}^{\bar{c}}$ in what follows. Owing to Assumption D, the $DF\text{-}GLS_{\mu}^{\bar{c}}$ and $DF\text{-}GLS_{\tau}^{\bar{c}}$ unit root statistics will be calculated from ADF-type regressions which include $p - 1$ lags.

value of c while remaining slightly undersized for other values of c. Consequently, two-sided tests will have size of at most α_2 across the specified range of c.

CY calibrate their constant-only test procedure by fixing $\alpha_2 = 0.1$ and considering $c \in [-50, 5]$ such that their resulting one-sided tests have a maximum (asymptotic) size of 0.05. The appropriate values of $\underline{\alpha}^Q_{1,\mu}$ and $\overline{\alpha}^Q_{1,\mu}$ are reported in Table 2 of CY, and are reproduced here in Table 1 for convenience. We denote the predictability test based on this confidence interval as Q^{GLS}_μ. We follow the approach taken by CY for the trend-augmented version of the Bonferroni Q test, with the appropriate values of $\underline{\alpha}^Q_{1,\tau}$ and $\overline{\alpha}^Q_{1,\tau}$ chosen such that one-sided tests for predictability also have a maximum asymptotic size of 0.05 for $c \in [-50, 5]$, with the asymptotic size of the test computed using the limiting distributions we subsequently outline in Section 4, and with these values of $\underline{\alpha}^Q_{1,\tau}$ and $\overline{\alpha}^Q_{1,\tau}$ also reported in Table 1. We denote the predictability test based on this confidence interval as Q^{GLS}_τ.

Remark 3.2. The appropriate values of $\underline{\alpha}^Q_{1,d}$ and $\overline{\alpha}^Q_{1,d}$ reported in Table 1 are only provided for $\delta < 0$. For $\delta > 0$, CY note that replacing x_t in (1) with $-x_t$ flips the sign of both β and δ (and, indeed of γ). Therefore, an equivalent right (left) tailed test for predictability when $\delta > 0$ can be performed as a left (right) tailed test for predictability based on (1) with x_t replaced by $-x_t$ using the values of $\underline{\alpha}^Q_{1,d}$ and $\bar{\alpha}^Q_{1,d}$ appropriate for a negative value of δ. This also holds for the Bonferroni t test discussed below. ◇

3.2 Bonferroni t Tests

The second test procedure we consider is the Bonferroni t test based approach of CES. Where $\gamma = 0$ in (2), this is based on the following (infeasible) OLS statistic for testing the null $\beta = \beta_0 = 0$, $t_\mu := \hat{\beta}_\mu / \sqrt{\sigma^2_u / \sum_{t=2}^T x^2_{\mu,t-1}}$, where $\hat{\beta}_\mu$ is obtained from the OLS estimated regression, $r_t = \hat{\alpha} + \hat{\beta}_\mu x_{t-1} + \hat{u}_t$. As with $Q_\mu(\beta, \rho)$, the behaviour of t_μ will be dependent on the trend coefficient γ, when $\gamma \neq 0$. Accordingly, CES suggest a with-trend variant of the OLS t statistic which is invariant to γ, specifically

$$t_\tau := \frac{\hat{\beta}_\tau}{\sqrt{\sigma^2_u / \sum_{t=2}^T x^2_{\tau,t-1}}} \tag{5}$$

where $\hat{\beta}_\tau$ is obtained from the with-trend estimated OLS regression, $r_t = \hat{\alpha} + \hat{\gamma} t + \hat{\beta}_\tau x_{t-1} + \hat{u}_t$.

Under Assumptions D and S the limiting null distribution of t_d for $d = \mu$ or $d = \tau$ is a function of the unknown parameter c. CES overcome this issue by constructing a confidence interval for β based on an initial confidence interval for c obtained by inverting the constant-only or with-trend ADF-OLS test (henceforth denoted $DF\text{-}OLS_\mu$ or $DF\text{-}OLS_\tau$ respectively) using pre-computed confidence belts.

Specifically, for a given value of δ, a $(1-\alpha_2)$ level confidence interval for β is obtained as $[\underline{\beta}^t_d(\overline{\alpha}^t_{1,d}, \alpha_2), \overline{\beta}^t_d(\underline{\alpha}^t_{1,d}, \alpha_2)]$, $d = \{\mu, \tau\}$, where:

$$\underline{\beta}^t_d(\overline{\alpha}^{t,d}_1, \alpha_2) = \hat{\beta}_d - \left\{ \max_{\underline{c}(\overline{\alpha}^t_{1,d}) \leq c \leq \overline{c}(\overline{\alpha}^t_{1,d})} cv^c_{1-\alpha_2/2,d} \right\} s_d, \quad \overline{\beta}^t_d(\underline{\alpha}^t_{1,d}, \alpha_2) = \hat{\beta}_d - \left\{ \min_{\underline{c}(\underline{\alpha}^t_{1,d}) \leq c \leq \overline{c}(\underline{\alpha}^t_{1,d})} cv^c_{\alpha_2/2,d} \right\} s_d$$

and where $cv^c_{\eta,d}$ denotes the η-level critical value of the limiting null distribution of t_d for a given value of c. The significance levels used to obtain the c confidence intervals, $\underline{\alpha}^t_{1,d}$ and $\overline{\alpha}^t_{1,d}$, are selected numerically to ensure that the implied one-sided tests for predictability constructed in this manner will have an asymptotic size of exactly $\alpha_2/2$ for some value of $c \in [-50, 5]$ while remaining slightly undersized for other values of c. For $\alpha_2 = 0.1$, the appropriate values of $\underline{\alpha}^t_{1,\mu}$ and $\overline{\alpha}^t_{1,\mu}$ are those of CY, and are reported in Table 1. We denote the predictability test based on this confidence interval as t^{OLS}_μ. The appropriate values of $\underline{\alpha}^t_{1,\tau}$ and $\overline{\alpha}^t_{1,\tau}$ in the with-trend case are also reported in Table 1 and were found by directly simulating the limit distributions that we subsequently detail in Section 4. We denote the predictability test based on this confidence interval as t^{OLS}_τ.

For full details on the practical implementation of the Q^{GLS}_μ and t^{OLS}_μ procedures, including consistent estimation of the parameters σ_e, σ_u, σ_v, σ_{ue}, ω_v and δ, implementation of the $DF\text{-}GLS^{\overline{c}}_\mu$ and $DF\text{-}OLS_\mu$ unit root tests, and the pre-computed confidence belts for the $DF\text{-}GLS^{\overline{c}}_\mu$ and $DF\text{-}OLS_\mu$ test statistics, see CY, CES and the corresponding supplementary material to CY.[2] We follow exactly the same steps as CY for the Q^{GLS}_τ and t^{OLS}_τ tests, but where any regression including an intercept is also augmented with a linear trend. Pre-computed confidence belts for the $DF\text{-}GLS^{\overline{c}}_\tau$ and $DF\text{-}OLS_\tau$ test statistics are included as part of the code used to implement all of the tests outlined in this paper which is available on request.

4 Asymptotic Behaviour of Tests under Strong Persistence and a Local Trend

In this section we outline the asymptotic behaviour of the constant-only Q^{GLS}_μ and t^{OLS}_μ tests, and the with-trend Q^{GLS}_τ and t^{OLS}_τ tests, when Assumption S holds, i.e. the case where the predictor is a strongly persistent process and contains a trend. While the Q^{GLS}_τ and t^{OLS}_τ tests are invariant to the trend coefficient γ, the Q^{GLS}_μ and t^{OLS}_μ tests are not; here, we consider the behaviour of Q^{GLS}_μ and t^{OLS}_μ under Assumption T, i.e. a local-to-zero

[2]The supplement to CY is available at https://scholar.harvard.edu/campbell/publications/implementing-econometric-methods-efficient-tests-stock-return-predictability-0. The confidence belts and also code for the procedures are available from Motohiro Yogo's personal website: https://sites.google.com/site/motohiroyogo/research/asset-pricing.

trend. We begin by outlining the limiting distributions of the relevant test statistics under the local alternative $H_b : \beta = b(\sigma_u/\omega_v)T^{-1}$, before proceeding to investigate the asymptotic size and power of the corresponding procedures both when a linear trend is present and when it is not. The following Theorem outlines the limiting distribution of the statistics, where in the context of Q_μ^{GLS} and Q_τ^{GLS}, $\tilde{\rho} = 1 + \tilde{c}T^{-1}$ for an arbitrary $\tilde{c}$.

Theorem 1. *Let data be generated according to (1)-(3). Let $W_1(s)$ and $W_2(s)$ be independent standard Brownian motion processes and let $W_{1c}(r) = \int_0^r e^{(r-s)c} dW_1(s)$. Under Assumptions D, S and T, and under the local alternative specified in Assumption B,*

$$(a) \quad t_\mu \xrightarrow{w} \frac{b\left\{\kappa \int_0^1 rW_{1c}^\mu(r)dr + \int_0^1 W_{1c}^\mu(r)^2 dr\right\} + \delta \int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)}{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}} + \sqrt{1-\delta^2}Z_\mu$$

$$(b) \quad Q_\mu(\beta_0, \tilde{\rho}) \xrightarrow{w} \frac{b\left[\kappa \int_0^1 rW_{1c}^\mu(r)dr + \int_0^1 W_{1c}^\mu(r)^2 dr\right] + \delta c\kappa \int_0^1 rW_{1c}^{\mu,\kappa}(r)dr + \delta(\tilde{c}-c)\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}{\sqrt{1-\delta^2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}} + Z_\mu$$

$$(c) \quad t_\tau \xrightarrow{w} b\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr} + \delta \frac{\int_0^1 W_{1c}^\tau(r)dW_1(r)}{\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}} + \sqrt{1-\delta^2}Z_\tau$$

$$(d) \quad Q_\tau(\beta_0, \tilde{\rho}) \xrightarrow{w} \frac{[b + \delta(\tilde{c}-c)]\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}}{\sqrt{1-\delta^2}} + Z_\tau.$$

where $\xrightarrow{w}$ denotes weak convergence of the associated probability measures, and where $W_{1c}^\mu(r) := W_{1c}(r) - \int_0^1 W_{1c}(s)ds$, $W_{1c}^\tau(r) := W_{1c}^\mu(r) - 12(r-0.5)\int_0^1 (s-0.5)W_{1c}(s)ds$, $W_{1c}^{\mu,\kappa}(r) := \{\kappa(r-0.5)\} + W_{1c}^\mu(r)$, $Z_\mu := \left(\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr\right)^{-1/2} \int_0^1 W_{1c}^{\mu,\kappa}(r)dW_2(r)$ and $Z_\tau := \left(\int_0^1 W_{1c}^\tau(r)^2 dr\right)^{-1/2} \int_0^1 W_{1c}^\tau(r)dW_2(r)$. Finally, Z_μ and Z_τ are two dependent standard normal random variables.

Remark 4.1. Representations for the limiting null distributions of the statistics obtain on setting $b = 0$ in the expressions in Theorem 1. If $b = 0$ and $\tilde{c} = c$, then $Q_\tau(\beta_0, \tilde{\rho})$ is asymptotically distributed as a $N(0,1)$ random variable, and if in addition $\kappa = 0$ then $Q_\mu(\beta_0, \tilde{\rho})$ is also asymptotically distributed as a $N(0,1)$ random variable. If $b = 0$ and $\delta = 0$ then t_τ is asymptotically distributed as a $N(0,1)$ random variable, and if in addition $\kappa = 0$ then t_μ is also asymptotically distributed as a $N(0,1)$ random variable. ◇

Remark 4.2. The representations in (a) and (b) of Theorem 1 show that the limiting null and local alternative distributions of both t_μ and $Q_\mu(\beta_0, \tilde{\rho})$ depend on the value of κ. It is immediately apparent from the representations in (c) and (d) of Theorem 1 that the limiting distributions of t_τ and $Q_\tau(\beta_0, \tilde{\rho})$ are invariant to the value of κ. Hence t_μ and

$Q_\mu(\beta_0, \tilde{\rho})$ are dependent on the magnitude of the trend in the predictor variable, while t_τ and $Q_\tau(\beta_0, \tilde{\rho})$ are not. ♢

4.1 Local Asymptotic Power of t_d^{OLS} and Q_d^{GLS} tests

We now report results of a Monte Carlo simulation experiment examining the asymptotic power of the t_d^{OLS} and Q_d^{GLS} tests under the local alternative given in Assumption B, when Assumptions D, S and T hold. We will focus on testing for predictability when $\delta < 0$ as the size and power of right (left) tailed tests for predictability when $\delta > 0$ are identical to left (right) tailed tests for predictability when $\delta < 0$, for the reasons outlined in Remark 3.2.

Before proceeding we require the limiting distributions of the $DF\text{-}OLS_\mu$, $DF\text{-}GLS_\mu^{\overline{c}}$, $DF\text{-}OLS_\tau$ and $DF\text{-}GLS_\tau^{\overline{c}}$ test statistics used to construct the initial confidence interval for ρ for the t_μ^{OLS}, Q_μ^{GLS}, t_τ^{OLS} and Q_τ^{GLS} tests, respectively. Under the conditions of Theorem 1, these limiting distributions are given by (see, for example, Harvey *et al.*, 2009):

$$DF\text{-}OLS_\mu \xrightarrow{w} \frac{\left(\kappa/2 + W_{1c}^\mu(1)\right)^2 - \left(-\kappa/2 + W_{1c}^\mu(0)\right)^2 - 1}{2\sqrt{\int_0^1 \left\{\kappa(r-1/2) + W_{1c}^\mu(r)\right\}^2 dr}} \tag{6}$$

$$DF\text{-}OLS_\tau \xrightarrow{w} \frac{W_{1c}^\tau(1)^2 - W_{1c}^\tau(0)^2 - 1}{2\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}} \tag{7}$$

$$DF\text{-}GLS_\mu^{\overline{c}} \xrightarrow{w} \frac{\left(\kappa + W_{1c}(1)\right)^2 - 1}{2\sqrt{\int_0^1 \left\{\kappa r + W_{1c}(r)\right\}^2 dr}}, \quad DF\text{-}GLS_\tau^{\overline{c}} \xrightarrow{w} \frac{W_{1c}^{\tau,\overline{c}}(1)^2 - 1}{2\sqrt{\int_0^1 W_{1c}^{\tau,\overline{c}}(r)^2 dr}} \tag{8}$$

where $W_{1c}^{\tau,\overline{c}}(r) := W_{1c}(r) - r\left\{\overline{c}^* W_{1c}(1) + 3(1-\overline{c}^*)\int_0^1 rW_{1c}(r)dr\right\}$ and $\overline{c}^* := (1-\overline{c})/(1-\overline{c}+\overline{c}^2/3)$.

Remark 4.3. The representations in (6) and (8) show that the limiting distributions of $DF\text{-}OLS_\mu$ and $DF\text{-}GLS_\mu^{\overline{c}}$ depend on κ, whereas (7) and (8) show that the limiting distributions of $DF\text{-}OLS_\tau$ and $DF\text{-}GLS_\tau^{\overline{c}}$ are invariant to the value of κ. Harvey *et al.* (2009) show that the impact of a neglected local trend in $DF\text{-}OLS_\mu$ and $DF\text{-}GLS_\mu^{\overline{c}}$ is to reduce both size and power of the unit root tests, implying a rightward shift in the tail of the distribution, resulting in a corresponding rightward shift in the confidence intervals for c. ♢

For clarity, we now outline how the local asymptotic power of the tests is computed for right tailed testing. Left tailed testing is handled similarly with obvious modifications.

For the Q_d^{GLS}, $d = \{\mu, \tau\}$, tests we first simulate draws from the limiting distribution of $DF\text{-}GLS_d^{\overline{c}}$. These draws are then used to compute the upper bound of the confidence interval for c which we denote $\overline{c}(\overline{\alpha}_{1,d}^Q)$ using pre-computed confidence belts implemented using the values of $\overline{\alpha}^Q_{1,d}$ appropriate for δ taken from Table 1.[3] Note that this value of c also corresponds to the upper bound of the confidence interval for ρ, i.e. $\overline{\rho}^d(\bar{\alpha}^Q_{1,d}) = 1 + \overline{c}(\bar{\alpha}^Q_{1,d})T^{-1}$. Testing in the right tail is equivalent to determining whether $\underline{\beta}^Q_d(\overline{\rho}_d(\overline{\alpha}^Q_{1,d}), \alpha_2) > 0$, and the asymptotic local power function associated with $Q_d(0, \overline{\rho}_d(\bar{\alpha}^Q_{1,d}))$ is given by $E[\Phi(h_d(\overline{\alpha}^Q_{1,d}, \alpha_2))]$ where $\Phi(.)$ denotes one minus the standard normal cdf and

$$h_d(\overline{\alpha}^Q_{1,d}, \alpha_2) := z_{1-\alpha_2/2} - (Q^\infty_d - Z_d) \tag{9}$$

where Q^∞_d denotes the limiting distribution of $Q_d(0, \overline{\rho}_d(\bar{\alpha}^Q_{1,d}))$, and Z_d is as defined in Theorem 1. Next we simulate a draw from Q^∞_d and construct $h_d(\overline{\alpha}^Q_{1,d}, \alpha_2)$ in (9). Finally, we evaluate whether a simulated draw from a Z_d exceeds this value of $h(\overline{\alpha}^Q_{1,d}, \alpha_2)$. The limiting power is then obtained as the average of these exceedances across replications.

For t^{OLS}_d, in each simulation replication we first simulate a draw from the limiting distribution of $DF\text{-}OLS_d$, and then obtain $[\underline{c}(\overline{\alpha}^t_{1,d}), \overline{c}(\overline{\alpha}^t_{1,d})]$ using the corresponding pre-computed confidence belts for the values of $\overline{\alpha}^t_{1,d}$ appropriate for δ obtained from Table 1. Then we simulate the limit of t_d using the results in Theorem 1, and compare this with the critical value $\max_{\underline{c}(\overline{\alpha}^t_{1,d}) \leq c \leq \overline{c}(\overline{\alpha}^t_{1,d})} cv^c_{1-\alpha_2/2,d}$. The limiting power is again calculated as the average of these exceedances across replications.

Figures 1-8 report the local asymptotic power of right-tailed test for predictability for $\delta = -0.95$ for $c = \{0, -2, -5, -10, -20, -30, -40, -50\}$ and for various values of κ.[4] Note that these figures also include results for the hybrid U^{hyb} and S^{hyb} test procedures we propose in Section 5 - these will be discussed later.

When $\kappa = 0$, so that no trend is present in the predictor (panel (a) of each figure), it is apparent that for small or moderate (negative) values of c the best power performance is offered by the Q^{GLS}_μ test, followed by the t^{OLS}_μ test. Also for this range of c, we observe that the Q^{GLS}_τ test has superior power to the t^{OLS}_τ test, although both have power that falls below the constant-only tests. These results when no trend is present are entirely expected, since the Q^{GLS}_τ and t^{OLS}_τ tests are based on regressions that unnecessarily include a trend.

As c becomes more negative, the Bonferroni t-tests start to display superior power to the Bonferroni Q tests, with t^{OLS}_μ displaying consistently superior power to Q^{GLS}_μ for $c \leq -30$.

[3]Here and throughout the paper results were obtained by direct simulation of the limiting distributions, with the Wiener processes approximated using NIID(0,1) random variates, and with the integrals approximated by normalized sums of 1,000 steps. All simulations were performed in Gauss 22.2 using 20,000 Monte Carlo replications. The confidence belts form part of the Gauss code used throughout the paper and are available on request.

[4]Additional results, available in the on-line supplementary appendix for $\delta = -0.75$ were found to be qualitatively similar to those discussed here for $\delta = -0.95$ for both right-tailed and left-tailed tests. This can be found at https://rtaylor-essex.droppages.com/esrc2/default.htm.

However, in this more negative c setting, the power differences between the competing tests are reduced compared to the small c cases, hence there is relatively little to choose between the constant-only procedures here. Overall, one would arguably wish to use the Q_μ^{GLS} test in the case of $\kappa = 0$ if allowing only for a constant in the predictor. For the trend-augmented tests there is little to choose between t_τ^{OLS} and Q_τ^{GLS} for $c = -30$, with t_τ^{OLS} offering superior power to Q_τ^{GLS} for lower values of b and vice-versa. For $c \leq -40$, however, t_τ^{OLS} clearly offers superior power to Q_τ^{GLS} across almost the full range of values of b. This relative power performance of the trend augmented tests is true for all values of κ given that the trend-augmented tests are exact invariant to the value of κ.

We now consider panels (b)-(f) of each figure, where κ is positive and increasing in magnitude. Here a different pattern emerges as the value of κ increases away from zero.[5] The asymptotic sizes of Q_μ^{GLS} and t_μ^{OLS} are now decreasing in κ, and as a consequence the powers of these tests are also decreasing in κ, with this effect more pronounced the more negative is the value of c. The power of the Q_τ^{GLS} and t_τ^{OLS} tests are, as previously discussed, invariant to the value of κ, with the consequence that for larger values of κ these tests outperform their constant-only counterparts, with the Q_τ^{GLS} test becoming the best performing procedure for small or moderate c, and the t_τ^{OLS} test displaying the best power for the larger c. Hence for larger κ, one would wish to use the Q_τ^{GLS} test when the c values are small or moderate, and the t_τ^{OLS} test otherwise.

Figures 9-13 report the local asymptotic power of left-tailed tests for predictability for $\delta = -0.95$ and $c = \{0, -2, -5, -10, -20\}$ for various values of κ. When $\kappa = 0$ the constant-only Q_μ^{GLS} and t_μ^{OLS} tests again outperform their with-trend Q_τ^{GLS} and t_τ^{OLS} counterparts for a given $d \in \{\mu, \kappa\}$, as expected. In the left-tailed testing environment it can also be seen that the range of values of c over which the Q_d^{GLS} tests display superior power to the t_d^{OLS} tests is smaller for a given $d \in \{\mu, \kappa\}$. There is little to choose between the t_μ^{OLS} and Q_μ^{GLS} tests for $c = -5$, but for $c \leq -10$ the t_μ^{OLS} test has superior power to Q_μ^{GLS}. Likewise, there is little to choose between the t_τ^{OLS} and Q_τ^{GLS} tests for $c = -10$, but for $c = -20$ the t_τ^{OLS} test has superior power to Q_τ^{GLS}. Additional results reported in the supplementary appendix show that the t_d^{OLS} tests continue to display superior power over the Q_d^{GLS} tests for $c < -20$. For $\kappa > 0$, however, the Q_μ^{GLS} and t_μ^{OLS} tests can suffer from substantial oversize, with the degree of this oversize increasing in κ and also as c becomes more negative. As such, the Q_μ^{GLS} and t_μ^{OLS} tests are inappropriate for testing for predictability in the left tail when $\delta < 0$ when uncertainty exists over the possible presence of a linear trend, and

[5] We also generated results for $\kappa < 0$ and found them to be broadly similar, although not perfectly symmetric, to those found for positive values of κ

reliable inference can only be made using Q_τ^{GLS} or t_τ^{OLS}. Here we would ideally use Q_τ^{GLS} for small or moderate c, and t_τ^{OLS} otherwise.

The results in Figures 1-13 show that, as might be expected, no single test is best suited to testing for predictability when uncertainty exists over both the values of c and κ. Instead, each of Q_μ^{GLS}, Q_τ^{GLS} and t_τ^{OLS} provides the best overall power for certain combinations of these parameters. Given that neither c nor κ can be consistently estimated, in the following section we propose hybrid tests for predictability that use combinations of the Q_μ^{GLS}, Q_τ^{GLS} and t_τ^{OLS} tests to deliver both controlled size and good power across the parameter space.

5 Hybrid Tests for Predictability

Based on the results in Section 4.1 we now propose tests for predictability when uncertainty exists over the possible presence of a linear trend in the predictor. We start with tests that are designed for strongly persistent predictors generated according to Assumption S, motivated by the results of the previous section, before outlining how these can be modified to also allow for weakly stationary predictors generated according to Assumption W.

We will outline our hybrid tests in what follows only for the case where $\delta < 0$. For $\delta > 0$, from the result in Remark 3.2, we may simply replace the predictor x_t in (1) with $-x_t$, thereby flipping the sign of δ such that our recommended procedures for negative values of δ which follow can then be applied. Given that this also flips the sign of β, for a right (left) tailed test for predictability one should perform a left (right) tailed test for predictability in the transformed predictive regression based on the predictor $-x_{t-1}$. So, for example, the right-tailed tests appropriate for $\delta < 0$ outlined in Section 5.1 are also recommended, on replacing x_{t-1} by $-x_{t-1}$ throughout, for use in the case where one wishes to perform left-tailed tests with $\delta > 0$. In practice, the true value of δ will be unknown, but the appropriate approach can be determined according to the sign of a consistent estimator of δ. Here we propose using the estimate of δ from the with-trend Bonferroni type test procedures, i.e. the sample correlation between $\hat{u}_t$ and $\hat{e}_t$, where $\hat{u}_t$ are the residuals from a regression of r_t on a constant, trend and x_{t-1} and $\hat{e}_t$ are the residuals from estimating an $AR(p)$ for the predictor variable allowing for a constant and trend.

5.1 Right-Tailed Tests when $\delta < 0$

The results in Section 4.1 suggest that for strongly persistent predictors, with $\delta < 0$, when $\kappa = 0$ the constant-only Q_μ^{GLS} test outperforms its with-trend counterpart Q_τ^{GLS}, while for larger $\kappa > 0$ the converse is true. As such, when testing in the right-tail the first test procedure we propose is a Union-of-Rejections strategy in which we reject the null of $\beta = 0$ in favour of the alternative that $\beta > 0$ when either the Q_μ^{GLS} or Q_τ^{GLS} tests reject, with the aim of capturing the relative power advantages of Q_μ^{GLS} and Q_τ^{GLS} for different magnitudes of κ. Taking a simple union-of-rejections in this manner, however, will inevitably result in an overall test with asymptotic size in excess of $(\alpha_2/2)$, given that inference from two tests is being combined, each individually calibrated to have a maximum asymptotic size of $(\alpha_2/2)$. To ensure that the union-of-rejections strategy has a maximum asymptotic size of $(\alpha_2/2)$ we modify the significance levels at which the confidence intervals for ρ are constructed for both the $DF\text{-}GLS_\mu$ and $DF\text{-}GLS_\tau$ tests, as well as the significance level used to construct the confidence interval for β for a given value of ρ. Recalling that the lower bound of the confidence interval for ρ used for right-tailed testing for the Q_d^{GLS} test is given by $\underline{\beta}_d^Q(\overline{\rho}_d(\overline{\alpha}_{1,d}^Q), \alpha_2)$ our proposed union-of-rejections test, U, is given by

$$U: \text{ Reject } H_0 \text{ if } \underline{U} > 0 \tag{10}$$

where

$$\underline{U} := \max\left(\underline{\beta}_\mu^Q(\overline{\rho}_\mu(\xi\overline{\alpha}_{1,\mu}^Q), \xi\alpha_2), \underline{\beta}_\tau^Q(\overline{\rho}_\tau(\xi\overline{\alpha}_{1,\tau}^Q), \xi\alpha_2)\right) \tag{11}$$

with $\xi < 1$ a scaling parameter chosen such that, for a given value of δ, the asymptotic size of U is no greater than $(\alpha_2/2)$ for the same grid of values of c considered by CY, i.e. $c \in [-50, 5]$. The appropriate values of ξ that lead to a right-tailed test with maximum asymptotic size of 0.05 are reported in Table 1.

While the union-of-rejections strategy outlined above will be shown to capture the superior power of Q_μ^{GLS} when κ is small, and that of Q_τ^{GLS} for larger values of κ when c is small or moderate, it is apparent from the results reported in Figures 1-8 that for the more negative values of c the power of both the Q_μ^{GLS} and Q_τ^{GLS} tests lag behind that of t_τ^{OLS}. As such, we consider an extra layer to our test procedure where for right-tailed tests the union-of-rejections test is employed when c is estimated to be "small", and the t_τ^{OLS} test is employed when c is estimated to be "large". To do so we propose using an estimate of c to choose which test to perform in practice. Specifically, we propose computing an estimate, $\hat{c}$, that is equal to the with-trend ADF-GLS normalised bias unit root test statistic, henceforth denoted $NB\text{-}GLS_\tau^{\overline{c}}$. Specifically, $\hat{c} = NB\text{-}GLS_\tau^{\overline{c}} := (T\hat{\phi})/(1 - \sum_{i=1}^{p-1}\hat{\psi}_i)$, where $\hat{\phi}$ and $\hat{\psi}_i$, $i = 1, ..., p-1$ are obtained by OLS estimation of

$$\Delta\tilde{x}_t = \phi\tilde{x}_{t-1} + \sum\nolimits_{i=1}^{p-1}\psi_i\Delta\tilde{x}_{t-i} + e_t$$

where, on setting $\bar{\rho}_T := 1 + \overline{c}T^{-1}$, $\tilde{x}_t := x_t - z_t'\tilde{\theta}$ with $\tilde{\theta}$ obtained from the quasi-differenced regression of $x_{\bar{c}} := (x_1, x_2 - \bar{\rho}_T x_1, ..., x_T - \bar{\rho}_T x_{T-1})'$ on $Z_{\bar{c}} := (z_1, z_2 - \bar{\rho}_T z_1, ..., z_T - \bar{\rho}_T z_{T-1})'$, where $z_t := (1, t)'$. The $NB\text{-}GLS_\tau^{\overline{c}}$ statistic is closely related to $DF\text{-}GLS_\tau^{\overline{c}}$, being obtained from the same regression, and in keeping with this link between the statistics, we use

$\overline{c} = -13.5$; cf. footnote 1. Under Assumption S, the limiting distribution of $\hat{c}$ is given by

$$\hat{c} \xrightarrow{w} \frac{W_{1c}^{\tau,\overline{c}}(1)^2 - 1}{2\int_0^1 W_{1c}^{\tau,\overline{c}}(r)^2 dr} \tag{12}$$

where $W_{1c}^{\tau,\overline{c}}$ is as previously defined under equation (8). While it is clear that $\hat{c}$ is not a consistent estimate of c, a near monotonic relationship nonetheless exists between the expected value of the limiting distribution of $\hat{c}$ and the true value of c. We therefore propose a cut-off rule where we employ the U test for $\hat{c} \geq c_R$, but switch to the t_τ^{OLS} test for $\hat{c} < c_R$ for some cut-off point c_R (R denoting right-tailed). Formally, our second proposed testing procedure, S, is therefore given by:

$$S: \text{ Reject } H_0 \text{ if } \underline{US} > 0 \tag{13}$$

where

$$\underline{US} := \mathbb{I}(\hat{c} \geq c_R)\underline{U} + \mathbb{I}(\hat{c} < c_R)\underline{\beta}_\tau^t(\overline{\alpha}_{1,\tau}^t, \alpha_2). \tag{14}$$

and where $\mathbb{I}(.)$ denotes the indicator function equal to 1(0) when its argument is true (false). Our choice of the cut-off value c_R to use in practice is motivated by the asymptotic local power functions in Figures 1-8, where we recall from the discussion in Section 4.1 that the local asymptotic power of the U test is superior to that of t_τ^{OLS} for $c \geq -30$, whereas for $c < -30$ the reverse is true. We found through extensive Monte Carlo simulation that the choice of $c_R = -35$ gave an overall test for predictability with the best overall power properties, tracking the power of U for small c and that of t_τ^{OLS} for large c. We also found that using the existing calibration for U and t_τ^{OLS} led to S maintaining a maximum asymptotic size of 0.05 for $c \in [-50, 5]$, so no further calibration was required for this particular test.

5.2 Left-Tailed Tests when $\delta < 0$

We now turn our attention to left tailed tests when $\delta < 0$ and Assumption S holds. We propose a simpler strategy for left-tailed tests as the asymptotic oversize of Q_μ^{GLS} and t_μ^{OLS} when $\kappa \neq 0$ prevents the implementation of an asymptotically size-controlled union-of-rejections procedure, such as that proposed in Section 5.1, as this relies on the constituent tests being (asymptotically) correctly sized or undersized both when $\kappa = 0$ and when $\kappa \neq 0$. The appropriate simplification for the U procedure is then to just use Q_τ^{GLS}, which recalling Section 3, entails rejecting the null of no predictability if $\overline{\beta}_d^Q(\underline{\rho}_d(\underline{\alpha}_{1,d}^Q), \alpha_2) < 0$.

Examining the relative power of Q_τ^{GLS} and t_τ^{OLS} in Figures 9-13 it is immediately apparent that the Q_τ^{GLS} test only offers superior power to t_τ^{OLS} when c is small, with the power of t_τ^{OLS} above that of Q_τ^{GLS} for even modest values of c. As such, for the switching strategy S we propose a simpler version to that in Section 5.1 where the Q_τ^{GLS} test is employed when $\hat{c} \geq c_L$ (L denoting left-tailed) and the t_τ^{OLS} test is used when $\hat{c} < c_L$. Specifically, for left tailed tests the decision rule for our test procedure S is given by.

$$S: \text{ Reject } H_0 \text{ if } \overline{S} < 0 \tag{15}$$

where

$$\overline{S} := \mathbb{I}(\hat{c} \geq c_L)\overline{\beta}_\tau^Q(\underline{\rho}_\tau(\underline{\alpha}_{1,\tau}^Q), \alpha_2) + \mathbb{I}(\hat{c} < c_L)\overline{\beta}_\tau^t(\underline{\alpha}_{1,\tau}^t, \alpha_2). \tag{16}$$

Our choice of the cut-off value c_L to use in practice is, again, motivated by the local asymptotic power functions presented in Figures 9-13 which, as discussed in Section 4.1, show that the local asymptotic power of the Q_τ^{GLS} test is superior to that of t_τ^{OLS} for $c > -10$ but inferior for $c < -10$, with little to choose between the two tests for $c = -10$. We again used Monte Carlo simulation to determine an appropriate value for c_L and found a value of $c_L = -15$ led to a test with the best overall power properties. As was the case for right-tailed testing, we found that the maximum asymptotic size of S was still maximised at 0.05 for $c \in [-50, 5]$ when testing in the left tail, so no further calibration was required.

5.3 Dealing with Weakly Persistent Predictors

The U and S tests outlined in Sections 5.1 and 5.2 are constructed under the assumption that the predictor is strongly persistent. When Assumption W holds, such that the predictor is weakly persistent, the Q_d^{GLS} and t_d^{OLS} tests, and hence the U and S tests, are asymptotically invalid. In contrast, under Assumption W a "conventional" OLS t-test, which compares the OLS t-statistic t_τ of (5) with standard normal critical values, is asymptotically valid and is optimal (among feasible tests) under Gaussianity, regardless of the value of δ; see Jansson and Moreira (2006,p.704).[6]

Based on these considerations, we propose an approach similar to that followed by Elliott *et al.* (2015) and Harvey *et al.* (2021), whereby we switch from the use of the U and S tests to a conventional t-test which compares t_τ of (5) with standard normal critical values, when there is sufficient evidence that the predictor is weakly persistent. We will use the with-trend variant of the ADF-OLS normalised bias statistic, $NB\text{-}OLS_\tau := (T\hat{\phi})/(1 - \sum_{i=1}^{p-1} \hat{\psi}_i)$,

[6]In contrast to the case of strongly persistent predictors, for weakly stationary predictors there is no loss of asymptotic local power, relative to a test based on t_μ (where the trend regressor is omitted), from basing the conventional t-test on the with-trend t_τ statistic when the trend is irrelevant (see, e.g., Grenander and Rosenblatt, 1957). We therefore always base the conventional t-test on t_τ because, unlike t_μ, it is exact invariant to the magnitude of the linear trend.

where $\hat{\phi}$ and $\hat{\psi}_i$, $i = 1, ..., p-1$ are obtained by OLS estimation of

$$\Delta x_t = \pi_0 + \pi_1 t + \phi x_{t-1} + \sum\nolimits_{i=1}^{p-1} \psi_i \Delta x_{t-i} + e_t,$$

to determine whether the predictor is weakly persistent. We use the OLS variant of the normalised bias unit root statistic, rather than the GLS variant used to estimate c in Section 5.1, because of its superior power properties for non-local departures from a unit root.

Under Assumption S, $NB\text{-}OLS_\tau = O_p(1)$, while under Assumption W, $NB\text{-}OLS_\tau$ diverges to $-\infty$ at a rate faster than $T^{1/2}$. If we classify a predictor as weakly persistent when $NB\text{-}OLS_\tau < cv_{NB}$ then, for any fixed value of cv_{NB}, a predictor generated according to Assumption W will be classified as weakly persistent asymptotically with probability one. However, employing a fixed critical value can result in a strongly persistent predictor generated according to Assumption S being classified as weakly persistent with non-zero probability (the usual type I error). To address this issue we instead propose the use of a (sample size dependent) diverging critical value given by

$$cv_{NB} = -\upsilon T^{1/2} \tag{17}$$

where $\upsilon > 0$ is a user-chosen tuning parameter, so that the conventional t-test is employed whenever $NB\text{-}OLS_\tau < -\upsilon T^{1/2}$. The divergence rate of $NB\text{-}OLS_\tau$ ensures that, in the limit, our Bonferroni-type U and S tests will always be performed when the predictor is strongly persistent, while the conventional t-test based on comparing t_τ of (5) with standard normal critical values will always be performed when the predictor is weakly persistent, regardless of the value of υ.

5.4 Overall Testing Approach

On the basis of Sections 5.1-5.3 we are now in a position to present our overall hybrid testing procedures for predictability, which we denote by U^{hyb} and S^{hyb}. We outline these test procedures for the case where $\delta < 0$. For $\delta > 0$, proceed as per the discussion at the start of Section 5 substituting x_{t-1} for $-x_{t-1}$ throughout. The decision rules for one-sided tests performed at the $\alpha/2$ nominal asymptotic level can be written as follows, where we again denote the α quantile of the normal distribution as z_α. All confidence intervals are constructed so that the resultant one-sided tests for predictability have maximum asymptotic size of $\alpha/2$.

Decision Rule for Hybrid Test Procedures ($\delta < 0$)

- **Decision Rule for U^{hyb}:**
  - **Right Tailed Tests:**
    - **If $NB\text{-}OLS_\tau \geq -\upsilon T^{1/2}$: Reject H_0 if $\underline{U} > 0$**
    - **If $NB\text{-}OLS_\tau < -\upsilon T^{1/2}$: Reject H_0 if $t_\tau > z_{1-\alpha/2}$**
  - **Left Tailed Tests:**
    - **If $NB\text{-}OLS_\tau \geq -\upsilon T^{1/2}$: Reject H_0 if $\overline{\beta}_\tau^Q(\underline{\rho}_\tau(\underline{\alpha}_{1,\tau}^Q), \alpha_2) < 0$**
    - **If $NB\text{-}OLS_\tau < -\upsilon T^{1/2}$: Reject H_0 if $t_\tau < z_{\alpha/2}$**
- **Decision Rule for S^{hyb}:**
  - **Right Tailed Tests:**
    - **If $NB\text{-}OLS_\tau \geq -\upsilon T^{1/2}$: Reject H_0 if $\underline{US} > 0$**
    - **If $NB\text{-}OLS_\tau < -\upsilon T^{1/2}$: Reject H_0 if $t_\tau > z_{1-\alpha/2}$**
  - **Left Tailed Tests:**
    - **If $NB\text{-}OLS_\tau \geq -\upsilon T^{1/2}$: Reject H_0 if $\overline{S} < 0$**
    - **If $NB\text{-}OLS_\tau < -\upsilon T^{1/2}$: Reject H_0 if $t_\tau < z_{\alpha/2}$**

Remark 5.1. Although the definitions of the U^{hyb} and S^{hyb} procedures given above are framed in terms of one-sided tests for predictability, in principle each of these procedures can also be used to perform two-sided tests for predictability. For a given test, if the right tailed and left tailed versions of the test are constructed such that they have asymptotic size no greater than $\alpha/2$, then combining inference from the two individual one-sided tests for predictability will lead to an overall two-sided test for predictability that will have asymptotic size no greater than α. ♢

6 Local Asymptotic Power of Hybrid Tests

We now return to Figures 1-8 to assess the power of our proposed U^{hyb} and S^{hyb} test procedures, concentrating first on right tailed tests for predictability.

On examining Figures 1-5 we see that when c is small or moderate, the powers of our hybrid U^{hyb} and S^{hyb} tests essentially coincide with each other, as for such values of c, drawings from the limit distribution of $\hat{c}$ in (12) rarely fall below c_R. For small κ, the powers of U^{hyb} and S^{hyb} lie between those of the Q_μ^{GLS} and Q_τ^{GLS} tests, as expected, but it can be seen that the U^{hyb} and S^{hyb} power profiles are reasonably close to that of the best performing Q_μ^{GLS} test and often well in excess of that for Q_τ^{GLS}. As κ increases, Q_τ^{GLS} becomes the most powerful individual test, and here we see that the U^{hyb} and S^{hyb} powers now move towards the (κ-invariant) Q_τ^{GLS} power profile. The consequence of this is that the hybrid tests are always among the best performing tests, having power close to that of Q_{μ}^{GLS} when κ is close to or equal to zero, and that of Q_{τ}^{GLS} for larger values of κ.

We next examine Figures 6-8 where c is large. When κ is small the power of the U^{hyb} test still tracks the power of the Q_{μ}^{GLS} test reasonably well, and for larger values of κ the power of U^{hyb} continues to track the power of the Q_{τ}^{GLS} test. However, for the larger c values we see that the power of U^{hyb} can lag behind that of the t_{τ}^{OLS} test regardless of the value of κ. For these larger c cases we see that the power of the S^{hyb} test now diverges from that of U^{hyb} since $\hat{c}$ is here much more likely to take a value below -35, causing S^{hyb} to switch into the t_{τ}^{OLS} test, more so as c becomes increasingly negative. The consequence is that the power of S^{hyb} is far superior to that of U^{hyb} for these values of c, and is almost identical to that of the best performing t_{τ}^{OLS} test for $c = \{-40, -50\}$.

We now turn our attention to Figures 9-13 which present the performance of S^{hyb} when testing in the left-tail. (Here, we recall that the U^{hyb} test reduces to Q_{τ}^{GLS} in the left tail under strong persistence.) The results show that for small c, $\hat{c}$ is almost never less than -15, hence the power of S^{hyb} coincides almost perfectly with that of Q_{τ}^{GLS}, which is the most powerful test in these scenarios that maintains size control across κ. As c becomes more negative, $\hat{c}$ increasingly drops below -15, with the consequence that inference for S^{hyb} increasingly becomes based on t_{τ}^{OLS}. As such, for large c the power of S^{hyb} more closely tracks that of t_{τ}^{OLS}, which is the best performing size-controlled test. As a consequence, the S^{hyb} test displays one of the best power profiles among size-controlled tests across all values of c, having power close to that of the Q_{τ}^{GLS} test for smaller c and close to that of t_{τ}^{OLS} for larger c. We note also that, due to being constructed using only the trend-invariant Q_{τ}^{GLS} and t_{τ}^{OLS} tests, the S^{hyb} test is itself invariant to the value of κ when testing in the left tail.

An additional consideration in evaluating the local asymptotic size and power of the tests is their behaviour when $c > 0$, such that the predictor series is locally explosive. In the supplementary appendix, we report additional results for the case $c = 2$, for both right-tailed and left-tailed testing. We find that in the right-tailed testing context, the best performing individual tests are Q_{μ}^{GLS} and t_{μ}^{OLS}, even when a large local trend is present (i.e. large κ), and the U^{hyb} and S^{hyb} procedures (which coincide here) track Q_{μ}^{GLS} fairly well across the different κ values considered. In the left-tailed testing scenario, of the two individual tests that achieve size control across c, i.e. Q_{τ}^{GLS} and t_{τ}^{OLS}, we find that Q_{τ}^{GLS} provides the better power profile, as in the case of $c = 0$ and small negative c. Here, the S^{hyb} test has a power profile that always follows this better performing Q_{τ}^{GLS} test, offering an attractive power profile across κ. Overall, the newly proposed hybrid tests also perform well in the locally explosive context.

In summary, our proposed hybrid test procedures display excellent asymptotic size and power properties regardless of the values of c and κ. The S^{hyb} test, in particular, has a power profile that is always close to the best performing size-controlled test in each scenario.

Finite sample simulation results reported in Section S.3 of the supplementary appendix show that the Q_{τ}^{GLS}, t_{τ}^{OLS}, U^{hyb} and S^{hyb} test procedures display excellent size control across a range of values of c and κ both when v_t in (3) is i.i.d. or serially correlated, with the only exception being Q_{τ}^{GLS} and U^{hyb} which display significant oversize for larger negative values of c. The oversize for Q_{τ}^{GLS} for less persistent predictors is expected given that this test is asymptotically invalid for weakly stationary predictors, while the oversize for U^{hyb} in this region arises from use of Q_{τ}^{GLS} through the union-of-rejections approach. Aside from this case, the relative power of the tests in finite samples is almost identical to that observed in the asymptotic power simulations, with the S^{hyb} test in particular displaying excellent power across the large range of simulation DGPs considered.

On the basis of our asymptotic and finite sample simulation results, we recommend basing inference on our proposed S^{hyb} predictability testing procedure as it has controlled size, and is always among the most powerful tests, over the full range of parameter settings considered.

7 Empirical Application

We now report results of an empirical exercise in which we apply the tests for predictability outlined in this paper to the US equity series analysed in Welch and Goyal (2008), using updated data at all available data frequencies (annual, quarterly and monthly) for the period 1926-2021 which can be obtained from http://www.hec.unil.ch/agoyal/. Our dependent variable, r_t, is the S&P500 value-weighted log-return, and for x_t we consider the same thirteen candidate predictors variables as Harvey *et al.* (2021): the dividend payout ratio, earnings-price ratio, dividend-price ratio, dividend yield, default yield spread, long-term yield, default return spread, net equity expansion, inflation rate, Treasury bill rate, term spread the book-to-market ratio and stock variance.

We first formally test for the presence of a linear trend in each predictor using a range of trend tests available in the literature that are designed to be robust to whether Assumption S or W holds; namely the $t_{\beta}^{RQF}(MU)$ test of Perron and Yabu (2009), the z_{λ}, z_{λ}^{m1} and z_{λ}^{m2} tests of Harvey *et al.* (2007), and the *Dan-J* test of Bunzel and Vogelsang (2005). We perform left-tailed trend tests for all predictors with the exception of the inflation rate and term spread for which right-tailed tests are performed, using the setting recommended by the authors in each case. Tables 2 reports the p-value of the $t_{\beta}^{RQF}(MU)$ and z_{λ} tests, as well as the significance level at which the remaining tests (which are designed to be performed only at discrete pre-assigned significance levels) reject the null hypothesis of no linear trend ($\gamma = 0$), for each predictor at each data frequency.

From the results reported in Table 2 it is seen that for each of the default yield spread, long term yield, default return spread, inflation rate, treasury bill rate and stock variance no trend is detected, regardless of data frequency. In contrast, for the dividend payout ratio, earnings-price ratio, dividend-price ratio, dividend yield and net equity expansion series a significant linear trend is detected regardless of the data frequency. For the remaining predictors the results of the trend tests are mixed, with the trend tests indicating the presence of a trend at some, but not all, data frequencies. In summary, there is at least some statistically significant evidence of a linear trend being present in the majority of the predictors considered.

We now discuss the results of applying right-tailed tests for predictability. All tests are performed at a nominal level of 0.05. Following CY, lag selection for all of the unit root tests utilised in the test procedures is performed using the Bayes Information Criterion (BIC) with a maximum number of lagged differences of 4. Finally we set $\upsilon = 10$ in (17) such that our hybrid S^{hyb} and U^{hyb} tests switch into a conventional t-test, comparing t_τ of (5) with standard normal critical values, whenever $NB\text{-}OLS_\tau < -10T^{1/2}$ as we found this choice of υ delivered good finite sample performance across a wide range of DGPs in the Monte Carlo simulation results reported in Section S.3. As Harvey *et al.* (2021) test for predictability in the left tail for the stock variance predictor we report results for a right-tail test for predictability when $(-1)\times$ stock variance is employed as the predictor, noting that this is equivalent to a left-tailed test using the original data (cf. Remark 3.2).

Table 2 also reports the lower bound of the confidence interval for β, denoted generically as $\underline{\beta}$, for each predictor at each frequency, and for each of the predictability tests discussed in this paper. Also reported is the estimator $\hat{\delta}$ from the with-trend Bonferroni type test procedures. We highlight any instances where this lower bound is greater than zero in bold to help identify instances where the null of $\beta = 0$ is rejected in favour of the alternative that $\beta > 0$. Finally, for the lower bound of β from the S^{hyb} and U^{hyb} tests we use the superscript z to identify instances where these tests have switched into the conventional t-test (i.e. comparing t_τ of (5) with standard normal critical values), and for S^{hyb} we use the superscript t to denote instances where this test is basing inference on the t_τ^{OLS} test.

For the dividend payout ratio, long term yield, net equity expansion, inflation rate, Treasury bill rate, term spread and stock variance predictors, no evidence of predictability is found by any of the tests for any data frequency and so we will not discuss results for these predictors further.

For the earnings-price ratio, as noted above, a linear trend was detected at all frequencies, giving reasonable evidence that a trend is present in this predictor. For this predictor the Q_τ^{GLS}, t_τ^{OLS}, U^{hyb} and S^{hyb} tests all reject the null of no predictability at each data frequency, while the Q_μ^{GLS} test fails to reject at any frequency and the t_μ^{OLS} test rejects only at the monthly and quarterly frequencies. These results suggest that for this predictor an unmodelled trend in the predictor may be negatively impacting the power of the constant-only tests, with the trend-augmented and hybrid tests retaining power to find significant evidence of predictability.

A similar story is seen for the dividend-price ratio where again a trend was detected at each data frequency, and where the Q_μ^{GLS} and t_μ^{OLS} tests provide no evidence of predictability at any frequency. The Q_τ^{GLS}, S^{hyb} and U^{hyb} tests, on the other hand, find evidence of predictability at both the annual and quarterly frequencies, although no predictability is detected by any test at the monthly frequency.

Turning to the dividend yield predictor a significant trend is detected at each data frequency by at least one of the trend tests and all of the predictability tests find significant evidence of predictability at all data frequencies, with the exception of the Q_μ^{GLS} and t_μ^{OLS} tests at the annual frequency. Interestingly, the annual frequency data provides the strongest evidence for the presence of a trend among the three data frequencies and so it is noteworthy that it is for the annual data that the Q_μ^{GLS} and t_μ^{OLS} fail to detect predictability, while our hybrid tests deliver rejections.

There appears to be no evidence of a trend in the default yield spread, and the only data frequency at which predictability is detected for this predictor is for quarterly data. For quarterly data rejections are found by all but the t_τ^{OLS} test, reflective of the fact that our hybrid S^{hyb} and U^{hyb} tests are competitive on power with the best performing individual tests when no (or a very small) trend is present in the predictor.

For the default return spread no trend is detected at any data frequency and only one rejection, at the monthly frequency, is observed for the Q_μ^{GLS} test. As the S^{hyb} and U^{hyb} tests have switched into the conventional t test for this predictor it is likely that this predictor is weakly persistent, and that the rejection from Q_μ^{GLS} may be reflecting the oversize of this test for weakly persistent predictors.

Finally, results for the book-to-market ratio are mixed. At the annual frequency no evidence of a trend is found and the only test to reject is Q_μ^{GLS} which is perhaps to be expected if no trend is present and the predictive power of this predictor is weak. At the quarterly and monthly frequencies, however, a trend is detected and all of the tests reject the null of no predictability with the exception of t_μ^{OLS} and Q_τ^{GLS} for monthly data.

Overall we find that for several predictor series, a trend appears to be present, and at the same time the constant-only Bonferroni Q and t-tests fail to reject the null of no predictability, indicating that the presence of omitted trends may be negatively impacting the power of the constant-only tests. In contrast, our proposed tests find evidence of predictability in many of these cases, highlighting the value of our hybrid procedures in detecting predictability when uncertainty exists regarding the presence of a linear trend in the predictor.

8 Conclusions

In this paper we have considered trend-augmented versions of the Bonferroni Q test of CY and the Bonferroni t-test of CES. We have shown that in the presence of an omitted trend in the predictor, when $\delta < 0$ ($\delta > 0$) the constant-only Bonferroni Q and t-tests can be severely undersized when testing in the right (left) tail, displaying a subsequent lack of power, and severely oversized when testing in the left (right) tail. The trend augmented Bonferroni Q and t-tests, while displaying power below their constant-only counterparts when no trend is present, are invariant to a trend in the predictor. We subsequently proposed union-of-rejections type hybrid testing procedures that are able to capture the power of the constant-only Bonferroni Q test when the predictor admits only a deterministic constant, and the power of the trend-augmented Bonferroni Q and t-tests when a trend is present in the predictor, with S^{hyb} being our recommended testing procedure given that it has controlled size, and is always among the most powerful tests, over the full range of parameter settings considered. An empirical illustration using an updated version of the dataset of Welch and Goyal (2008) demonstrated that our proposed approach finds evidence of predictability in several instances where a trend appears to be present in the predictor where the constant-only Bonferroni Q and t-tests fail to reject, indicating that the presence of omitted trends may be negatively impacting the power of the constant-only tests in this commonly used dataset.

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Table 1: Parameters to Deliver One-Sided Tests with Maximum 5% Asymptotic Size.

	Q_μ^{GLS}		t_μ^{OLS}		Q_τ^{GLS}		t_τ^{OLS}		U
δ	$\underline{\alpha}_{1,\mu}^Q$	$\bar{\alpha}_{1,\mu}^Q$	$\underline{\alpha}_{1,\mu}^t$	$\bar{\alpha}_{1,\mu}^t$	$\underline{\alpha}_{1,\tau}^Q$	$\bar{\alpha}_{1,\tau}^Q$	$\underline{\alpha}_{1,\tau}^t$	$\bar{\alpha}_{1,\tau}^t$	ξ
-0.999	0.050	0.055	0.020	0.035	0.050	0.050	0.040	0.035	0.500
-0.975	0.055	0.080	0.025	0.035	0.055	0.055	0.040	0.025	0.630
-0.950	0.055	0.100	0.025	0.040	0.060	0.065	0.040	0.020	0.660
-0.925	0.055	0.115	0.025	0.040	0.065	0.070	0.035	0.020	0.710
-0.900	0.060	0.130	0.025	0.035	0.070	0.075	0.050	0.020	0.730
-0.875	0.060	0.140	0.025	0.035	0.070	0.085	0.050	0.015	0.710
-0.850	0.060	0.150	0.025	0.035	0.075	0.090	0.050	0.015	0.730
-0.825	0.060	0.160	0.025	0.035	0.075	0.095	0.055	0.010	0.740
-0.800	0.065	0.170	0.025	0.035	0.080	0.100	0.060	0.010	0.750
-0.775	0.065	0.180	0.030	0.035	0.080	0.105	0.065	0.010	0.760
-0.750	0.065	0.190	0.025	0.035	0.085	0.110	0.065	0.010	0.760
-0.725	0.065	0.195	0.025	0.035	0.085	0.115	0.065	0.010	0.760
-0.700	0.070	0.205	0.025	0.035	0.090	0.120	0.065	0.010	0.750
-0.675	0.070	0.215	0.025	0.035	0.090	0.125	0.065	0.005	0.750
-0.650	0.070	0.225	0.025	0.035	0.095	0.130	0.080	0.005	0.740
-0.625	0.075	0.230	0.025	0.035	0.095	0.135	0.080	0.005	0.740
-0.600	0.075	0.240	0.030	0.035	0.100	0.140	0.085	0.005	0.740
-0.575	0.075	0.250	0.035	0.035	0.100	0.140	0.085	0.005	0.740
-0.550	0.080	0.260	0.035	0.035	0.105	0.145	0.090	0.005	0.730
-0.525	0.080	0.270	0.045	0.035	0.110	0.150	0.095	0.005	0.730
-0.500	0.080	0.280	0.060	0.035	0.115	0.150	0.095	0.010	0.730
-0.475	0.085	0.285	0.050	0.035	0.120	0.150	0.095	0.010	0.730
-0.450	0.085	0.295	0.055	0.040	0.120	0.155	0.095	0.010	0.730
-0.425	0.090	0.310	0.035	0.040	0.125	0.165	0.095	0.010	0.710
-0.400	0.090	0.320	0.060	0.040	0.130	0.165	0.150	0.010	0.710
-0.375	0.095	0.330	0.040	0.040	0.135	0.165	0.150	0.010	0.710
-0.350	0.100	0.345	0.030	0.040	0.140	0.170	0.150	0.010	0.690
-0.325	0.100	0.355	0.015	0.045	0.145	0.170	0.150	0.010	0.690
-0.300	0.105	0.360	0.010	0.050	0.150	0.175	0.150	0.010	0.680
-0.275	0.110	0.370	0.005	0.040	0.155	0.175	0.200	0.010	0.680
-0.250	0.115	0.375	0.005	0.035	0.165	0.175	0.200	0.010	0.680
-0.225	0.125	0.380	0.005	0.025	0.170	0.175	0.200	0.010	0.680
-0.200	0.130	0.390	0.005	0.025	0.175	0.175	0.200	0.005	0.670
-0.175	0.140	0.395	0.005	0.010	0.185	0.175	0.200	0.005	0.650
-0.150	0.150	0.400	0.005	0.010	0.200	0.175	0.200	0.005	0.650
-0.125	0.160	0.405	0.005	0.010	0.200	0.165	0.200	0.005	0.630
-0.100	0.175	0.415	0.005	0.005	0.210	0.145	0.200	0.005	0.610
-0.075	0.190	0.420	0.005	0.005	0.220	0.130	0.200	0.005	0.610
-0.050	0.215	0.425	0.005	0.005	0.225	0.100	0.150	0.005	0.590
-0.025	0.250	0.435	0.005	0.005	0.185	0.035	0.150	0.005	0.570

Table 2: Trend Tests and $\underline{\beta}$ for Updated Welch and Goyal (2008) Dataset

Annual Data

Predictor	$p(t_\beta^{RQF})$	$p(Z_\lambda)$	Z_λ^{m1}	Z_λ^{m2}	$DAN\text{-}J$	$\hat{\delta}$	$\underline{\beta}$: t_μ^{OLS}	Q_μ^{GLS}	t_τ^{OLS}	Q_τ^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	**0.000**	**0.000**	***	***	***	-0.313	-0.1650	-0.1439	-0.1635	-0.1353	-0.1605	-0.1635^t
Earnings-Price Ratio	**0.083**	0.223				-0.301	-0.0018	-0.0095	**0.0192**	**0.0391**	**0.0188**	**0.0188**
Dividend-Price Ratio	**0.000**	**0.008**	**	**	*	-0.843	-0.0624	-0.0443	-0.0280	**0.0406**	**0.0251**	**0.0251**
Dividend Yield	0.200	**0.028**	**	**	*	0.133	-0.0024	-0.0024	**0.0838**	**0.0753**	**0.0753**	**0.0753**
Default Yield Spread	0.314	0.397				-0.570	-0.1221	-0.1002	-0.1705	-0.1242	-0.1192	-0.1192
Long Term Yield	0.376	0.384				-0.028	-0.0467	-0.0460	-0.0547	-0.0539	-0.0544	-0.0544
Default Return Spread	0.267	0.227				0.315	-0.2064	-0.2143	-0.1993	-0.4130	-1.0099^z	-1.0099^z
Net Equity Expansion	**0.000**	**0.000**	***	***	**	0.086	-0.3479	-0.3280	-0.3408	-0.3512	-0.3512	-0.3408^t
Inflation Rate	0.259	0.262				-0.032	-0.0724	-0.0738	-0.0834	-0.0872	-0.0943	-0.0943
Treasury Bill Rate	0.355	0.363				0.093	-0.0802	-0.0859	-0.0817	-0.0886	-0.0886	-0.0886
Term Spread	0.361	0.363				-0.111	-0.0821	-0.0837	-0.0943	-0.0905	-0.1065	-0.0943^t
Book-to-market Ratio	0.196	0.282				-0.806	-0.0388	**0.0036**	-0.0313	-0.0229	-0.0087	-0.0087
Stock Variance	0.306	0.472				0.398	-0.1521	-0.2169	-0.1559	-0.2288	-0.2288	-0.1559^t

Quarterly Data

Predictor	$p(t_\beta^{RQF})$	$p(Z_\lambda)$	Z_λ^{m1}	Z_λ^{m2}	$DAN\text{-}J$	$\hat{\delta}$	$\underline{\beta}$: t_μ^{OLS}	Q_μ^{GLS}	t_τ^{OLS}	Q_τ^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	**0.000**	**0.064**	**	**	***	-0.120	-0.0229	-0.0235	-0.0250	-0.0307	-0.0277	-0.0250^t
Earnings-Price Ratio	**0.000**	0.241			*	-0.614	**0.0043**	-0.0157	**0.0068**	**0.0049**	**0.0006**	$\mathbf{0.0068^t}$
Dividend-Price Ratio	0.183	**0.072**	**	**	**	-0.949	-0.0085	-0.0061	-0.0002	**0.0160**	**0.0110**	**0.0110**
Dividend Yield	0.204	0.106	*	**	*	0.113	**0.0023**	**0.0024**	**0.0240**	**0.0224**	**0.0224**	$\mathbf{0.0240^t}$
Default Yield Spread	0.196	0.416				-0.512	**0.0095**	**0.0126**	-0.0051	**0.0062**	**0.0071**	**0.0071**
Long Term Yield	0.363	0.391				-0.054	-0.0159	-0.0155	-0.0171	-0.0169	-0.0177	-0.0177
Default Return Spread	0.355	0.265				0.303	-0.1263	-0.1792	-0.1247	-0.3683	-0.6196^z	-0.6196^z
Net Equity Expansion	**0.000**	**0.004**	***	***	**	0.115	-0.0755	-0.0676	-0.0909	-0.0898	-0.0898	-0.0909^t
Inflation Rate	0.151	0.314				0.030	-0.1078	-0.1020	-0.1093	-0.1033	-0.1033	-0.1093^t
Treasury Bill Rate	0.393	0.414				-0.067	-0.0280	-0.0269	-0.0298	-0.0295	-0.0303	-0.0303
Term Spread	**0.074**	0.345				0.031	-0.0273	-0.0256	-0.0279	-0.0279	-0.0279	-0.0279^t
Book-to-market Ratio	0.409	0.360			*	-0.793	**0.0212**	**0.0185**	**0.0392**	**0.0122**	**0.0153**	**0.0153**
Stock Variance	0.227	0.455				0.290	-0.1263	-0.1100	-0.1272	-0.1320	-0.1320	-0.1272^t

Monthly Data

Predictor	$p(t_\beta^{RQF})$	$p(Z_\lambda)$	Z_λ^{m1}	Z_λ^{m2}	$DAN\text{-}J$	$\hat{\delta}$	$\underline{\beta}$: t_μ^{OLS}	Q_μ^{GLS}	t_τ^{OLS}	Q_τ^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	0.412	**0.089**	**	***	**	-0.049	-0.0052	-0.0053	-0.0059	-0.0065	-0.0059	-0.0059^t
Earnings-Price Ratio	0.374	0.353		**		-0.799	**0.0009**	-0.0060	**0.0015**	**0.0017**	**0.0006**	$\mathbf{0.0015^t}$
Dividend-Price Ratio	0.200	0.198	*	**	*	-0.975	-0.0042	-0.0031	-0.0026	-0.0008	-0.0026	-0.0026
Dividend Yield	0.201	0.199	*	**	*	-0.068	**0.0008**	**0.0008**	**0.0096**	**0.0099**	**0.0085**	**0.0085**
Default Yield Spread	0.438	0.434				-0.248	-0.0015	-0.0012	-0.0036	-0.0017	-0.0028	-0.0028
Long Term Yield	0.354	0.384				-0.087	-0.0051	-0.0049	-0.0057	-0.0057	-0.0056	-0.0056
Default Return Spread	0.321	0.374				0.182	-0.0162	**0.0636**	-0.0158	-0.0259	-0.0678^z	-0.0678^z
Net Equity Expansion	**0.000**	**0.020**	**	***	**	-0.030	-0.0222	-0.0225	-0.0268	-0.0269	-0.0240	-0.0240
Inflation Rate	0.189	0.456				0.035	-0.0747	-0.0652	-0.0760	-0.0720	-0.0720	-0.0760^t
Treasury Bill Rate	0.387	0.402				-0.056	-0.0072	-0.0070	-0.0077	-0.0077	-0.0079	-0.0079
Term Spread	**0.001**	0.400		*		0.008	-0.0079	-0.0078	-0.0084	-0.0084	-0.0084	-0.0084^t
Book-to-market Ratio	0.438	0.435			*	-0.807	-0.0001	**0.0035**	**0.0007**	-0.0023	**0.0025**	**0.0025**
Stock Variance	0.204	0.497				0.267	-0.0429	-0.0018	-0.0431	-0.0130	-0.0130	-0.0431^t

Notes:

(i) The entries in the columns headed $p(t_\beta^{RQF})$ and $p(Z_\lambda)$ denote p-values for the t_β^{RQF} and Z_λ tests. Bold entries highlight p-values below 0.1.

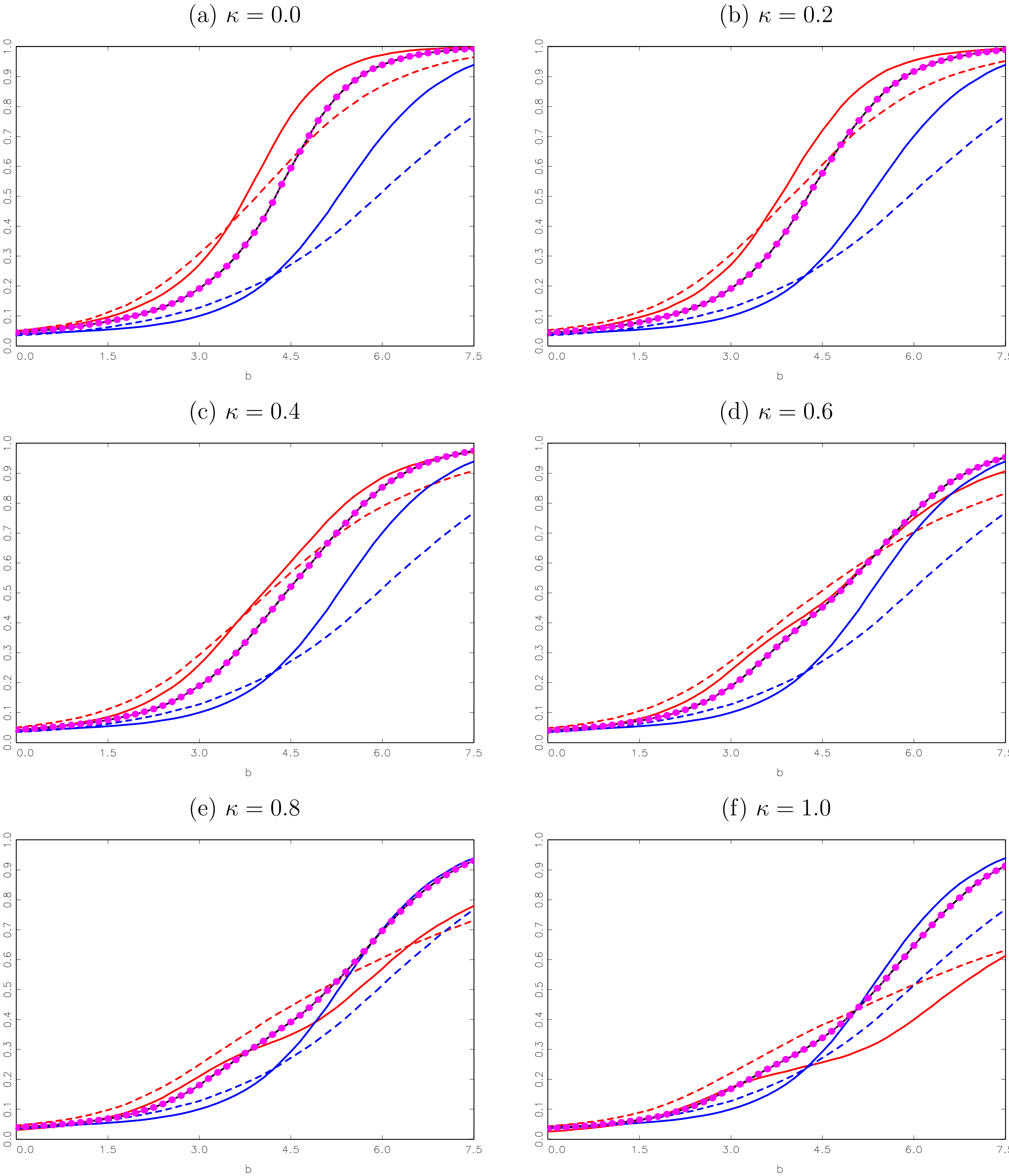
(ii) For Z_λ^{m1}, Z_λ^{m2} and $DAN\text{-}J$, $*$ denotes rejection at the 10% level, $**$ denotes rejection at the 5% level, and $***$ denotes rejection at the 1% level.

(iii) Bold entries in the $\underline{\beta}$ columns highlight cases where the null hypothesis of no predictability can be rejected at the 5% level.

(iv) For entries in the U^{hyb} and S^{hyb} columns, a z superscript denotes that the test compares t_τ with $N(0,1)$ critical values, while a t superscript denotes that the test bases inference on the t_τ^{OLS} test.

(v) In the case of Stock Variance, we report $\hat{\delta}$ and $\underline{\beta}$ for (-1)$\times$ Stock Variance as the predictor. A right-tailed test from this regression is equivalent to a left-tailed test using the original data.

Figure 1: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = 0$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure 2: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -2$

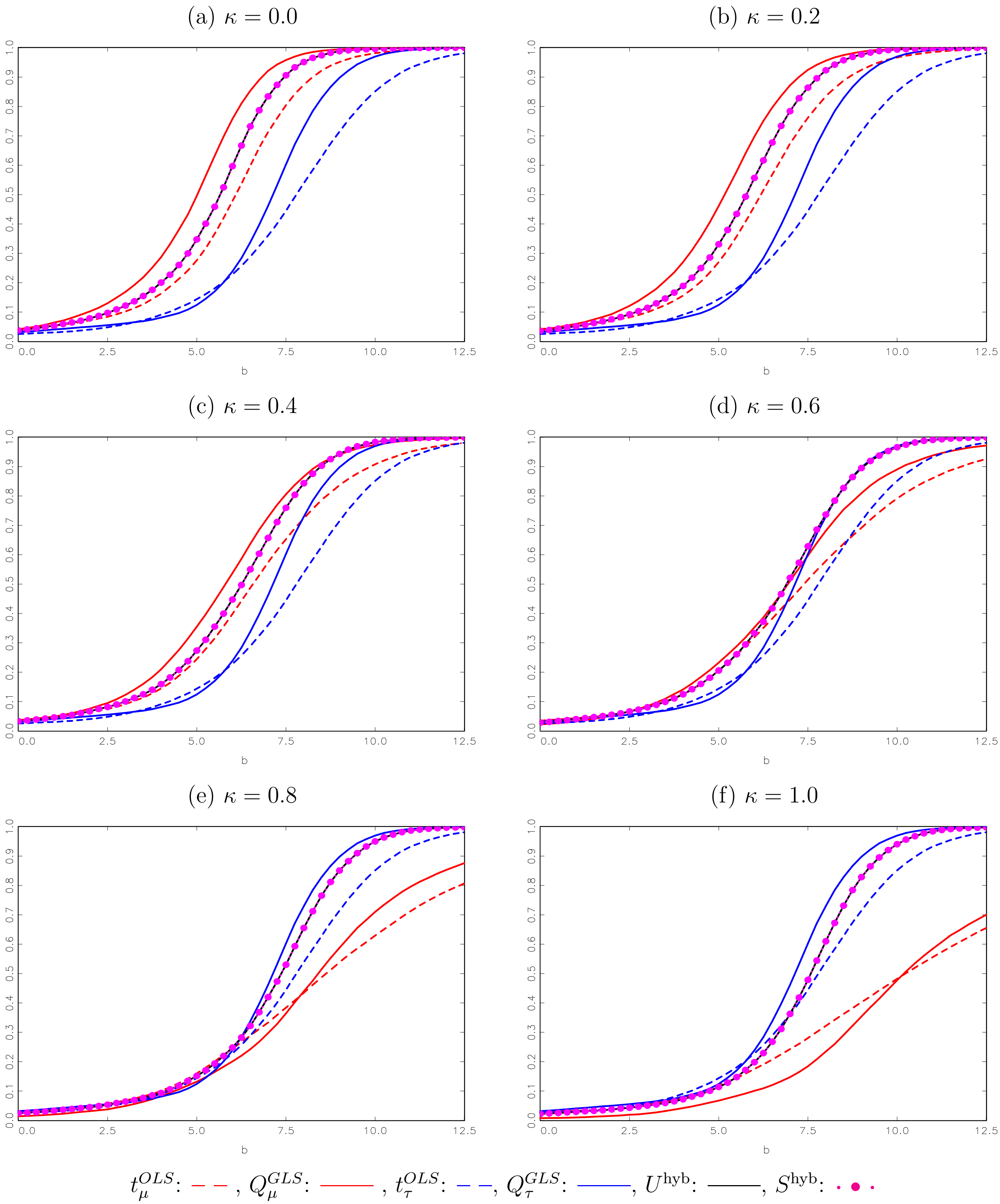
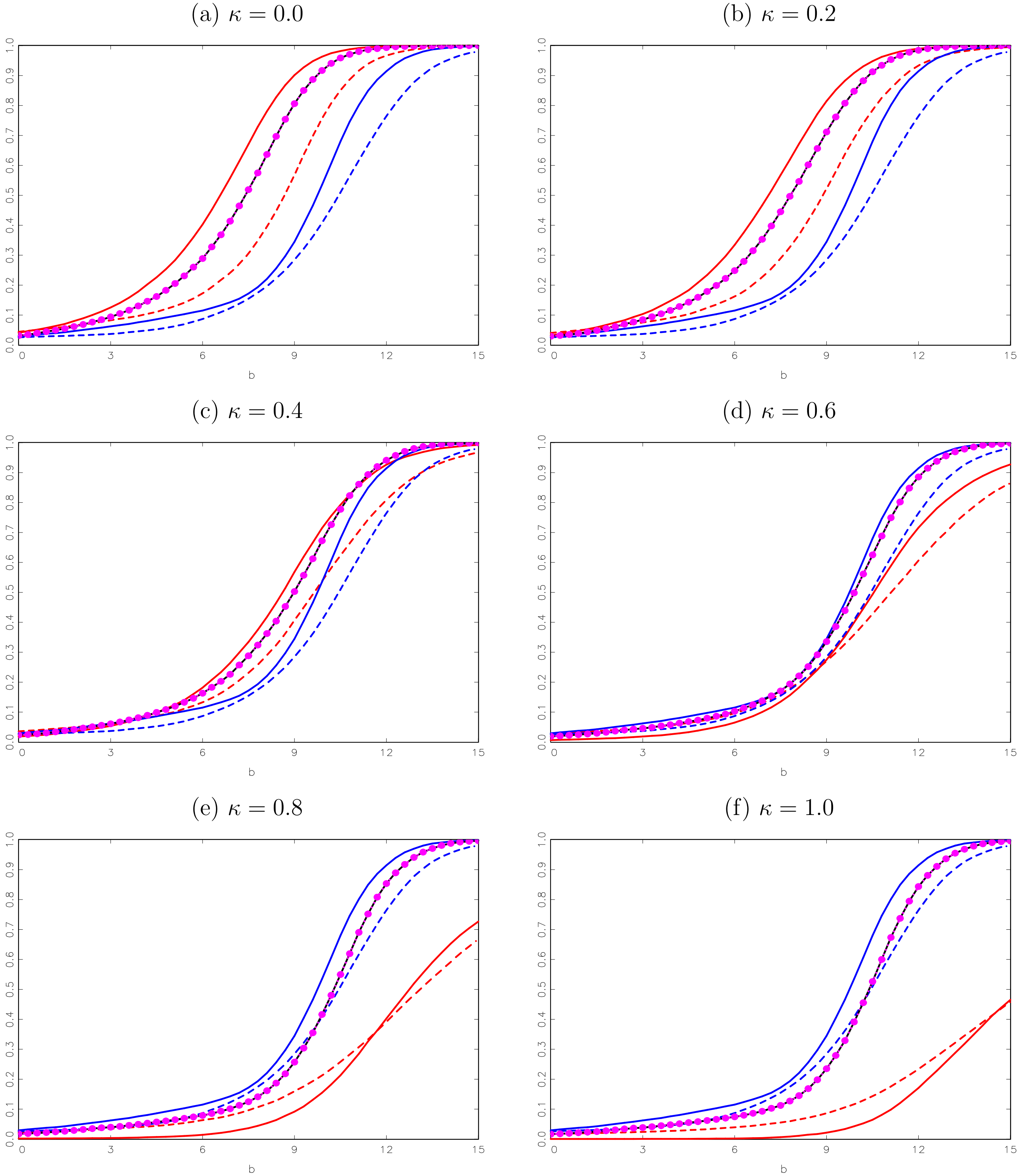


Figure 3: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -5$



t_{μ}^{OLS}: – –, Q_{μ}^{GLS}: ——, t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure 4: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -10$

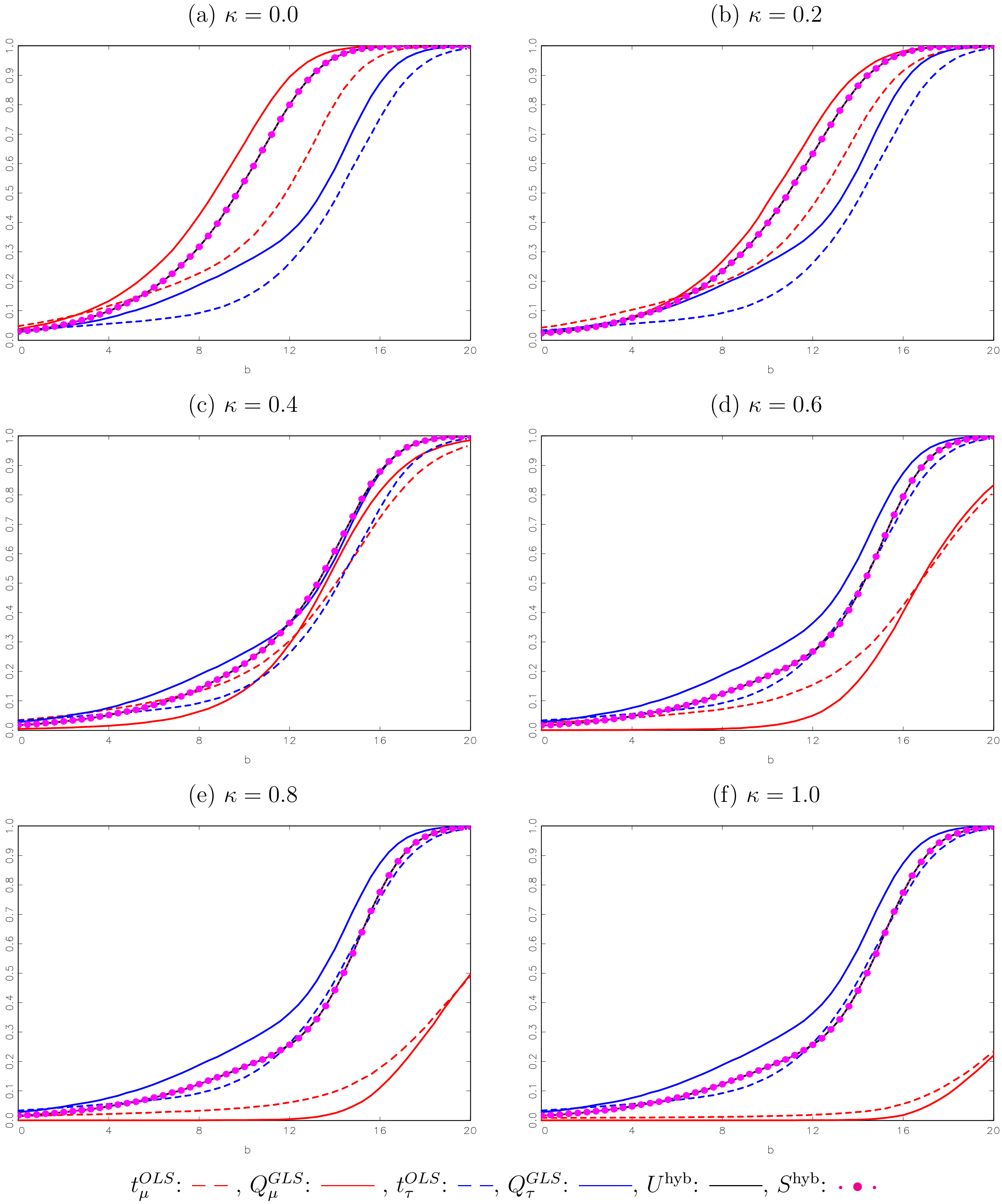


Figure 5: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -20$

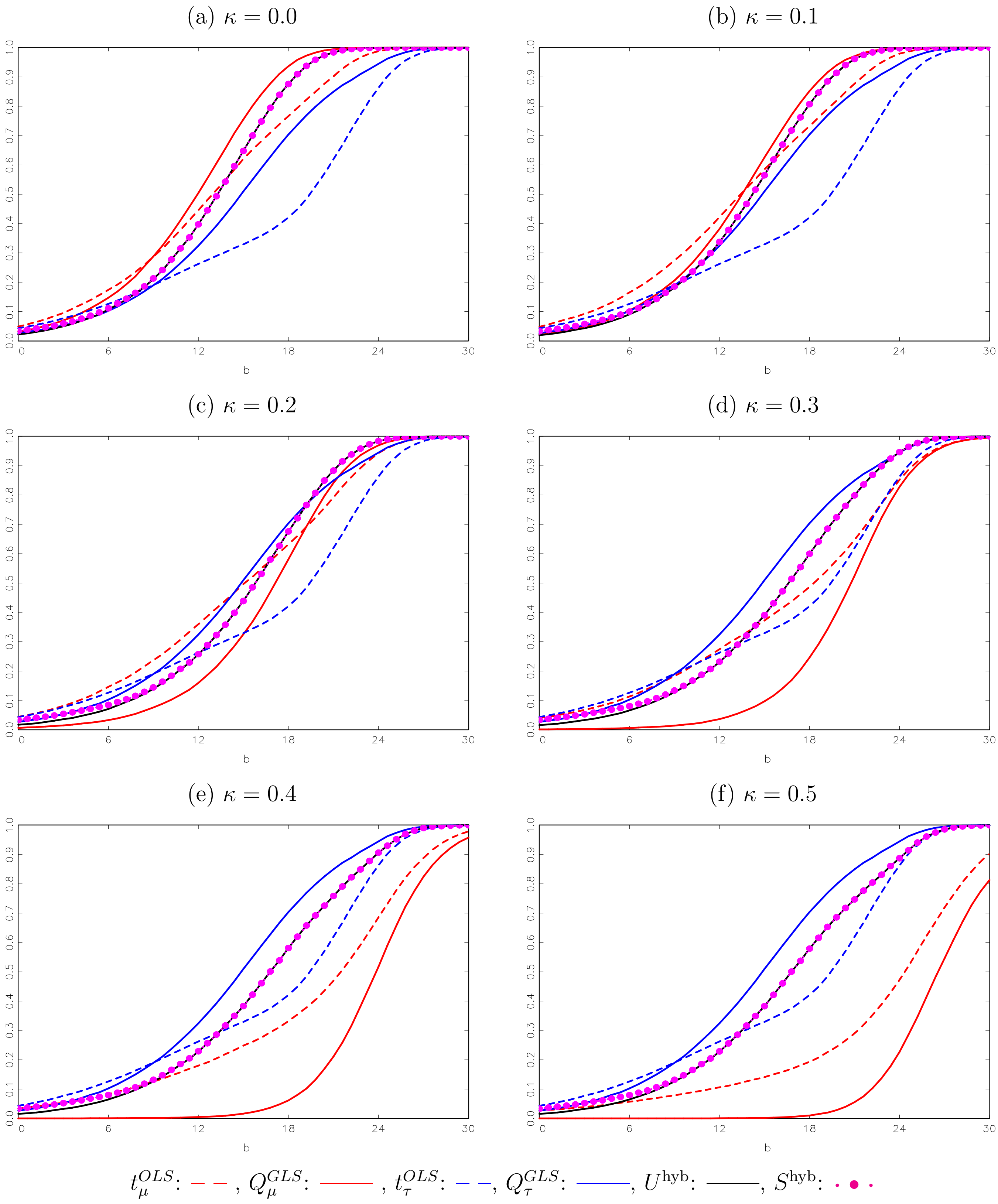
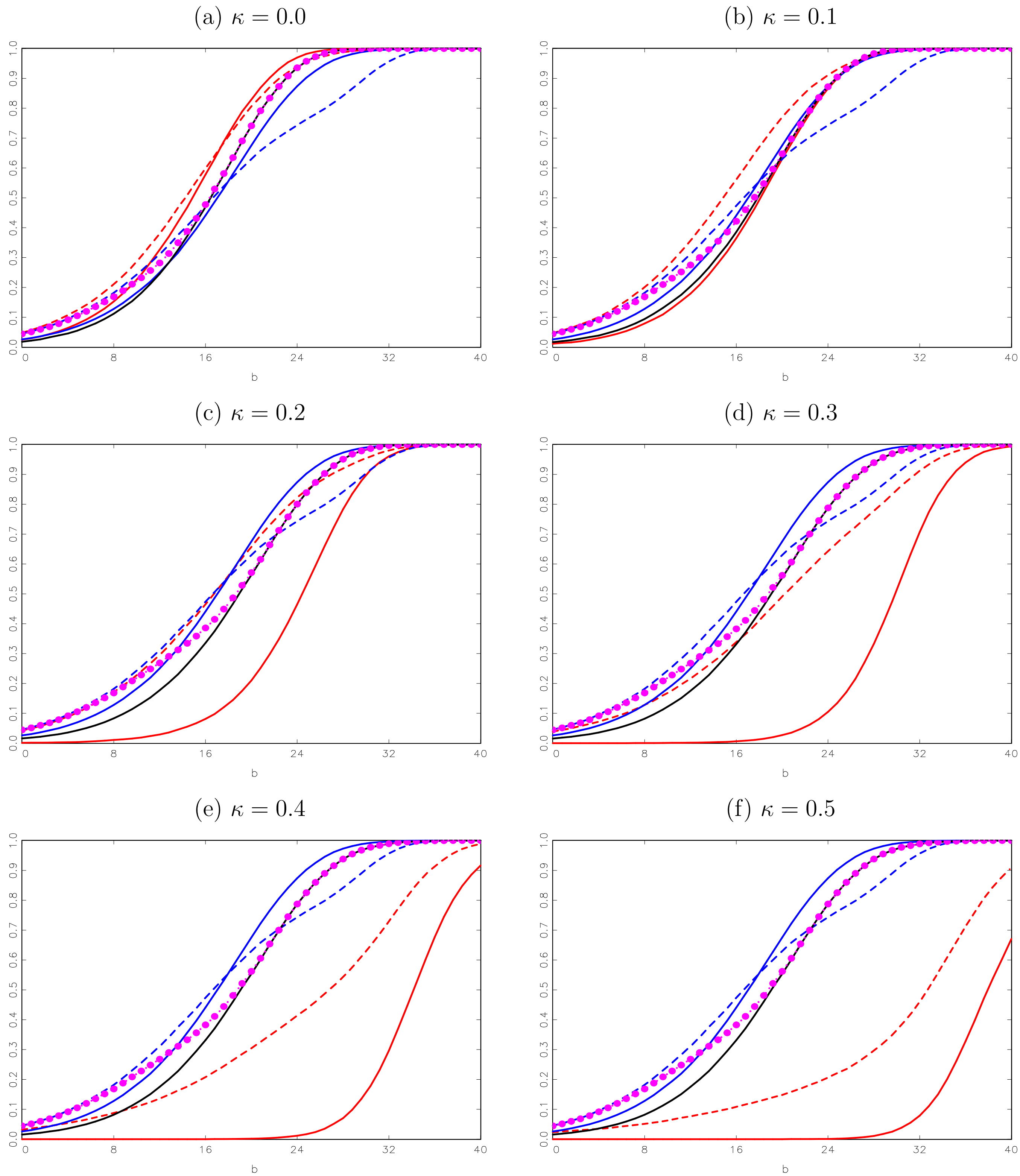


Figure 6: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -30$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure 7: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -40$

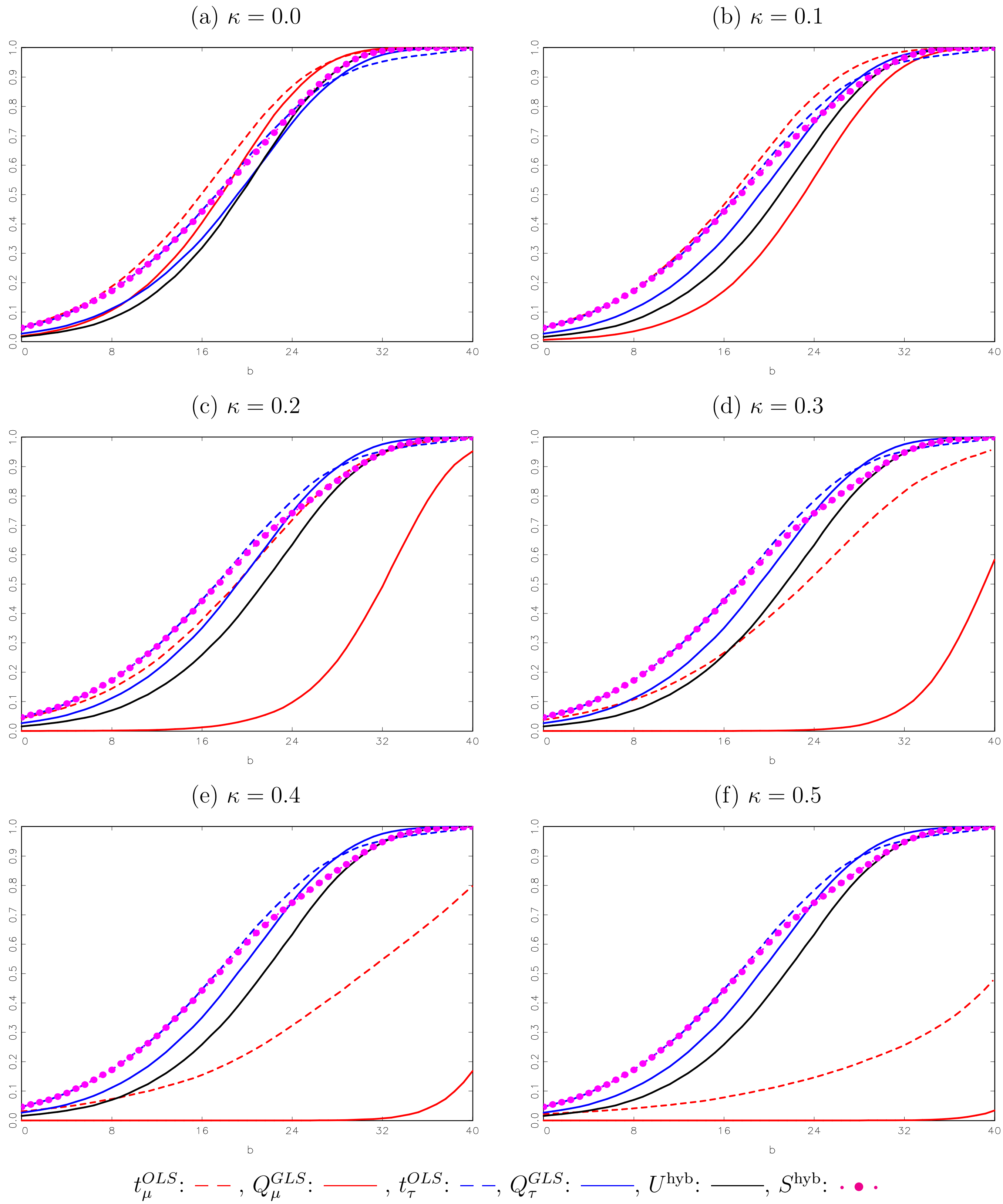
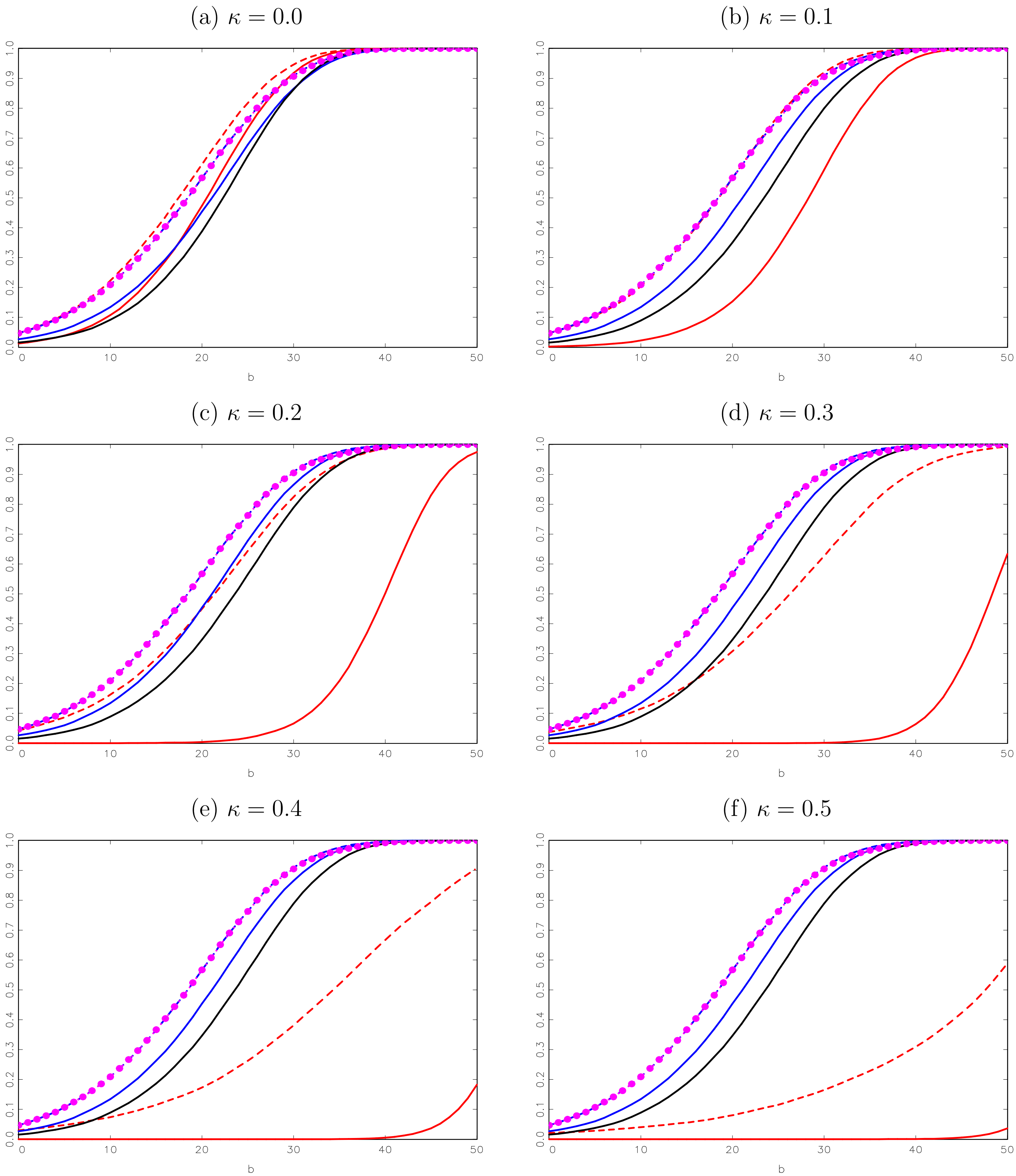


Figure 8: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = -50$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – – , Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure 9: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = 0$

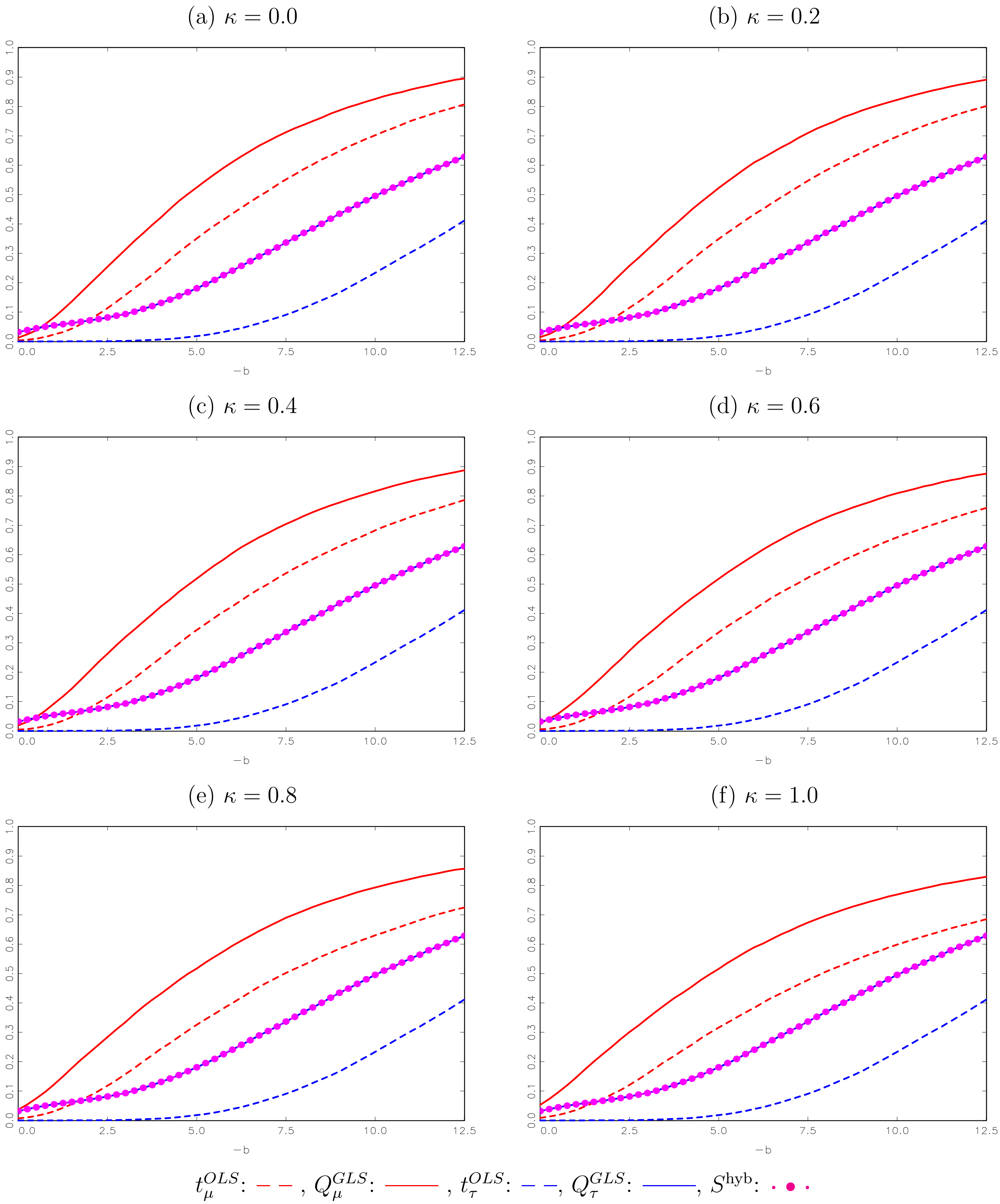
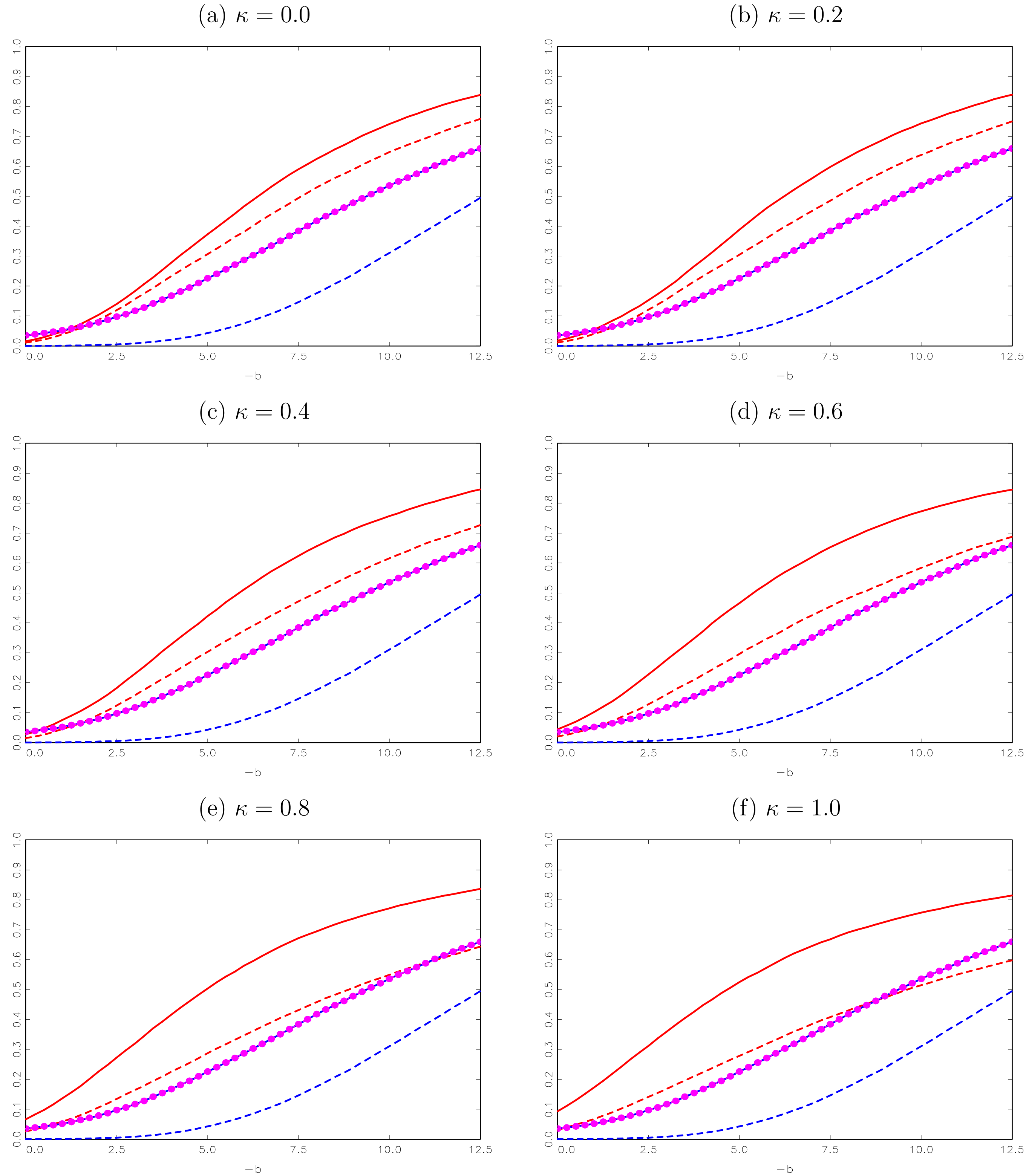
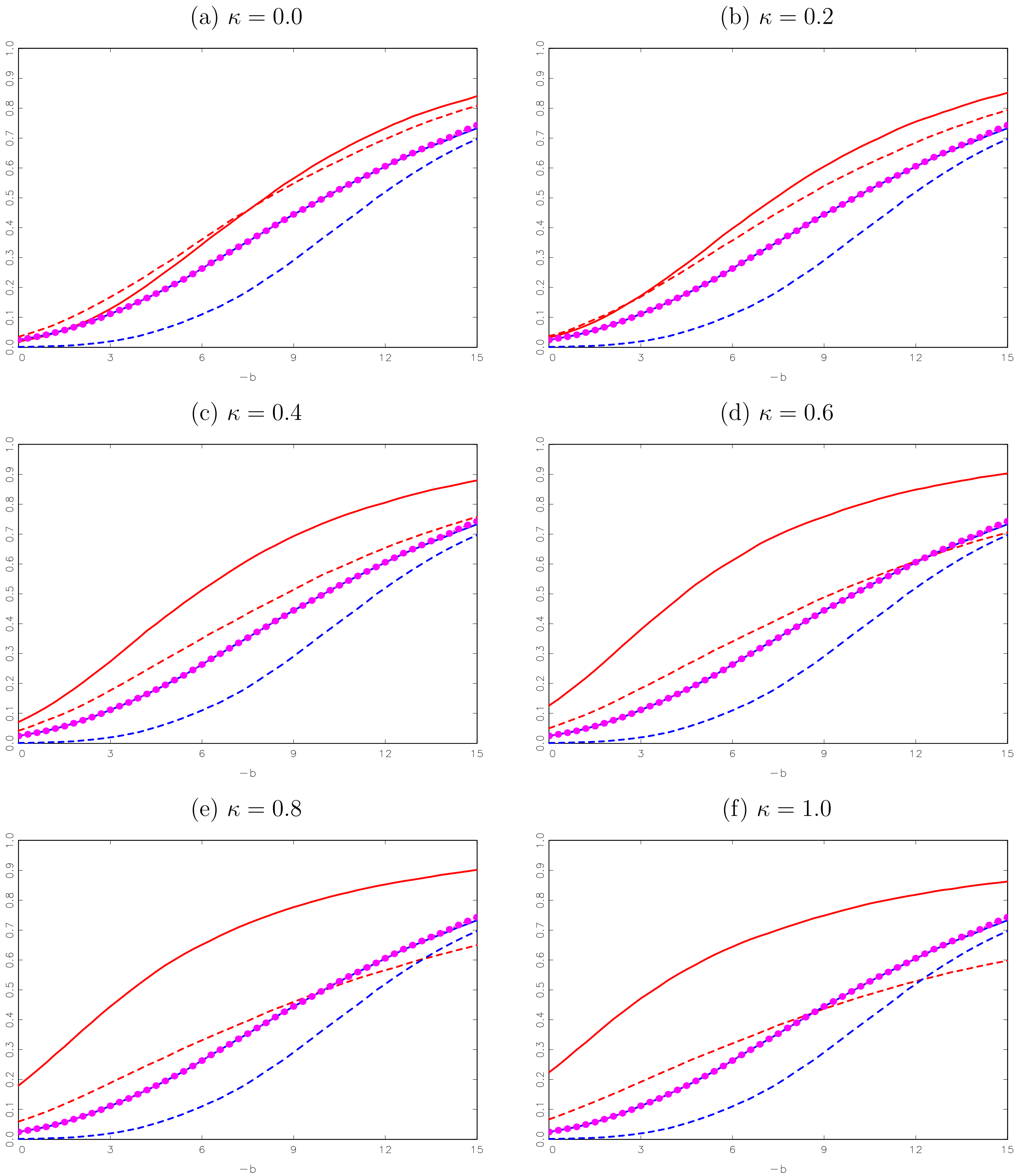


Figure 10: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -2$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, S^{hyb}: • • •

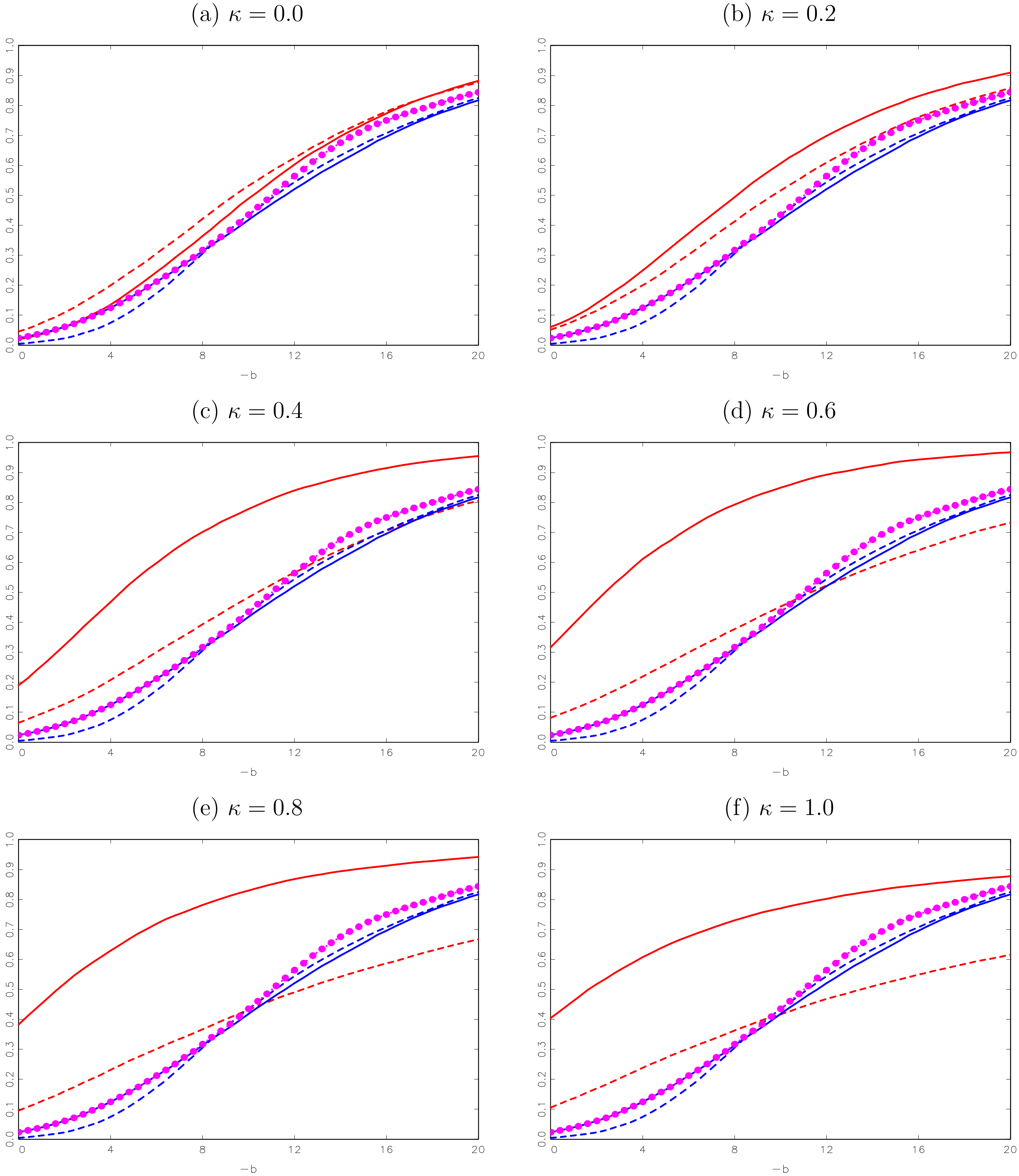
Figure 11: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -5$



(a) $\kappa = 0.0$ (b) $\kappa = 0.2$ (c) $\kappa = 0.4$ (d) $\kappa = 0.6$ (e) $\kappa = 0.8$ (f) $\kappa = 1.0$

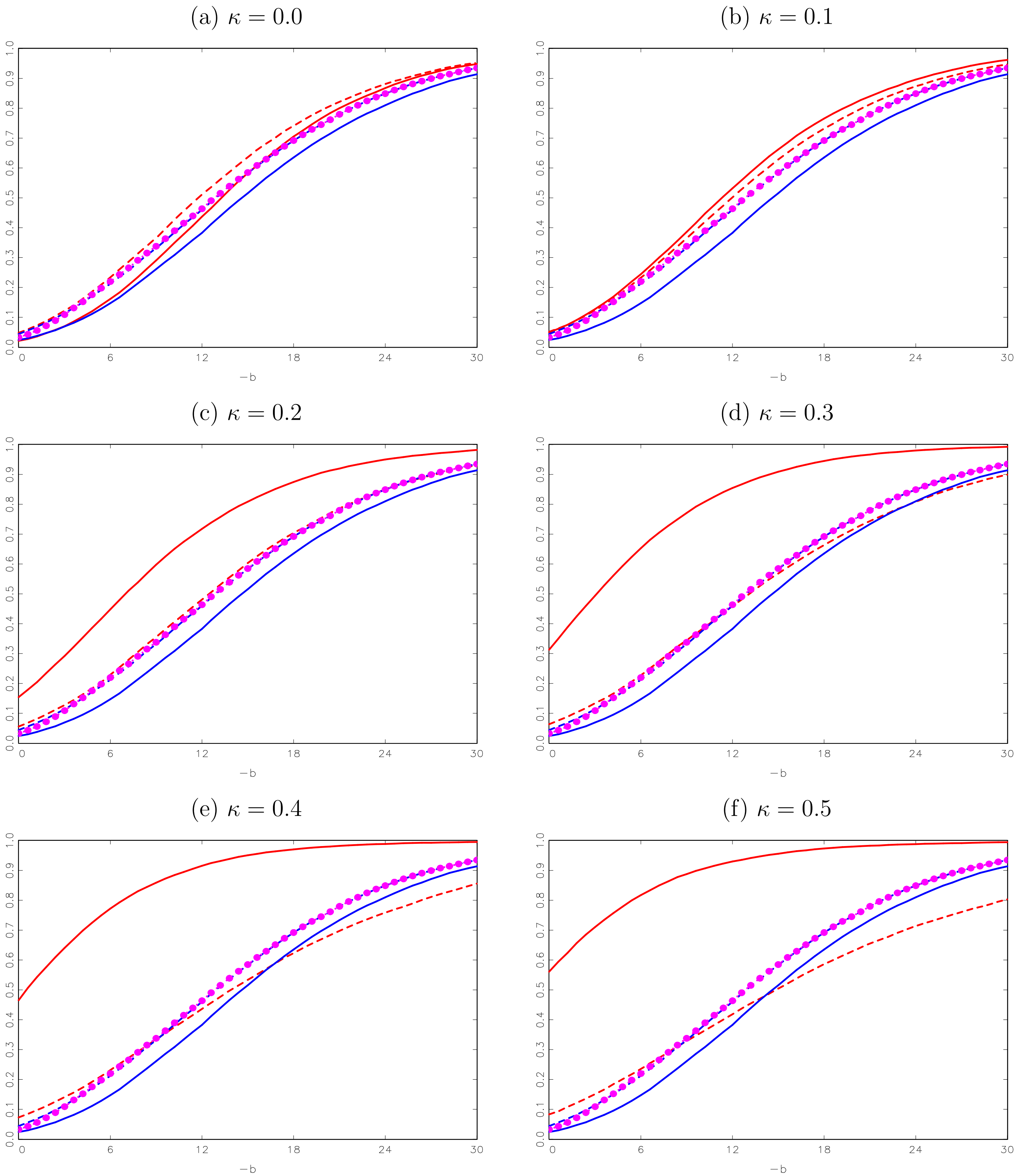
t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, S^{hyb}: · • ·

Figure 12: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -10$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, S^{hyb}: • • •

Figure 13: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -20$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, S^{hyb}: • • •

Supplementary Material - Bonferroni Type Tests for Return Predictability with Possibly Trending Predictors

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Abstract

The outline of this supplementary paper is as follows. Section S.1 provides proofs of the Theorems in the paper. Section S.2 reports the local asymptotic power of our proposed tests across additional scenarios to those considered in the main paper. Finally, Section S.3 reports results from a Monte Carlo simulation exercise examining the finite sample size and power performance of our proposed tests.

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S.1 Proof of Theorems

It is useful to use the Cholesky decomposition to write

$$\begin{aligned} e_t &= \sigma_e \varepsilon_{1t} \\ u_t &= \sigma_u\{\delta\varepsilon_{1t} + (1-\delta^2)^{1/2}\varepsilon_{2t}\} \end{aligned}$$

where e_t denotes the innovation to v_t, and ε_{1t} and ε_{2t} are independent martingale difference sequences with unit variance. Also, note that we can write as

$$r_t = \alpha^* + \beta_T w_{t-1} + u_t \tag{S.1}$$

with $\alpha^* = \alpha + \beta_T\mu$. Since a constant term is fitted in the regression, for the purposes of the theory we can set $\alpha = \mu = 0$ (and therefore $\alpha^* = 0$) without loss of generality in what follows.

S.1.1 Proof of Theorem 1(a)

First write

$$\begin{aligned} T^{-1/2}\sum_{t=2}^{rT}\varepsilon_{1t} &\overset{w}{\to} W_1(r) \\ T^{-1/2}\sum_{t=2}^{rT}\varepsilon_{2t} &\overset{w}{\to} W_2(r) \end{aligned}$$

where $W_1(r)$ and $W_2(r)$ are independent Brownian motions.

The t_μ statistic can be written as

$$t_\mu = \frac{T^{-1}\sum_{t=2}^{T} x_{t-1}(r_t - \bar{r})}{\sqrt{\sigma_u^2 T^{-2}\sum_{t=2}^{T}(x_{t-1} - \bar{x}_{-1})^2}}$$

Consider first the numerator of t_μ. Using (S.1),

$$\sum_{t=2}^{T} x_{t-1}(r_t - \bar{r}) = \beta_T \sum_{t=2}^{T} x_{t-1}(w_{t-1} - \bar{w}_{-1}) + \sum_{t=2}^{T} x_{t-1}(u_t - \bar{u})$$

Here,

$$
\begin{aligned}
T^{-2}\sum_{t=2}^{T} x_{t-1}(w_{t-1}-\bar{w}_{-1}) &= \sum_{t=2}^{T}\{\gamma_T(t-1)+w_{t-1}\}(w_{t-1}-\bar{w}_{-1}) \\
&= \kappa\omega_v T^{-5/2}\sum_{t=2}^{T}(t-1)(w_{t-1}-\bar{w}_{-1}) + T^{-2}\sum_{t=2}^{T}(w_{t-1}-\bar{w}_{-1})^2 \\
&\stackrel{w}{\to} \kappa\omega_v^2\int_0^1 rW_{1c}^{\mu}(r)dr + \omega_v^2\int_0^1 W_{1c}^{\mu}(r)^2 dr
\end{aligned}
$$

Also,

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{T} x_{t-1}(u_t-\bar{u}) &= T^{-1}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})u_t \\
&= \kappa\omega_v T^{-3/2}\sum_{t=2}^{T}\{(t-1)-\overline{t-1}\}u_t + T^{-1}\sum_{t=2}^{T}(w_{t-1}-\bar{w}_{-1})u_t \\
&= \kappa\omega_v\sigma_u T^{-3/2}\sum_{t=2}^{T}\{(t-1)-\overline{t-1}\}\{\delta\varepsilon_{1t}+(1-\delta^2)^{1/2}\varepsilon_{2t}\} \\
&\quad +\sigma_u T^{-1}\sum_{t=2}^{T}(w_{t-1}-\bar{w}_{-1})\{\delta\varepsilon_{1t}+(1-\delta^2)^{1/2}\varepsilon_{2t}\} \\
&\stackrel{w}{\to} \kappa\omega_v\sigma_u\{\delta\int_0^1(r-0.5)dW_1(r)+(1-\delta^2)^{1/2}\int_0^1(r-0.5)dW_2(r)\} \\
&\quad +\omega_v\sigma_u\{\delta\int_0^1 W_{1c}^{\mu}(r)dW_1(r)+(1-\delta^2)^{1/2}\int_0^1 W_{1c}^{\mu}(r)dW_2(r)\} \\
&= \omega_v\sigma_u\int_0^1 W_{1c}^{\mu,\kappa}(r)\{\delta dW_1(r)+(1-\delta^2)^{1/2}dW_2(r)\}
\end{aligned}
$$

So,

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{T} x_{t-1}(r_t-\bar{r}) &\stackrel{w}{\to} \sigma_u\omega_v b\{\kappa\int_0^1 rW_{1c}^{\mu}(r)dr+\int_0^1 W_{1c}^{\mu}(r)^2dr\}+\sigma_u\omega_v\{\delta\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r) \\
&+ \omega_v\sigma_u(1-\delta^2)^{1/2}\int_0^1 W_{1c}^{\mu,\kappa}dW_2(r)\}
\end{aligned}
$$

Next consider the denominator of t^μ

$$
\begin{aligned}
T^{-2}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2 &= T^{-2}\sum_{t=2}^{T}\{\gamma_T(t-1-\overline{t-1})+(w_{t-1}-\bar{w}_{-1})\}^2 \\
&= \kappa^2\omega_v^2 T^{-3}\sum_{t=2}^{T}(t-1-\overline{t-1})^2+T^{-2}\sum_{t=2}^{T}(w_{t-1}-\bar{w}_{-1})^2 \\
&\quad +2\kappa\omega_v T^{-5/2}\sum_{t=2}^{T}(t-1-\overline{t-1})(w_{t-1}-\bar{w}_{-1}) \\
&\xrightarrow{w} \kappa^2\omega_v^2/12+\omega_v^2\int_0^1 W_{1c}^{\mu}(r)^2dr+2\kappa\omega_v^2\int_0^1 rW_{1c}^{\mu}(r)dr \\
&= \omega_v^2\int_0^1\{\kappa(r-0.5)+W_{1c}^{\mu}(r)\}^2\,dr \\
&= \omega_v^2\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
t_\mu &= \frac{T^{-1}\sum_{t=2}^{T}x_{t-1}(r_t-\bar{r})}{\sqrt{\sigma_u^2T^{-2}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2}} \\
&\xrightarrow{w} \frac{b\{\kappa\int_0^1 rW_{1c}^{\mu}(r)dr+\int_0^1 W_{1c}^{\mu}(r)^2dr\}+\delta\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)+(1-\delta^2)^{1/2}\int_0^1 W_{1c}^{\mu,\kappa}dW_2(r)}{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
&= \frac{b\{\kappa\int_0^1 rW_{1c}^{\mu}(r)dr+\int_0^1 W_{1c}^{\mu}(r)^2dr\}+\delta\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)}{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}}+(1-\delta^2)^{1/2}Z_\mu
\end{aligned}
$$

where $Z_\mu := \{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr\}^{-1/2}\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_2(r)$ is a $N(0,1)$ random variable.

S.1.2 Proof of Theorem 1(b)

Assuming $\tilde{\rho}_T = 1+\tilde{c}/T$ and letting $y_t := (r_t-(\sigma_{u,e}/\sigma_e\omega_v)(x_t-\tilde{\rho}x_{t-1}))$, we can write $Q_\mu(\beta_0,\tilde{\rho})$ with $\beta_0=0$ as

$$
Q_\mu(\beta_0,\tilde{\rho}) = \frac{\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})y_t+\frac{T}{2}(\sigma_{u,e}/\sigma_e\omega_v)(\omega_v^2-\sigma_v^2)}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2}} \tag{S.2}
$$

Turning first to the numerator of (S.2) first note that we can write

$$
\begin{aligned}
y_t &= \beta_T w_{t-1} + u_t - (\sigma_{u,e}/\sigma_e\omega_v)(x_t - \tilde{\rho}x_{t-1}) \\
&= \beta_T w_{t-1} + u_t - (\sigma_{u,e}/\sigma_e\omega_v)(x_t - \rho x_{t-1}) \\
&\quad +(\sigma_{u,e}/\sigma_e\omega_v)T^{-1}(\tilde{c} - c)x_{t-1} \\
&= \beta_T w_{t-1} + u_t - (\sigma_{u,e}/\sigma_e\omega_v)\{v_t - \gamma_T\{cT^{-1}(t-1) - 1\} \\
&\quad +(\sigma_{u,e}/\sigma_e\omega_v)T^{-1}(\tilde{c} - c)x_{t-1}
\end{aligned}
$$

using

$$
\begin{aligned}
x_t - \rho x_{t-1} &= \gamma_T t + w_t - \rho w_{t-1} - \rho\gamma_T(t-1) \\
&= v_t + \gamma_T\{t - (1 + cT^{-1})(t-1)\} \\
&= v_t - \gamma_T\{cT^{-1}(t-1) - 1\}
\end{aligned}
$$

So,

$$
\begin{aligned}
y_t &= \beta_T w_{t-1} + \{u_t - (\sigma_{u,e}/\sigma_e\omega_v)v_t\} + (\sigma_{u,e}/\sigma_e\omega_v)\gamma_T\{cT^{-1}(t-1) - 1\} \\
&\quad +(\sigma_{u,e}/\sigma_e\omega_v)\{T^{-1}(\tilde{c} - c)x_{t-1}\} \qquad \text{(S.3)}
\end{aligned}
$$

Hence we find

$$
\begin{aligned}
Q_\mu(\beta_0, \tilde{\rho}) &= \frac{\beta_T \sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})w_{t-1}}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})^2}} \qquad \text{(S.4)} \\
&\quad + \frac{\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})(u_t - (\sigma_{u,e}/\sigma_e\omega_v)v_t) + \frac{T}{2}(\sigma_{u,e}/\sigma_e\omega_v)(\omega_v^2 - \sigma_v^2)}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})^2}} \qquad \text{(S.5)} \\
&\quad + \frac{(\sigma_{u,e}/\sigma_e\omega_v)\gamma_T cT^{-1}\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})(t-1)}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})^2}} \qquad \text{(S.6)} \\
&\quad + \frac{(\sigma_{u,e}/\sigma_e\omega_v)T^{-1}(\tilde{c} - c)\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})^2}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T (x_{t-1} - \bar{x}_{-1})^2}} \qquad \text{(S.7)}
\end{aligned}
$$

We now examine the limit of each of the terms (S.4)-(S.7) in turn. Beginning with (S.4), we can write it as

$$\begin{aligned}
&\frac{(b\sigma_u/\omega_v)T^{-2}\sum_{t=2}^{T}x_{t-1}(w_t-\bar{w}_{-1})}{(1-\delta^2)^{1/2}\sigma_u\sqrt{T^{-2}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2}} \\
&\xrightarrow{w} \frac{(b\sigma_u/\omega_v)\{\kappa\omega_v^2\int_0^1 rW_{1c}^{\mu}(r)dr+\omega_v^2\int_0^1 W_{1c}^{\mu}(r)^2dr\}}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\omega_v^2\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
= \;& \frac{b\{\kappa\int_0^1 rW_{1c}^{\mu}(r)dr+\int_0^1 W_{1c}^{\mu}(r)^2dr\}}{(1-\delta^2)^{1/2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \qquad \text{(S.8)}
\end{aligned}$$

For (S.5) we note that

$$T^{-1}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})v_t \xrightarrow{w} \omega_v^2\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)+\frac{1}{2}(\omega_v^2-\sigma_v^2)$$

Then,

$$\begin{aligned}
&\frac{T^{-1}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})(u_t-\frac{\sigma_{ue}}{\sigma_e\omega_v}v_t)+\frac{1}{2}\frac{\sigma_{ue}}{\sigma_e\omega_v}(\omega_v^2-\sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}(T^{-2}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2)^{1/2}} \\
= \;& \frac{T^{-1}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})u_t-\frac{\sigma_{ue}}{\sigma_e\omega_v}T^{-1}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})v_t+\frac{1}{2}\frac{\sigma_{ue}}{\sigma_e\omega_v}(\omega_v^2-\sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}(T^{-2}\sum_{t=2}^{T}(x_{t-1}-\bar{x}_{-1})^2)^{1/2}} \\
&\xrightarrow{w} \frac{\omega_v\sigma_u\int_0^1 W_{1c}^{\mu,\kappa}(r)\{\delta dW_1(r)+(1-\delta^2)^{1/2}dW_2(r)\}-\frac{\sigma_{ue}}{\sigma_e\omega_v}\omega_v^2\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)}{\sigma_u(1-\delta^2)^{1/2}\omega_v\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
+ \;& \frac{-\frac{1}{2}\frac{\sigma_{ue}}{\sigma_e\omega_v}(\omega_v^2-\sigma_v^2)+\frac{1}{2}\frac{\sigma_{ue}}{\sigma_e\omega_v}(\omega_v^2-\sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}\omega_v\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
= \;& \frac{\delta\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)+(1-\delta^2)^{1/2}\int_0^1 W_{1c}^{\mu,\kappa}dW_2(r)-\delta\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)}{(1-\delta^2)^{1/2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
= \;& \frac{\int_0^1 W_{1c}^{\mu,\kappa}(r)dW_2(r)}{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \qquad \text{(S.9)} \\
= \;& Z_{\mu}\sim N(0,1) \qquad \text{(S.10)}
\end{aligned}$$

For (S.6),

$$
\begin{aligned}
&\frac{(\sigma_{u,e}/\sigma_e\omega_v)\gamma_T c T^{-1}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})(t-1)}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}} \\
=\ &\frac{(\sigma_{u,e}/\sigma_e\omega_v)c\kappa\omega_v T^{-3/2}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})(t-1)}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}} \\
=\ &\frac{\delta c\kappa T^{-5/2}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})(t-1)}{(1-\delta^2)^{1/2}\sqrt{T^{-2}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}} \\
\overset{w}{\to}\ &\frac{\delta c\kappa\omega_v\int_0^1 rW_{1c}^{\mu,\kappa}(r)dr}{(1-\delta^2)^{1/2}\omega_v\{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
=\ &\frac{\delta c\kappa\int_0^1 rW_{1c}^{\mu,\kappa}(r)dr}{(1-\delta^2)^{1/2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \qquad \text{(S.11)}
\end{aligned}
$$

Finally, for (S.7),

$$
\begin{aligned}
&\frac{(\sigma_{u,e}/\sigma_e\omega_v)T^{-1}(\tilde{c}-c)\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}{(1-\delta^2)^{1/2}\sigma_u\sqrt{\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}} \\
=\ &\frac{\delta\omega_v^{-1}(\tilde{c}-c)T^{-2}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}{(1-\delta^2)^{1/2}\sqrt{T^{-2}\sum_{t=2}^T(x_{t-1}-\bar{x}_{-1})^2}} \\
\overset{w}{\to}\ &\frac{\delta\omega_v^{-1}(\tilde{c}-c)\omega_v^2\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}{(1-\hat{\delta}^2)^{1/2}\omega_v\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \\
=\ &\frac{\delta(\tilde{c}-c)\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}{(1-\delta^2)^{1/2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}} \qquad \text{(S.12)}
\end{aligned}
$$

Combining results we therefore have that

$$
Q_\mu(\beta_0,\tilde{\rho}) \overset{w}{\to} \frac{b\{\kappa\int_0^1 rW_{1c}^{\mu}(r)dr+\int_0^1 W_{1c}^{\mu}(r)^2dr\}+\delta c\kappa\int_0^1 rW_{1c}^{\mu,\kappa}(r)dr+\delta(\tilde{c}-c)\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr}{(1-\delta^2)^{1/2}\{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2dr\}^{1/2}}+Z_\mu
$$

S.1.3 Proof of Theorem 1(c)

Write the t_τ statistic as

$$
\begin{aligned}
t^\tau &= \frac{b\sigma_u\omega_v^{-1}(T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2)^{1/2}}{\sigma_u} + \frac{T^{-1}\sum_{t=2}^T x_{\tau,t-1}u_t}{\sigma_u\sqrt{T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2}} \\
&\xrightarrow{w} b(\int_0^1 W_{1c}^\tau(r)^2 dr)^{1/2} + \frac{\delta\int_0^1 W_{1c}^\tau(r)dW_1(r) + (1-\delta^2)^{1/2}\int_0^1 W_{1c}^\tau(r)dW_2(r)}{\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}} \\
&= b(\int_0^1 W_{1c}^\tau(r)^2 dr)^{1/2} + \delta\frac{\int_0^1 W_{1c}^\tau(r)dW_1(r)}{\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}} + (1-\delta^2)^{1/2}Z_\tau
\end{aligned}
$$

where $Z_\tau := \left(\int_0^1 W_{1c}^\tau(r)^2 dr\right)^{-1/2}\int_0^1 W_{1c}^\tau(r)dW_2(r)$ is a $N(0,1)$ random variable.

S.1.4 Proof of Theorem 1(d)

The $Q_\tau(\beta_0,\tilde{\rho})$ statistic with $\beta_0 = 0$ can be written as

$$
\begin{aligned}
Q_\tau(\beta_0,\tilde{\rho}) &= \frac{b\omega_v^{-1}(T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2)^{1/2}}{(1-\delta^2)^{1/2}} + \frac{\delta(\tilde{c}-c)(T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2)^{1/2}}{\omega_v(1-\delta^2)^{1/2}} \\
&\quad + \frac{T^{-1}\sum_{t=2}^T x_{\tau,t-1}(u_t - \frac{\sigma_{ue}}{\sigma_e\omega_v}v_t) + \frac{1}{2}\frac{\sigma_{ue}}{\sigma_e\omega_v}(\omega_v^2-\sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}\sqrt{T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2}} \qquad \text{(S.13)}
\end{aligned}
$$

We will derive limiting expressions for each of the three terms on the right hand side of (S.13) Here

$$
\frac{b\omega_v^{-1}(T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2)^{1/2}}{(1-\delta^2)^{1/2}} \xrightarrow{w} \frac{b\{\int_0^1 W_{1c}^\tau(r)^2 dr\}^{1/2}}{(1-\delta^2)^{1/2}} \qquad \text{(S.14)}
$$

and

$$
\frac{\delta(\tilde{c}-c)(T^{-2}\sum_{t=2}^T x_{\tau,t-1}^2)^{1/2}}{\omega_v(1-\delta^2)^{1/2}} \xrightarrow{w} \frac{\delta(\tilde{c}-c)\left(\int_0^1 W_{1c}^\tau(r)^2 dr\right)^{1/2}}{(1-\delta^2)^{1/2}} \qquad \text{(S.15)}
$$

Finally,

$$
\begin{aligned}
& \frac{T^{-1}\sum_{t=2}^{T} x_{\tau,t-1}(u_t - \frac{\sigma_{ue}}{\sigma_e \omega_v} v_t) + \frac{1}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}\sqrt{T^{-2}\sum_{t=2}^{T} x_{\tau,t-1}^2}} \\
= \; & \frac{T^{-1}\sum_{t=2}^{T} x_{\tau,t-1}u_t - \frac{\sigma_{ue}}{\sigma_e \omega_v} T^{-1}\sum_{t=2}^{T} x_{\tau,t-1}v_t + \frac{1}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}\sqrt{T^{-2}\sum_{t=2}^{T} x_{\tau,t-1}^2}} \\
& \xrightarrow{w} \frac{\sigma_u \omega_v \int_0^1 W_{1c}^{\tau}(s)\{\delta dW_1(r) + (1-\delta^2)^{1/2} dW_2(r)\} - \frac{\sigma_{ue}}{\sigma_e \omega_v}\omega_v^2 \int_0^1 W_{1c}^{\tau}(s) dW_1(s)}{\sigma_u(1-\delta^2)^{1/2}\omega_v\sqrt{\int_0^1 W_{1c}^{\tau}(r)^2 dr}} \\
+ \; & \frac{-\frac{1}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2) + \frac{1}{2}\frac{\sigma_{ue}}{\sigma_e \omega_v}(\omega_v^2 - \sigma_v^2)}{\sigma_u(1-\delta^2)^{1/2}\omega_v\sqrt{\int_0^1 W_{1c}^{\tau}(r)^2 dr}} \\
= \; & \frac{\int_0^1 W_{1c}^{\tau}(s) dW_2(s)}{\sqrt{\int_0^1 W_{1c}^{\tau}(r)^2 dr}} \\
= \; & Z_{\tau} \sim N(0,1)
\end{aligned}
\tag{S.16}
$$

Combining results we obtain

$$
\begin{aligned}
Q_{\tau}(\beta_0, \tilde{\rho}) \; & \xrightarrow{w} \; \frac{b\{\int_0^1 W_{1c}^{\tau}(r)^2 dr\}^{1/2}}{(1-\delta^2)^{1/2}} + \frac{\delta(\tilde{c}-c)\{\int_0^1 W_{,1c}^{\tau}(r)^2 dr\}^{1/2}}{(1-\delta^2)^{1/2}} + Z_{\tau} \\
& = \; \frac{\{b + \delta(\tilde{c}-c)\}\{\int_0^1 W_{1c}^{\tau}(r)^2 dr)^{1/2}\}}{(1-\delta^2)^{1/2}} + Z_{\tau}.
\end{aligned}
$$

S.2 Additional Local Asymptotic Power Simulations

In this section we report additional asymptotic simulation results to those reported in the main paper. Figures S.1 - S.3 report local asymptotic power for left-tailed tests for predictability with $\delta = -0.95$ and $c = -30, -40, -50$. Finally, the local asymptotic power of the tests for predictability for an explosive predictor with $c = 2$ and $\delta = -0.95$ are reported in Figures S.4 - S.5, with Figure S.4 reporting the power of right-tailed tests, and Figure S.5 reporting the power of left-tailed tests. Additional results for $\delta = -0.75$ can be found in the on-line supplementary appendix which can be found at https://rtaylor-essex.droppages.com/esrc2/default.htm.

Figure S.1: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -30$

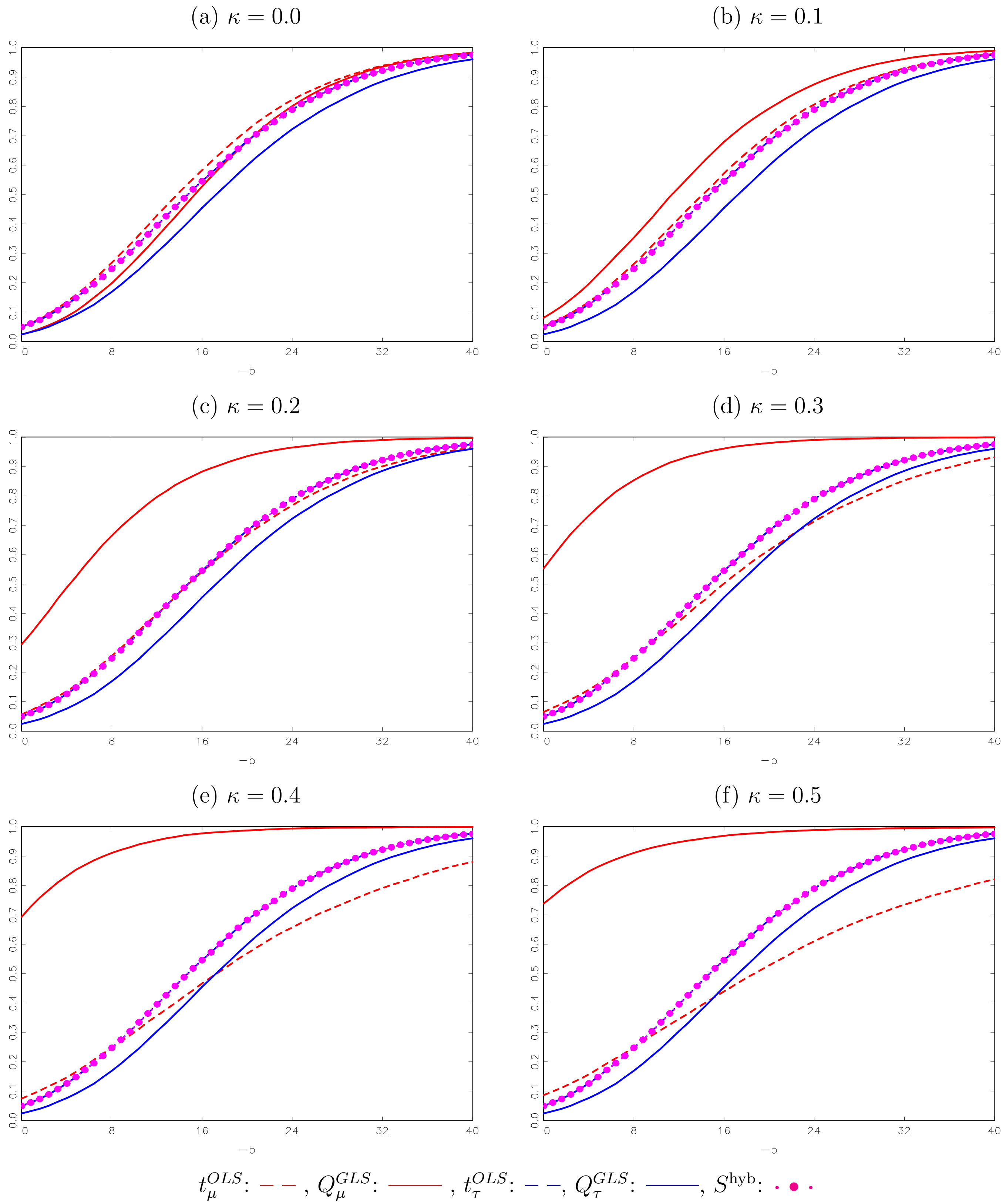
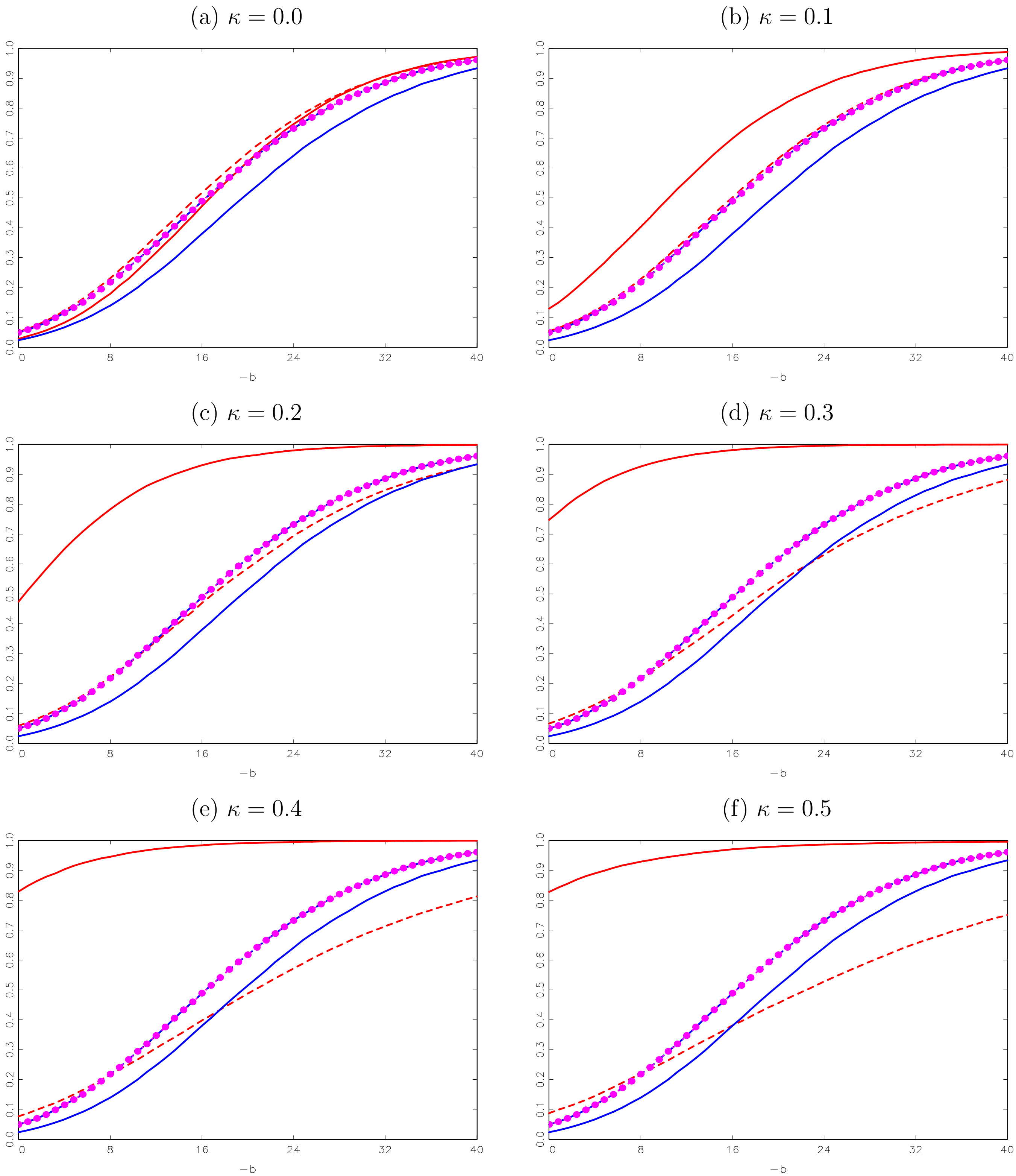
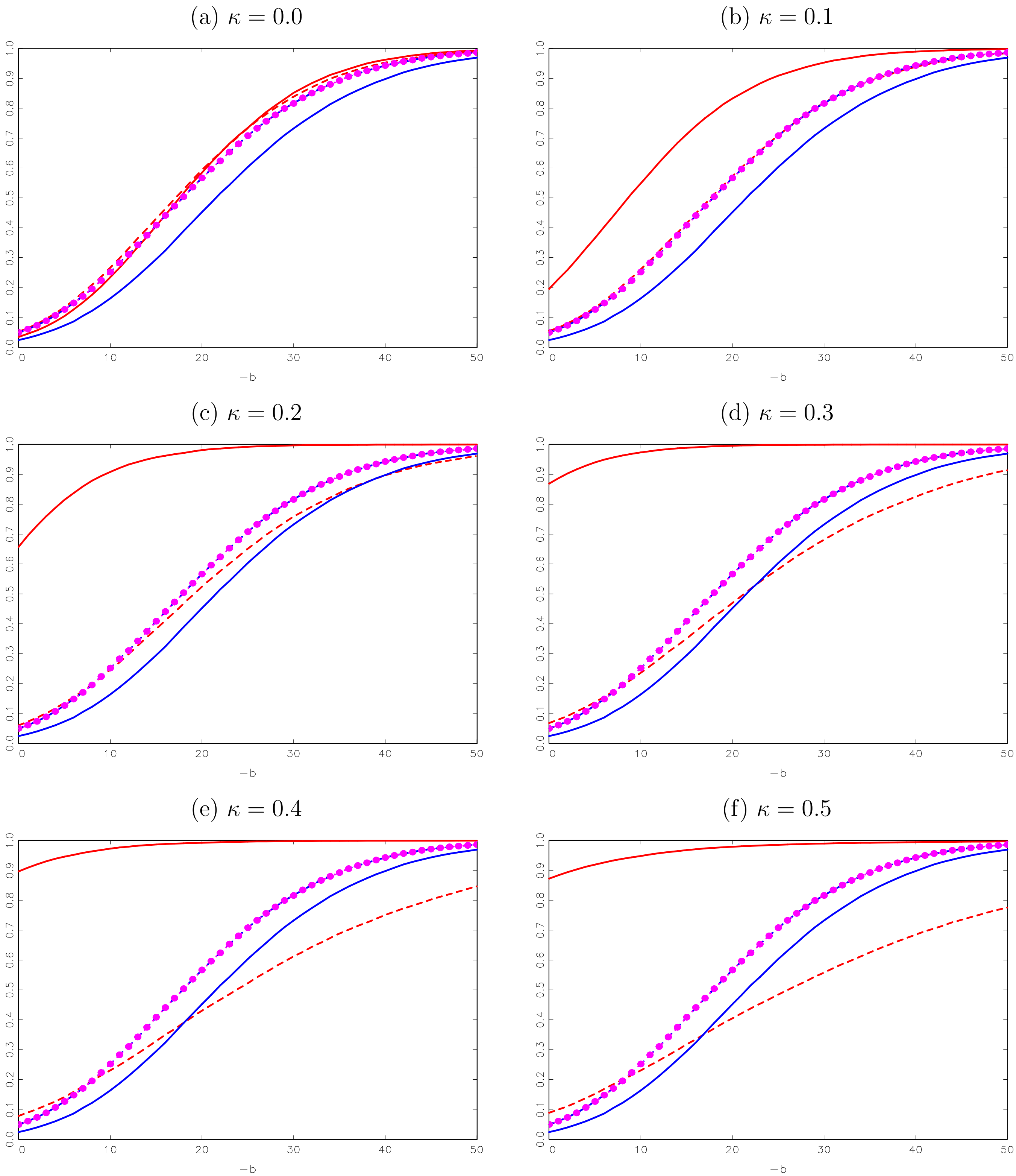


Figure S.2: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -40$



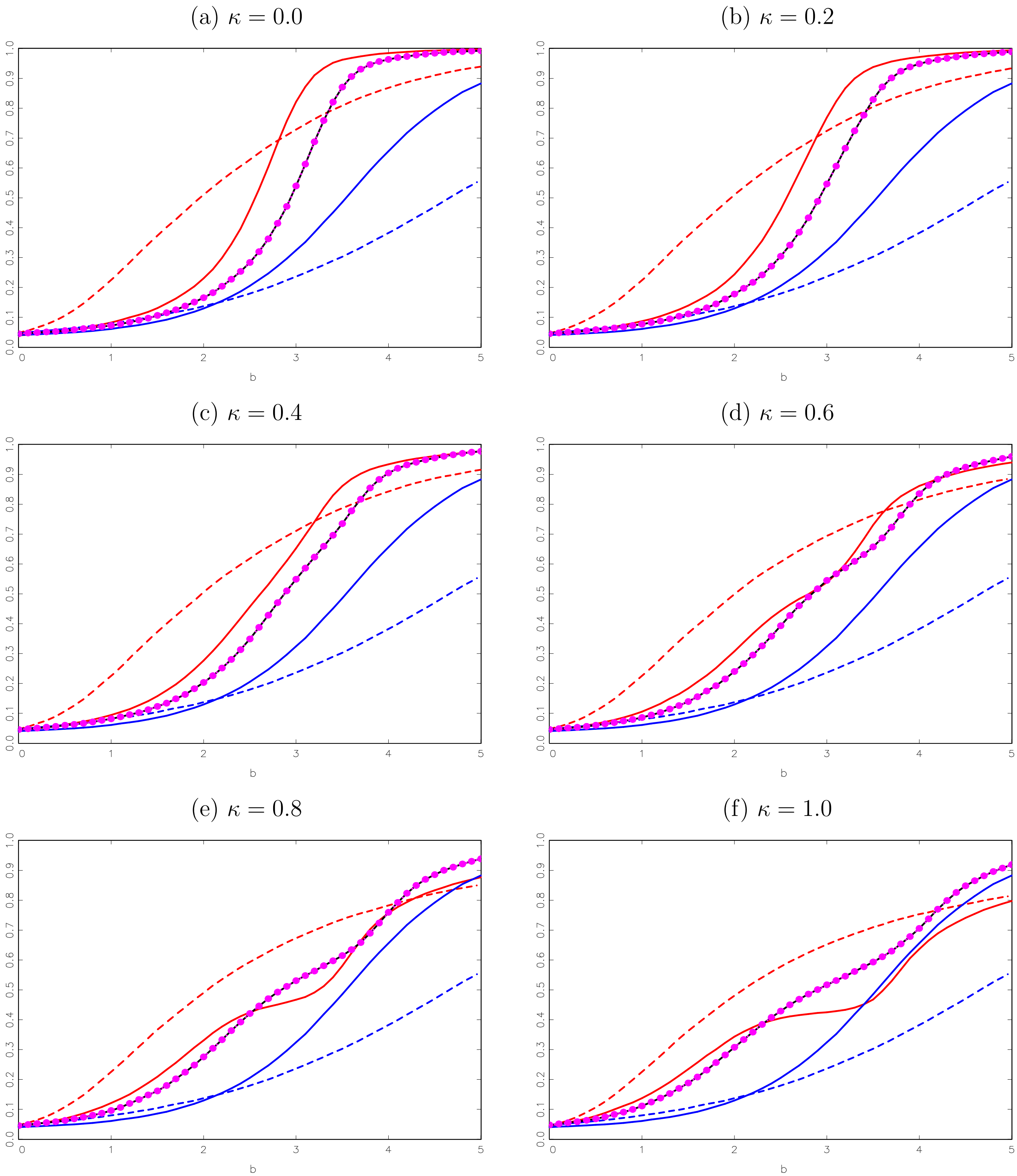
t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, S^{hyb}: • ● •

Figure S.3: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = -50$



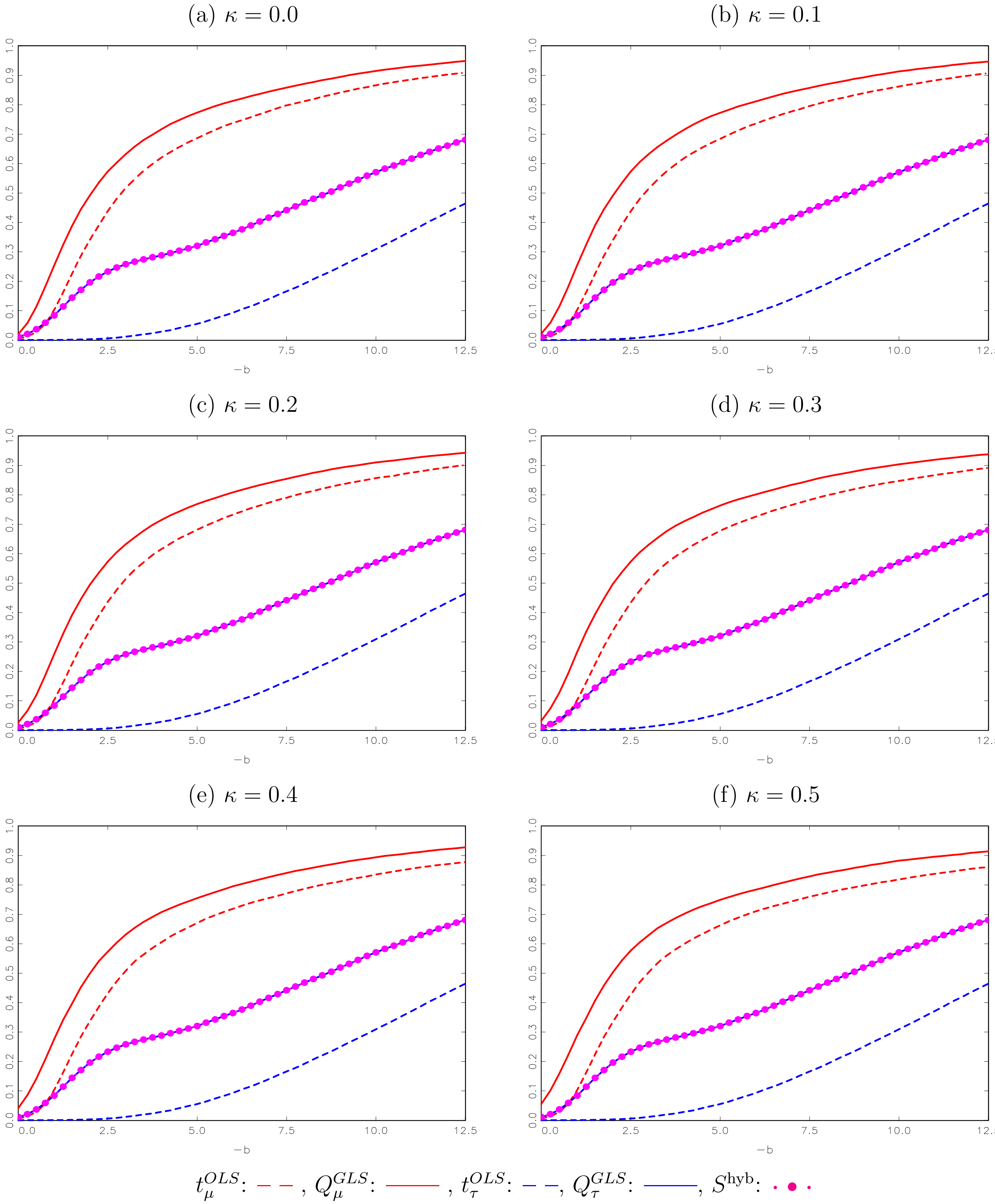
t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, S^{hyb}: • • •

Figure S.4: Local Asymptotic Power of Right Tailed Tests - $\delta = -0.95$, $c = 2$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.5: Local Asymptotic Power of Left Tailed Tests - $\delta = -0.95$, $c = 2$



S.3 Finite Sample Size and Power

In this section we evaluate the finite sample size and power of tests for predictability discussed in this paper. To do so data were generated according to (1) - (3) with $v_t = \phi v_{t-1} + e_t$ where $e_t \sim NIID(0,1)$, setting $w_1 = v_1 = e_1$. We set $T = 250$ and generate data according to Assumptions S and T such that $\rho = 1 + cT^{-1}$ and $\gamma_T = \kappa\omega_v T^{-1/2}$, noting that for larger negative values of c the predictor will behave more like a weakly stationary process in finite samples. All tests are performed at a nominal level of 0.05. Following CY, lag selection for all of the unit root tests utilised in the test procedures is performed using the Bayes Information Criterion (BIC) with a maximum number of lagged differences of 4. Finally we set $\upsilon = 10$ in (17) such that our hybrid S^{hyb} and U^{hyb} tests switch into the conventional t-test whenever $NB\text{-}OLS_\tau < -10T^{1/2}$ as we found this choice of υ delivered good finite sample performance across a wide range of DGPs.

S.3.1 Finite Sample Size

We being by examining the finite sample size of the tests. We first report result for $\phi = 0.0$ such that v_t is an i.i.d. process, and for $c = 2, 0, -2, -5, -10, -20, -30, -40, -50, -100, -250$, with the final setting clearly corresponding to weak persistence ($\rho = 0$ when $c = -250$). We report results for $\kappa = 0$ in Table S.1 and $\kappa = 0.5 + 0.5I(c > -20)$ in Table S.2, where we make κ dependent on c in the latter scenario due to the impact of κ on the size of the tests being greater the more negative is the value of c.

Turning first to Table S.1 we see that for right-tailed tests with $\kappa = 0$ and $\phi = 0.0$ all tests are well size controlled for $c > -50$, with this result unsurprising given that all tests are designed to be asymptotically size controlled when ρ is local-to-unity, with the t_μ^{OLS} and t_τ^{OLS} tests retaining size control across all other values of c. As c becomes more negative we do see some size distortions for the Q_d^{GLS} tests, as for these values of c the predictor will be behaving more like a weakly stationary process, in which case these tests are asymptotically invalid. While the Q_μ^{GLS} test displays severe size distortions only for $c = -250$, the Q_τ^{GLS} test also suffers severe size distortions for $c = -100$. As a consequence, the U^{hyb} test does suffer from severe size distortions for $c = -100$ as while U^{hyb} is correctly switching into the conventional t-test in almost all (99.9%) of replications for $c = -250$, and is therefore correctly sized, this will not be true for $c = -100$ and the oversize of Q_τ^{GLS} in this scenario feeds through into the size of U^{hyb}. This renders the U^{hyb} test potentially unreliable. The size of the S^{hyb} test, on the other hand, is well controlled across all values of c. This is due to the fact for intermediate values of c that span the gap between strongly persistent and weakly persistent predictors this test will be switching into the size controlled t_τ^{OLS} test with very high probability.

For left tailed tests with $\kappa = 0$ and $\phi = 0.0$ we observe that, with the exception of the Q_μ^{GLS} test for larger negative values of c, all tests display reasonable size control across all values of c. While the tests have very low size for values of c closer to zero this is in line with the asymptotic size of the tests when maximising size at 0.05 across a large range of c. We see that the size of S^{hyb} is identical to that of Q_τ^{GLS} for c close to 0, and to that of t_τ^{OLS} for more negative values of c (with the exception of $c = -250$ where S^{hyb} is almost always switching into conventional t) demonstrating that the switching rule in (16) is effective in finite samples.

We now turn to Table S.2 which reports results for a large positive value of κ. First we observe that the size of the Q_τ^{GLS} and t_τ^{OLS} tests are identical to those in Table S.1 for both right and left tailed tests due to these tests being invariant to the value of κ. For right (left) tail tests we see that both the Q_μ^{GLS} and t_μ^{GLS} tests can be severely undersized (oversized), with this undersize (oversize) more pronounced the more negative is the value of c for a given value of κ. For right tailed tests we see that for c close to zero the size of both U^{hyb} and S^{hyb} is slightly lower relative to the case where $\kappa = 0$, and we will see that this translates into a loss of power for these tests relative to when $\kappa = 0$, although the power of these tests for $\kappa \neq 0$ will be shown to still be close to that of the most powerful test in each scenario. For left tailed tests the S^{hyb} test has identical size to that seen in Table S.1 as this test is a function of two tests that are both invariant to κ.

We now briefly discuss the results for the size of the tests when $\phi = 0.5$ so that the predictor is generated as an AR(2) process. We report results only for $c = 2, 0, -2, -5, -10, -20, -30, -40, -50$ so that the serial correlation induced by the value of $\rho = 1 + cT^{-1}$ remains the dominant driver of the persistence of the predictor. Table S.3 reports the size of the tests when $\kappa = 0$ and Table S.4 reports results for when

$\kappa = 0.5 + 0.5I(c > -20)$. While the size of the tests is not identical to the case when $\phi = 0.0$, the difference in size between $\phi = 0.5$ and $\phi = 0.0$ is minimal in a vast majority of cases. This is likely due to the fact that we are using the BIC to select the AR order for the predictor, which selects the true order in a vast majority of instances.

S.3.2 Finite Sample Power

We now examine the finite sample power properties of all tests. We begin by reporting power for both right and left tailed tests for $c = 0, -2, -5, -10, -20, -30, -40, -50, -100, -250$ and $\delta = -0.95$[1], all across various values of κ. We then briefly discuss the relative power performance of the tests for an explosive predictor with $c = 2$.

We first examine the finite sample power of right-tailed tests for predictability reported in Figures S.6 - S.15. The power of the tests when $\kappa = 0$ is reported in panel (a) of each figure, with these results mirroring those found for local asymptotic power in Sections 4.1 and 6 where the best overall power performance for c close to zero is displayed by the Q_μ^{GLS} test. For the more negative values of $c <= -30$ the best power is displayed by the t_μ^{OLS} test. For c close to zero we see that, much like when examining local asymptotic power, the finite sample power of the hybrid U^{hyb} and S^{hyb} tests is very close to that of the best performing Q_μ^{GLS} test. For larger negative values of c the power of the U^{hyb} test is less competitive, and for $c = -100$ the test is oversized, as noted above. The S^{hyb} test, on the other hand, is among the better performing tests for all values of c, owing to this test basing inference on the t_τ^{OLS} test with increasing probability as c becomes more negative. For the largest value of c considered ($c = -250$), both U^{hyb} and S^{hyb} have switched into the standard t_τ test in almost all replications, and consequently display an attractive power profile.

The power of right-tailed tests when $\kappa > 0$ is reported in panels (b) - (f) of Figures S.6 - S.15. Again, these results closely mirror those found for the local asymptotic power of the tests, with the power of the Q_μ^{GLS} and t_μ^{OLS} tests falling away as the value of κ increases, and the power of Q_τ^{GLS} and t_τ^{OLS} invariant to the value of κ. For c close to zero the power of the hybrid U^{hyb} and S^{hyb} tests tracks close to the most powerful Q_μ^{GLS} test for small κ, while for larger κ, the hybrid tests closely track the power of the better performing Q_τ^{GLS}

[1]Additional results for $\delta = -0.75$ can be found in the on-line supplementary appendix which can be found at https://rtaylor-essex.droppages.com/esrc2/default.htm.

test, hence the U^{hyb} and S^{hyb} tests are among the most powerful tests regardless of the value of κ. For larger negative values of c we see that S^{hyb} continues to be among the most powerful tests given that it is increasingly likely to switch into the t_τ^{OLS} test, which performs well in this region, as the value of c decreases.

Results for left tailed tests are reported in Figures S.16 - S.25. As with the local asymptotic power results, although the Q_μ^{GLS} and t_μ^{OLS} tests perform well when $\kappa = 0$, these tests suffer from substantial oversize when $\kappa \neq 0$. Among the with-trend tests, the Q_τ^{GLS} test displays the best overall power for c close to zero, and the t_τ^{OLS} test peforms best for more negative values of c. With the exception of the case $c = -250$, the U^{hyb} test here reduces to Q_τ^{GLS}, and therefore does not perform well unless c is close to zero. On the other hand, the hybrid S^{hyb} test is able to capture the superior power of the best performing test in each scenario, tracking closely the power of Q_τ^{GLS} for c close to zero, and that of t_τ^{OLS} for other values of c.

Finally, the finite sample power of the tests for an explosive predictor with $c = 2$ and $\delta = -0.95$ are reported in Figures S.26 - S.27. Figure S.26 reports power of right-tailed tests. The main differences that we see compared to the previous values of c considered is that in the explosive predictor case the best overall power performance is, in fact, delivered by the t_μ^{OLS} test, with the impact of an omitted trend on the constant-only tests less pronounced than for $c \leq 0$. Figure S.27 reports power of left-tailed tests. Much like with right-tailed tests with an explosive predictor, the presence of an omitted trend has minimal impact on the constant-only tests such that Q_μ^{GLS} and t_μ^{OLS} are the best performing tests. While for an explosive predictor the constant-only tests appear to remain the better performing tests even for relatively large values of κ, this does not change our recommendation to use our proposed hybrid tests in practice given that a predictor that is explosive for the entire sample period is extremely unlikely to be observed in empirical practice.

Overall, we have demonstrated that the S^{hyb} test, in particular, is very well suited to testing for predictability when uncertainty exists over the presence of a trend. For both right and left tailed tests S^{hyb} displays excellent size control, and has power that is never far behind that of the best performing test in each scenario considered across a very wide range of values of c and κ.

Table S.1: Finite Sample Size, $T = 250$, $\phi = 0.0$, $\kappa = 0$.

(a) Right Tailed Tests								(b) Left Tailed Tests							
c	δ	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}	c	δ	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.049	0.048	0.047	0.053	0.055	0.055	2	-0.95	0.017	0.005	0.009	0.000	0.009	0.009
	-0.75	0.046	0.049	0.055	0.050	0.056	0.056		-0.75	0.013	0.010	0.013	0.000	0.013	0.013
	-0.50	0.045	0.050	0.054	0.047	0.052	0.052		-0.50	0.021	0.023	0.018	0.005	0.018	0.018
	-0.25	0.047	0.049	0.053	0.050	0.053	0.053		-0.25	0.035	0.030	0.029	0.027	0.029	0.029
0	-0.95	0.049	0.051	0.047	0.036	0.052	0.052	0	-0.95	0.010	0.004	0.023	0.000	0.023	0.023
	-0.75	0.052	0.052	0.045	0.038	0.054	0.054		-0.75	0.010	0.010	0.013	0.000	0.013	0.013
	-0.50	0.055	0.051	0.046	0.040	0.052	0.052		-0.50	0.018	0.021	0.013	0.007	0.013	0.013
	-0.25	0.057	0.052	0.048	0.046	0.049	0.049		-0.25	0.030	0.028	0.025	0.028	0.025	0.026
-2	-0.95	0.050	0.044	0.039	0.025	0.044	0.044	-2	-0.95	0.010	0.011	0.024	0.000	0.024	0.024
	-0.75	0.051	0.038	0.033	0.024	0.044	0.044		-0.75	0.009	0.018	0.019	0.002	0.019	0.019
	-0.50	0.054	0.041	0.035	0.029	0.041	0.042		-0.50	0.016	0.029	0.019	0.014	0.019	0.019
	-0.25	0.055	0.045	0.041	0.039	0.041	0.041		-0.25	0.029	0.034	0.029	0.036	0.029	0.030
-5	-0.95	0.049	0.044	0.036	0.028	0.041	0.042	-5	-0.95	0.013	0.034	0.016	0.001	0.016	0.016
	-0.75	0.049	0.034	0.031	0.018	0.038	0.038		-0.75	0.010	0.035	0.013	0.007	0.013	0.013
	-0.50	0.051	0.034	0.031	0.021	0.036	0.036		-0.50	0.015	0.041	0.017	0.026	0.017	0.017
	-0.25	0.052	0.038	0.036	0.031	0.036	0.036		-0.25	0.028	0.041	0.031	0.043	0.031	0.032
-10	-0.95	0.045	0.047	0.039	0.039	0.038	0.038	-10	-0.95	0.019	0.046	0.017	0.006	0.017	0.017
	-0.75	0.044	0.039	0.033	0.020	0.035	0.036		-0.75	0.012	0.045	0.013	0.026	0.013	0.013
	-0.50	0.047	0.034	0.031	0.018	0.031	0.032		-0.50	0.017	0.045	0.018	0.042	0.018	0.021
	-0.25	0.047	0.034	0.034	0.026	0.032	0.032		-0.25	0.028	0.045	0.029	0.049	0.029	0.035
-20	-0.95	0.038	0.049	0.043	0.047	0.034	0.038	-20	-0.95	0.035	0.046	0.017	0.046	0.017	0.026
	-0.75	0.036	0.045	0.036	0.034	0.031	0.034		-0.75	0.020	0.046	0.014	0.050	0.014	0.033
	-0.50	0.039	0.042	0.033	0.026	0.028	0.030		-0.50	0.021	0.047	0.017	0.049	0.017	0.040
	-0.25	0.045	0.042	0.037	0.028	0.030	0.031		-0.25	0.031	0.046	0.029	0.050	0.029	0.047
-30	-0.95	0.034	0.050	0.053	0.049	0.037	0.047	-30	-0.95	0.067	0.048	0.019	0.048	0.019	0.048
	-0.75	0.032	0.045	0.041	0.043	0.031	0.040		-0.75	0.035	0.049	0.014	0.049	0.014	0.049
	-0.50	0.036	0.046	0.037	0.038	0.028	0.037		-0.50	0.028	0.047	0.017	0.049	0.017	0.049
	-0.25	0.043	0.047	0.039	0.038	0.029	0.036		-0.25	0.036	0.048	0.029	0.051	0.029	0.051
-40	-0.95	0.032	0.049	0.067	0.050	0.046	0.050	-40	-0.95	0.107	0.049	0.020	0.047	0.020	0.047
	-0.75	0.030	0.047	0.050	0.046	0.036	0.045		-0.75	0.057	0.048	0.014	0.048	0.014	0.048
	-0.50	0.034	0.047	0.042	0.044	0.029	0.042		-0.50	0.041	0.049	0.017	0.049	0.017	0.049
	-0.25	0.042	0.049	0.041	0.044	0.029	0.042		-0.25	0.041	0.048	0.027	0.051	0.027	0.051
-50	-0.95	0.032	0.049	0.085	0.051	0.060	0.051	-50	-0.95	0.150	0.048	0.022	0.047	0.022	0.047
	-0.75	0.030	0.048	0.064	0.047	0.046	0.047		-0.75	0.084	0.048	0.016	0.049	0.016	0.049
	-0.50	0.033	0.049	0.051	0.047	0.033	0.046		-0.50	0.054	0.048	0.017	0.049	0.017	0.049
	-0.25	0.041	0.051	0.045	0.048	0.030	0.048		-0.25	0.046	0.049	0.028	0.051	0.028	0.051
-100	-0.95	0.050	0.047	0.316	0.048	0.259	0.048	-100	-0.95	0.315	0.050	0.040	0.052	0.037	0.052
	-0.75	0.042	0.047	0.248	0.049	0.203	0.049		-0.75	0.232	0.051	0.026	0.052	0.024	0.052
	-0.50	0.036	0.049	0.161	0.049	0.118	0.049		-0.50	0.148	0.051	0.018	0.051	0.017	0.051
	-0.25	0.037	0.051	0.085	0.051	0.055	0.051		-0.25	0.079	0.051	0.020	0.052	0.020	0.052
-250	-0.95	0.281	0.040	0.857	0.036	0.065	0.065	-250	-0.95	0.447	0.065	0.048	0.074	0.039	0.039
	-0.75	0.274	0.041	0.837	0.038	0.061	0.061		-0.75	0.407	0.062	0.039	0.067	0.042	0.041
	-0.50	0.207	0.045	0.793	0.041	0.057	0.057		-0.50	0.345	0.059	0.030	0.062	0.044	0.043
	-0.25	0.099	0.048	0.594	0.047	0.053	0.054		-0.25	0.205	0.053	0.018	0.055	0.046	0.046

Table S.2: Finite Sample Size, $T = 250$, $\phi = 0.0$ $\kappa = 0.5 + 0.5I(c > -20)$.

(a) Right Tailed Tests								(b) Left Tailed Tests							
c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}	c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.050	0.048	0.047	0.053	0.055	0.055	2	-0.95	0.046	0.006	0.009	0.000	0.009	0.009
	-0.75	0.047	0.047	0.055	0.050	0.057	0.057		-0.75	0.030	0.011	0.013	0.000	0.013	0.013
	-0.50	0.045	0.047	0.054	0.047	0.053	0.053		-0.50	0.030	0.026	0.018	0.005	0.018	0.018
	-0.25	0.045	0.047	0.053	0.050	0.051	0.051		-0.25	0.040	0.033	0.029	0.027	0.029	0.029
0	-0.95	0.030	0.044	0.047	0.036	0.044	0.044	0	-0.95	0.042	0.011	0.023	0.000	0.023	0.023
	-0.75	0.034	0.045	0.045	0.038	0.043	0.043		-0.75	0.027	0.017	0.013	0.000	0.013	0.013
	-0.50	0.039	0.044	0.046	0.040	0.043	0.043		-0.50	0.030	0.029	0.013	0.007	0.013	0.013
	-0.25	0.046	0.047	0.048	0.046	0.046	0.046		-0.25	0.038	0.033	0.025	0.028	0.025	0.026
-2	-0.95	0.010	0.029	0.039	0.025	0.027	0.027	-2	-0.95	0.069	0.032	0.024	0.000	0.024	0.024
	-0.75	0.014	0.028	0.033	0.024	0.027	0.027		-0.75	0.038	0.036	0.019	0.002	0.019	0.019
	-0.50	0.023	0.032	0.035	0.029	0.030	0.030		-0.50	0.039	0.043	0.019	0.014	0.019	0.019
	-0.25	0.038	0.041	0.041	0.039	0.039	0.039		-0.25	0.043	0.039	0.029	0.036	0.029	0.030
-5	-0.95	0.001	0.020	0.036	0.028	0.023	0.023	-5	-0.95	0.161	0.064	0.016	0.001	0.016	0.016
	-0.75	0.004	0.020	0.031	0.018	0.021	0.021		-0.75	0.077	0.061	0.013	0.007	0.013	0.013
	-0.50	0.013	0.025	0.031	0.021	0.024	0.024		-0.50	0.058	0.060	0.017	0.026	0.017	0.017
	-0.25	0.031	0.035	0.036	0.031	0.033	0.033		-0.25	0.055	0.045	0.031	0.043	0.031	0.032
-10	-0.95	0.000	0.011	0.039	0.039	0.024	0.025	-10	-0.95	0.302	0.101	0.017	0.006	0.017	0.017
	-0.75	0.002	0.012	0.033	0.020	0.022	0.023		-0.75	0.138	0.085	0.013	0.026	0.013	0.013
	-0.50	0.010	0.019	0.031	0.018	0.023	0.024		-0.50	0.086	0.074	0.018	0.042	0.018	0.021
	-0.25	0.027	0.030	0.034	0.026	0.032	0.032		-0.25	0.066	0.052	0.029	0.049	0.029	0.035
-20	-0.95	0.001	0.029	0.043	0.047	0.027	0.037	-20	-0.95	0.341	0.079	0.017	0.046	0.017	0.026
	-0.75	0.003	0.027	0.036	0.034	0.023	0.029		-0.75	0.162	0.069	0.014	0.050	0.014	0.033
	-0.50	0.009	0.027	0.033	0.026	0.022	0.027		-0.50	0.095	0.061	0.017	0.049	0.017	0.040
	-0.25	0.025	0.034	0.037	0.028	0.031	0.032		-0.25	0.068	0.052	0.029	0.050	0.029	0.047
-30	-0.95	0.001	0.028	0.053	0.049	0.035	0.047	-30	-0.95	0.460	0.080	0.019	0.048	0.019	0.048
	-0.75	0.002	0.026	0.041	0.043	0.028	0.040		-0.75	0.229	0.071	0.014	0.049	0.014	0.049
	-0.50	0.007	0.027	0.037	0.038	0.025	0.036		-0.50	0.125	0.063	0.017	0.049	0.017	0.049
	-0.25	0.022	0.032	0.039	0.038	0.030	0.037		-0.25	0.079	0.055	0.029	0.051	0.029	0.051
-40	-0.95	0.001	0.026	0.067	0.050	0.045	0.050	-40	-0.95	0.535	0.081	0.020	0.047	0.020	0.047
	-0.75	0.002	0.026	0.050	0.046	0.035	0.045		-0.75	0.282	0.072	0.014	0.048	0.014	0.048
	-0.50	0.006	0.027	0.042	0.044	0.028	0.043		-0.50	0.149	0.064	0.017	0.049	0.017	0.049
	-0.25	0.020	0.033	0.041	0.044	0.032	0.042		-0.25	0.088	0.057	0.027	0.051	0.027	0.051
-50	-0.95	0.000	0.024	0.085	0.051	0.060	0.051	-50	-0.95	0.577	0.082	0.022	0.047	0.022	0.047
	-0.75	0.001	0.024	0.064	0.047	0.046	0.047		-0.75	0.321	0.072	0.016	0.049	0.016	0.049
	-0.50	0.005	0.027	0.051	0.047	0.033	0.046		-0.50	0.167	0.064	0.017	0.049	0.017	0.049
	-0.25	0.019	0.033	0.045	0.048	0.034	0.048		-0.25	0.092	0.056	0.028	0.051	0.028	0.051
-100	-0.95	0.001	0.007	0.316	0.048	0.259	0.048	-100	-0.95	0.609	0.098	0.040	0.052	0.037	0.052
	-0.75	0.002	0.011	0.248	0.049	0.203	0.049		-0.75	0.377	0.086	0.026	0.052	0.024	0.052
	-0.50	0.006	0.018	0.161	0.049	0.120	0.049		-0.50	0.200	0.074	0.018	0.051	0.017	0.051
	-0.25	0.018	0.029	0.085	0.051	0.062	0.051		-0.25	0.105	0.057	0.020	0.052	0.020	0.052
-250	-0.95	0.000	0.003	0.857	0.036	0.065	0.065	-250	-0.95	0.830	0.158	0.048	0.074	0.039	0.039
	-0.75	0.000	0.006	0.837	0.038	0.061	0.061		-0.75	0.594	0.125	0.039	0.067	0.042	0.041
	-0.50	0.002	0.012	0.793	0.041	0.057	0.057		-0.50	0.314	0.101	0.030	0.062	0.044	0.043
	-0.25	0.012	0.024	0.594	0.047	0.053	0.054		-0.25	0.143	0.062	0.018	0.055	0.046	0.046

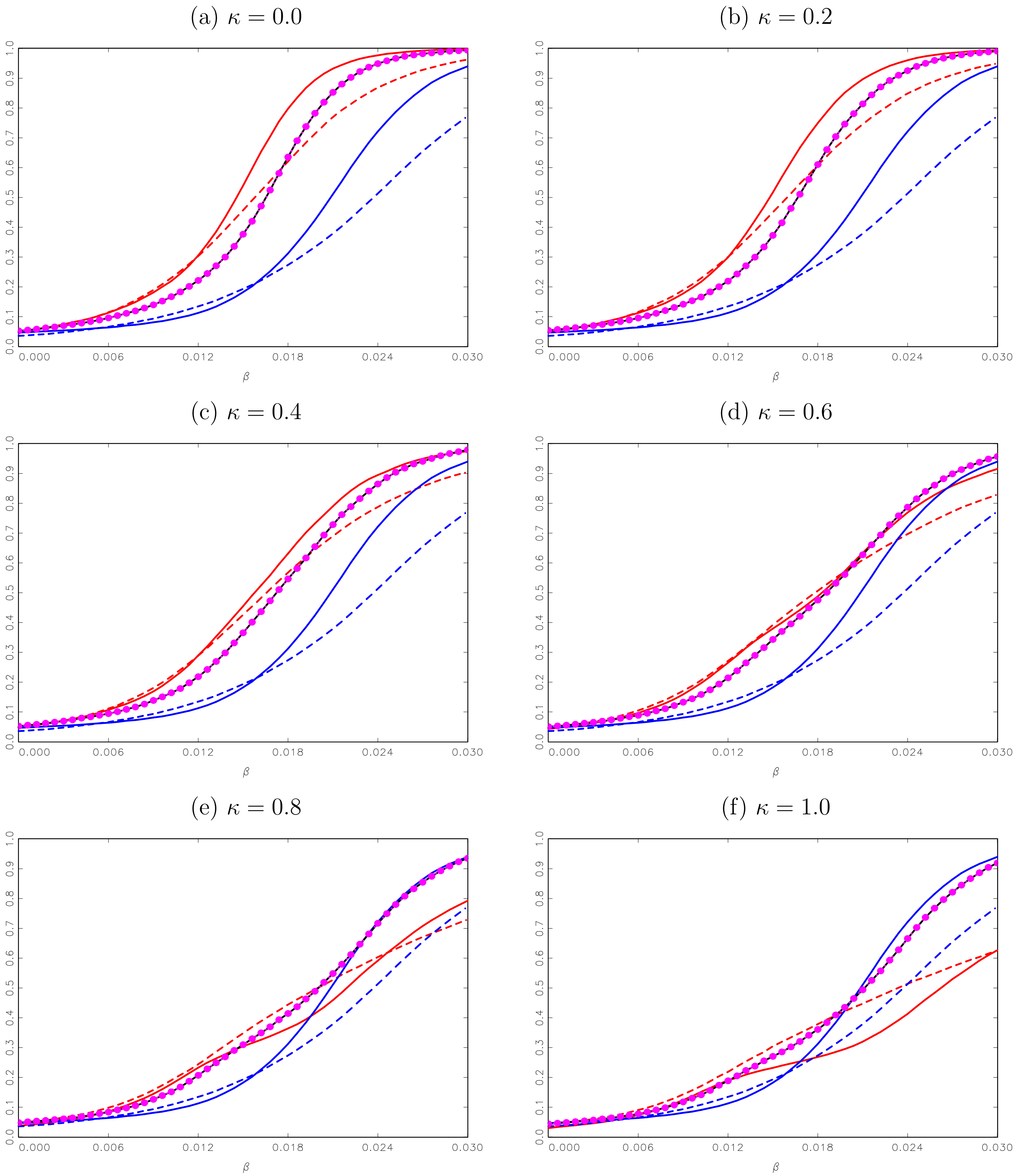
Table S.3: Finite Sample Size, $T = 250$, $\phi = 0.5$, $\kappa = 0$.

(a) Right Tailed Tests								(b) Left Tailed Tests							
c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}	c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.050	0.050	0.048	0.055	0.054	0.054	2	-0.95	0.000	0.006	0.010	0.000	0.010	0.010
	-0.75	0.045	0.051	0.053	0.053	0.056	0.056		-0.75	0.005	0.010	0.013	0.000	0.013	0.013
	-0.50	0.045	0.049	0.052	0.048	0.052	0.052		-0.50	0.015	0.023	0.018	0.005	0.018	0.018
	-0.25	0.048	0.049	0.051	0.049	0.053	0.053		-0.25	0.033	0.031	0.029	0.027	0.029	0.029
0	-0.95	0.048	0.051	0.046	0.038	0.051	0.052	0	-0.95	0.010	0.004	0.021	0.000	0.021	0.021
	-0.75	0.051	0.051	0.044	0.040	0.054	0.054		-0.75	0.010	0.010	0.011	0.001	0.011	0.011
	-0.50	0.055	0.052	0.044	0.042	0.051	0.051		-0.50	0.018	0.021	0.013	0.007	0.013	0.013
	-0.25	0.055	0.052	0.048	0.047	0.048	0.048		-0.25	0.031	0.028	0.026	0.027	0.026	0.026
-2	-0.95	0.048	0.043	0.037	0.026	0.042	0.043	-2	-0.95	0.010	0.011	0.023	0.000	0.023	0.023
	-0.75	0.050	0.040	0.033	0.025	0.043	0.043		-0.75	0.009	0.018	0.018	0.002	0.018	0.018
	-0.50	0.053	0.042	0.034	0.031	0.043	0.043		-0.50	0.016	0.029	0.019	0.014	0.019	0.019
	-0.25	0.055	0.045	0.041	0.040	0.042	0.042		-0.25	0.029	0.035	0.029	0.034	0.029	0.030
-5	-0.95	0.046	0.045	0.033	0.028	0.037	0.037	-5	-0.95	0.013	0.033	0.015	0.001	0.015	0.015
	-0.75	0.046	0.036	0.029	0.018	0.036	0.036		-0.75	0.010	0.034	0.013	0.006	0.013	0.013
	-0.50	0.049	0.034	0.031	0.022	0.035	0.036		-0.50	0.015	0.040	0.018	0.025	0.018	0.018
	-0.25	0.052	0.040	0.037	0.032	0.036	0.036		-0.25	0.029	0.041	0.030	0.043	0.030	0.032
-10	-0.95	0.040	0.048	0.033	0.035	0.031	0.033	-10	-0.95	0.017	0.046	0.017	0.005	0.017	0.017
	-0.75	0.040	0.039	0.029	0.019	0.030	0.031		-0.75	0.012	0.043	0.013	0.023	0.013	0.014
	-0.50	0.043	0.033	0.030	0.019	0.029	0.030		-0.50	0.017	0.044	0.019	0.040	0.019	0.022
	-0.25	0.047	0.036	0.034	0.027	0.031	0.031		-0.25	0.029	0.045	0.031	0.049	0.031	0.036
-20	-0.95	0.028	0.049	0.032	0.046	0.027	0.037	-20	-0.95	0.033	0.046	0.020	0.036	0.020	0.026
	-0.75	0.029	0.045	0.028	0.029	0.026	0.030		-0.75	0.020	0.047	0.015	0.047	0.015	0.030
	-0.50	0.035	0.041	0.030	0.023	0.025	0.028		-0.50	0.022	0.047	0.020	0.048	0.020	0.038
	-0.25	0.043	0.039	0.036	0.027	0.028	0.030		-0.25	0.033	0.047	0.031	0.049	0.031	0.045
-30	-0.95	0.021	0.048	0.034	0.047	0.024	0.045	-30	-0.95	0.063	0.047	0.021	0.049	0.021	0.049
	-0.75	0.023	0.047	0.028	0.037	0.022	0.036		-0.75	0.034	0.047	0.016	0.049	0.016	0.047
	-0.50	0.029	0.045	0.030	0.032	0.023	0.034		-0.50	0.030	0.047	0.019	0.049	0.019	0.048
	-0.25	0.039	0.044	0.036	0.032	0.027	0.033		-0.25	0.038	0.049	0.030	0.050	0.030	0.049
-40	-0.95	0.016	0.049	0.035	0.049	0.024	0.048	-40	-0.95	0.104	0.048	0.023	0.049	0.023	0.049
	-0.75	0.019	0.047	0.029	0.042	0.022	0.042		-0.75	0.055	0.048	0.016	0.047	0.016	0.047
	-0.50	0.025	0.046	0.029	0.037	0.022	0.037		-0.50	0.041	0.047	0.019	0.050	0.019	0.050
	-0.25	0.037	0.047	0.036	0.038	0.025	0.038		-0.25	0.043	0.048	0.030	0.050	0.030	0.050
-50	-0.95	0.013	0.050	0.037	0.049	0.024	0.049	-50	-0.95	0.150	0.049	0.024	0.049	0.024	0.049
	-0.75	0.015	0.047	0.030	0.045	0.021	0.045		-0.75	0.081	0.048	0.017	0.048	0.017	0.048
	-0.50	0.022	0.046	0.030	0.041	0.020	0.041		-0.50	0.055	0.048	0.020	0.049	0.020	0.049
	-0.25	0.035	0.049	0.036	0.043	0.025	0.043		-0.25	0.048	0.047	0.030	0.050	0.030	0.050

Table S.4: Finite Sample Size, $T = 250$, $\phi = 0.5$ $\kappa = 0.5 + 0.5I(c > -20)$.

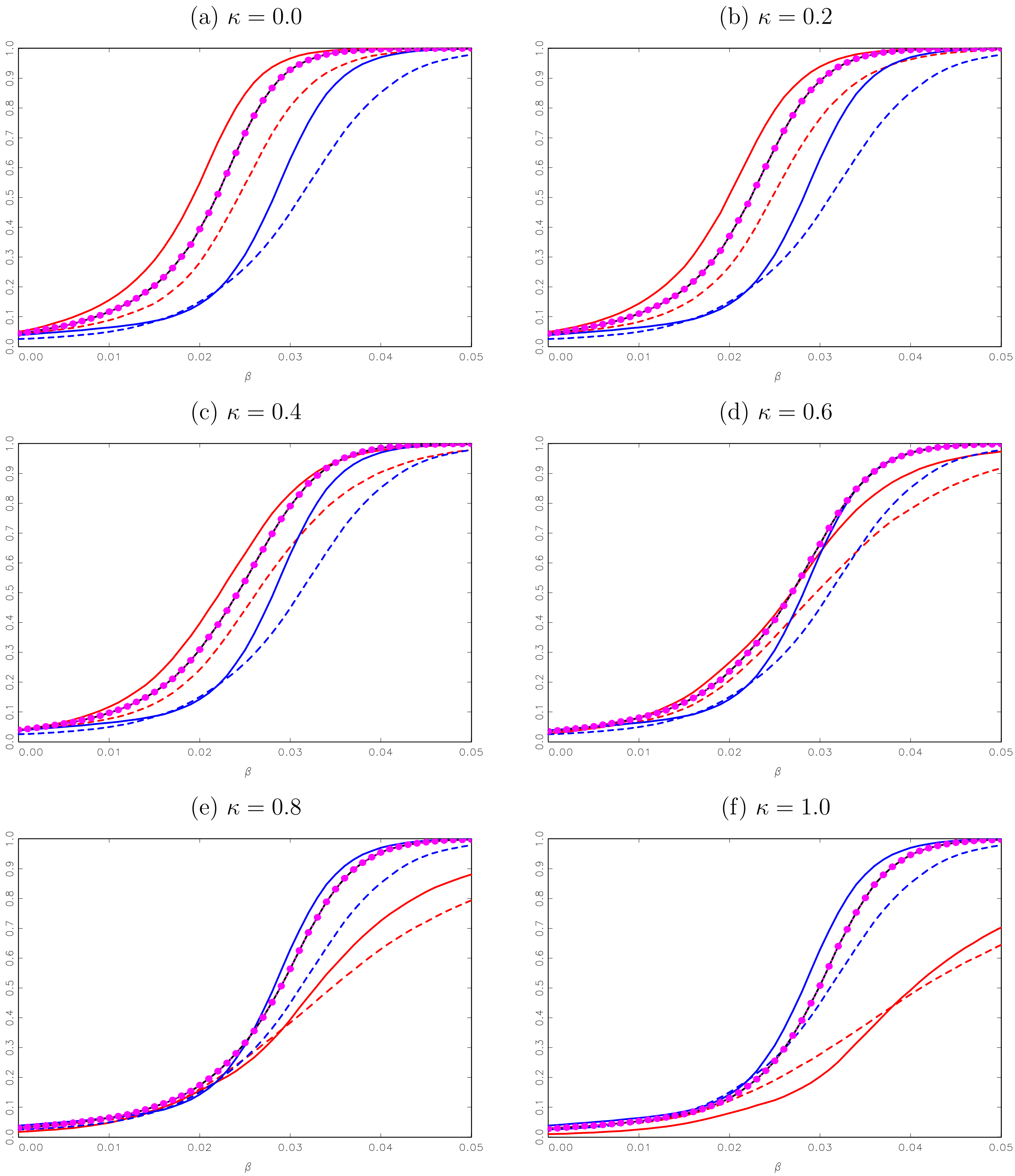
(a) Right Tailed Tests								(b) Left Tailed Tests							
c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}	c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.049	0.048	0.048	0.055	0.056	0.056	2	-0.95	0.007	0.006	0.010	0.000	0.010	0.010
	-0.75	0.046	0.047	0.053	0.053	0.057	0.057		-0.75	0.012	0.012	0.013	0.000	0.013	0.013
	-0.50	0.045	0.047	0.052	0.048	0.052	0.052		-0.50	0.022	0.026	0.018	0.005	0.018	0.018
	-0.25	0.045	0.048	0.051	0.049	0.051	0.051		-0.25	0.036	0.033	0.029	0.027	0.029	0.029
0	-0.95	0.030	0.044	0.046	0.038	0.044	0.044	0	-0.95	0.039	0.011	0.021	0.000	0.021	0.021
	-0.75	0.033	0.043	0.044	0.040	0.044	0.044		-0.75	0.026	0.016	0.011	0.001	0.011	0.011
	-0.50	0.038	0.045	0.044	0.042	0.044	0.044		-0.50	0.031	0.030	0.013	0.007	0.013	0.013
	-0.25	0.047	0.048	0.048	0.047	0.045	0.045		-0.25	0.040	0.033	0.026	0.027	0.026	0.026
-2	-0.95	0.009	0.030	0.037	0.026	0.026	0.026	-2	-0.95	0.067	0.033	0.023	0.000	0.023	0.023
	-0.75	0.012	0.028	0.033	0.025	0.026	0.027		-0.75	0.039	0.036	0.018	0.002	0.018	0.018
	-0.50	0.023	0.030	0.034	0.031	0.030	0.030		-0.50	0.039	0.043	0.019	0.014	0.019	0.019
	-0.25	0.038	0.039	0.041	0.040	0.039	0.039		-0.25	0.044	0.039	0.029	0.034	0.029	0.030
-5	-0.95	0.001	0.020	0.033	0.028	0.020	0.021	-5	-0.95	0.155	0.064	0.015	0.001	0.015	0.015
	-0.75	0.004	0.019	0.029	0.018	0.020	0.021		-0.75	0.077	0.059	0.013	0.006	0.013	0.013
	-0.50	0.014	0.024	0.031	0.022	0.024	0.024		-0.50	0.058	0.060	0.018	0.025	0.018	0.018
	-0.25	0.031	0.034	0.037	0.032	0.035	0.035		-0.25	0.054	0.044	0.030	0.043	0.030	0.032
-10	-0.95	0.001	0.011	0.033	0.035	0.020	0.022	-10	-0.95	0.286	0.098	0.017	0.005	0.017	0.017
	-0.75	0.003	0.012	0.029	0.019	0.019	0.021		-0.75	0.134	0.084	0.013	0.023	0.013	0.014
	-0.50	0.011	0.018	0.030	0.019	0.021	0.023		-0.50	0.086	0.073	0.019	0.040	0.019	0.022
	-0.25	0.027	0.030	0.034	0.027	0.033	0.033		-0.25	0.065	0.051	0.031	0.049	0.031	0.036
-20	-0.95	0.000	0.025	0.032	0.046	0.021	0.035	-20	-0.95	0.339	0.084	0.020	0.036	0.020	0.026
	-0.75	0.002	0.022	0.028	0.029	0.019	0.026		-0.75	0.161	0.073	0.015	0.047	0.015	0.030
	-0.50	0.008	0.025	0.030	0.023	0.019	0.025		-0.50	0.096	0.063	0.020	0.048	0.020	0.038
	-0.25	0.025	0.032	0.036	0.027	0.029	0.031		-0.25	0.070	0.052	0.031	0.049	0.031	0.045
-30	-0.95	0.000	0.020	0.034	0.047	0.021	0.045	-30	-0.95	0.452	0.090	0.021	0.049	0.021	0.049
	-0.75	0.002	0.018	0.028	0.037	0.019	0.035		-0.75	0.222	0.078	0.016	0.049	0.016	0.047
	-0.50	0.007	0.020	0.030	0.032	0.020	0.033		-0.50	0.122	0.067	0.019	0.049	0.019	0.048
	-0.25	0.023	0.030	0.036	0.032	0.030	0.035		-0.25	0.079	0.055	0.030	0.050	0.030	0.049
-40	-0.95	0.000	0.015	0.035	0.049	0.023	0.048	-40	-0.95	0.514	0.095	0.023	0.049	0.023	0.049
	-0.75	0.002	0.014	0.029	0.042	0.020	0.042		-0.75	0.262	0.082	0.016	0.047	0.016	0.047
	-0.50	0.007	0.017	0.029	0.037	0.021	0.037		-0.50	0.141	0.071	0.019	0.050	0.019	0.050
	-0.25	0.021	0.029	0.036	0.038	0.030	0.039		-0.25	0.086	0.057	0.030	0.050	0.030	0.050
-50	-0.95	0.001	0.011	0.037	0.049	0.024	0.049	-50	-0.95	0.542	0.099	0.024	0.049	0.024	0.049
	-0.75	0.002	0.012	0.030	0.045	0.020	0.045		-0.75	0.286	0.084	0.017	0.048	0.017	0.048
	-0.50	0.007	0.016	0.030	0.041	0.021	0.041		-0.50	0.152	0.072	0.020	0.049	0.020	0.049
	-0.25	0.021	0.027	0.036	0.043	0.031	0.044		-0.25	0.088	0.059	0.030	0.050	0.030	0.050

Figure S.6: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = 0$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – – , Q_τ^{GLS}: —— , U^{hyb}: —— , S^{hyb}: • • •

Figure S.7: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -2$



t_μ^{OLS}: - - , Q_μ^{GLS}: —— , t_τ^{OLS}: - - , Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.8: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -5$

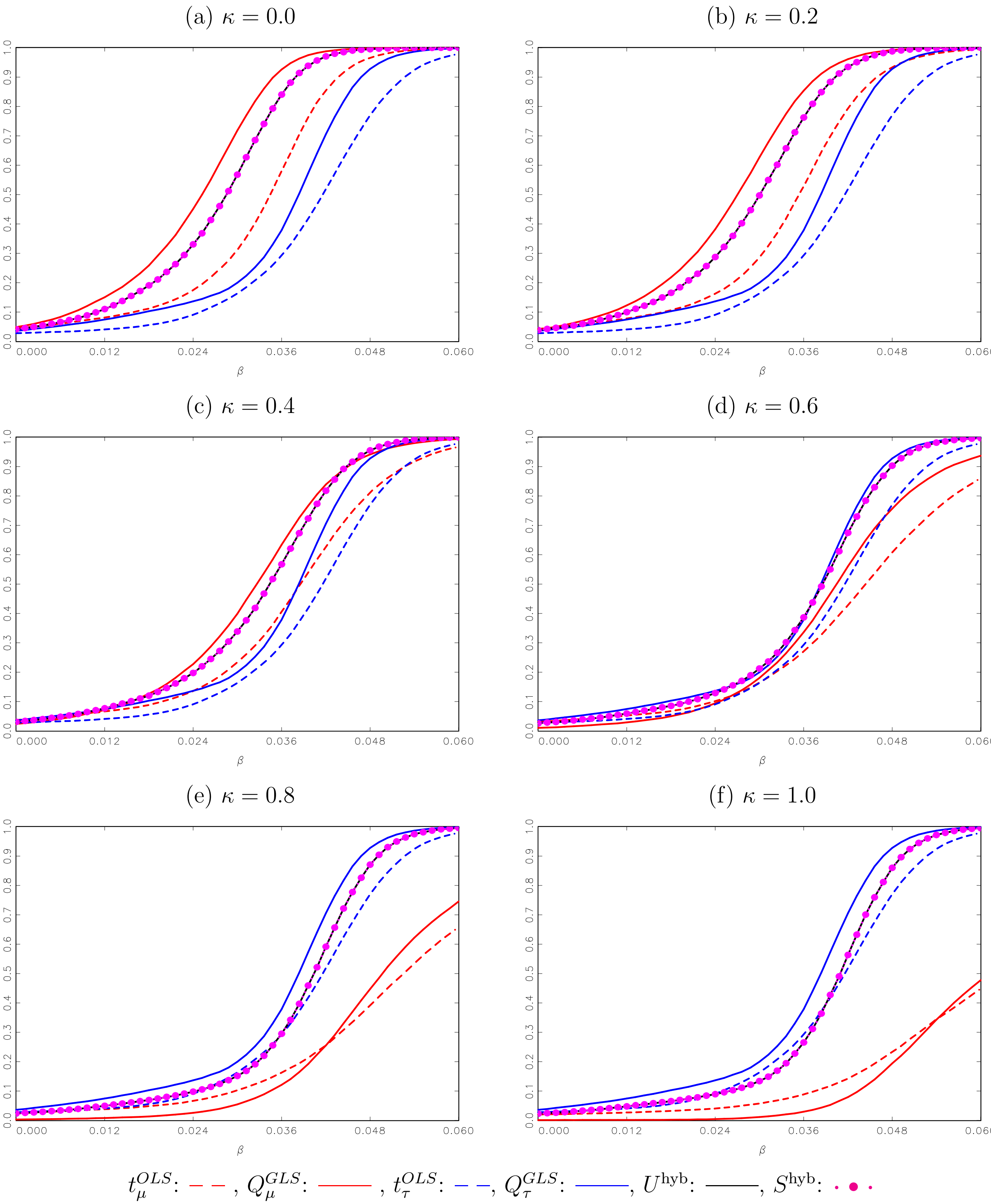
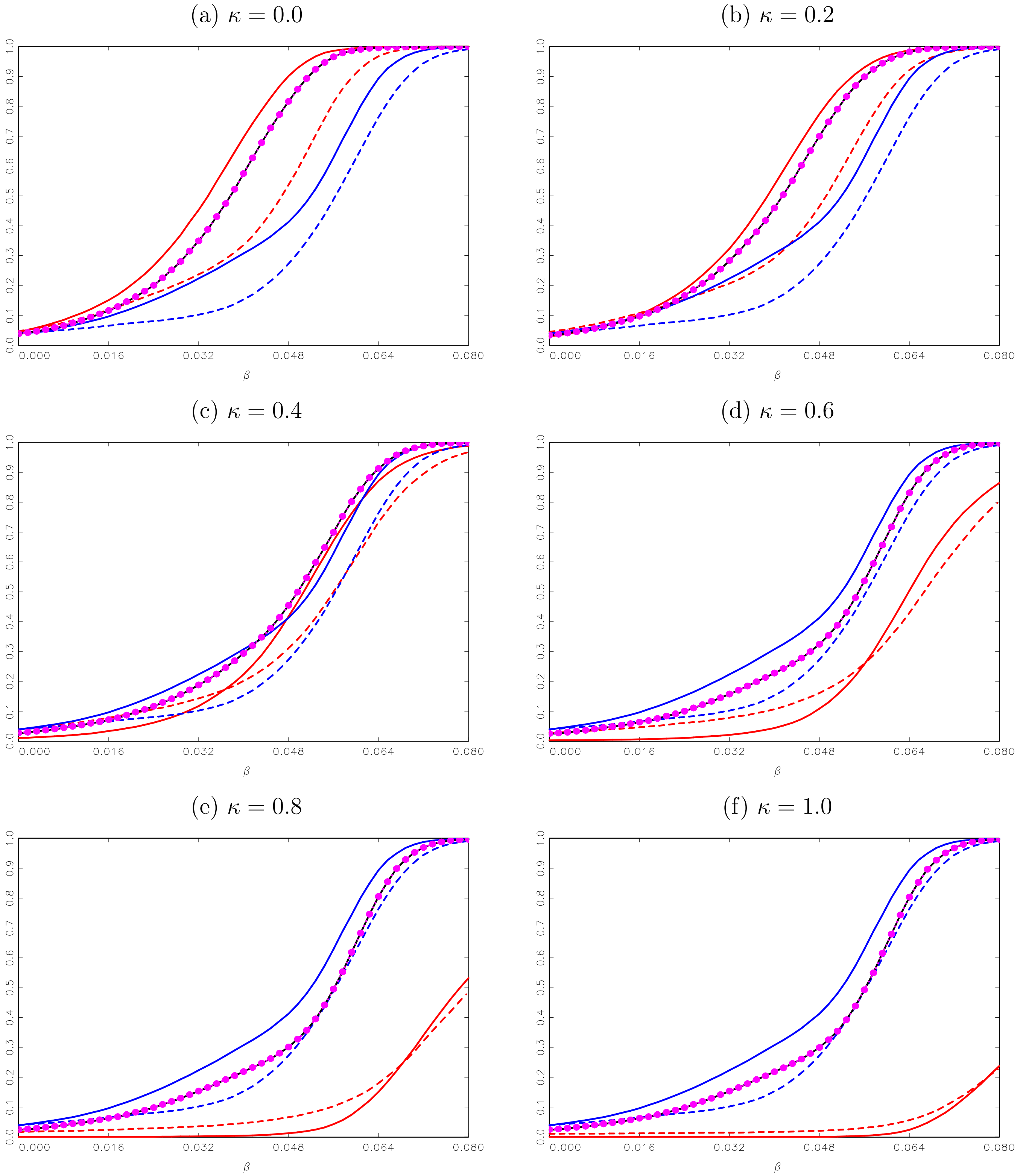


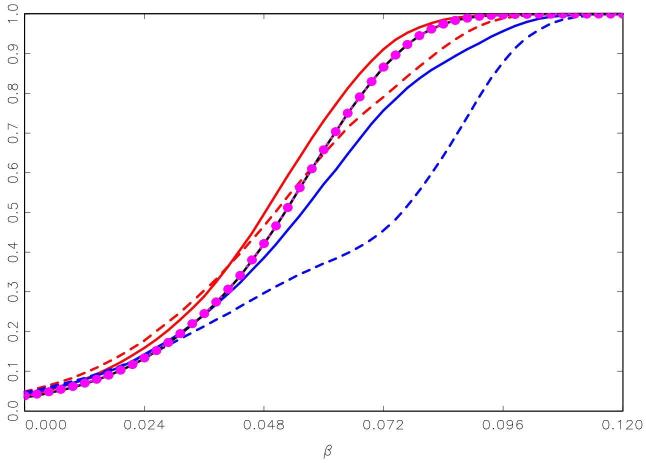
Figure S.9: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -10$



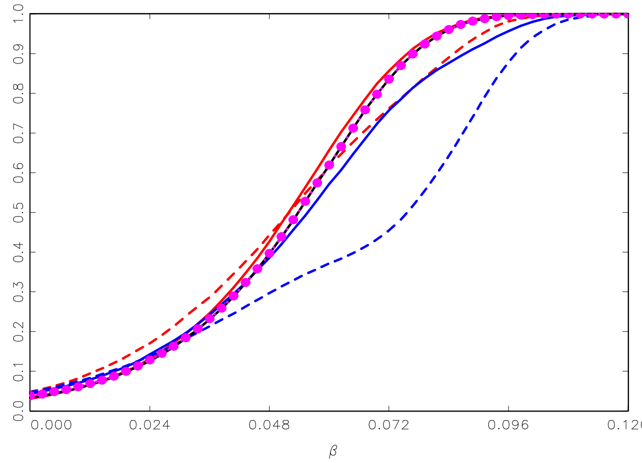
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – – , Q_τ^{GLS}: —— , U^{hyb}: —— , S^{hyb}: · • ·

Figure S.10: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -20$

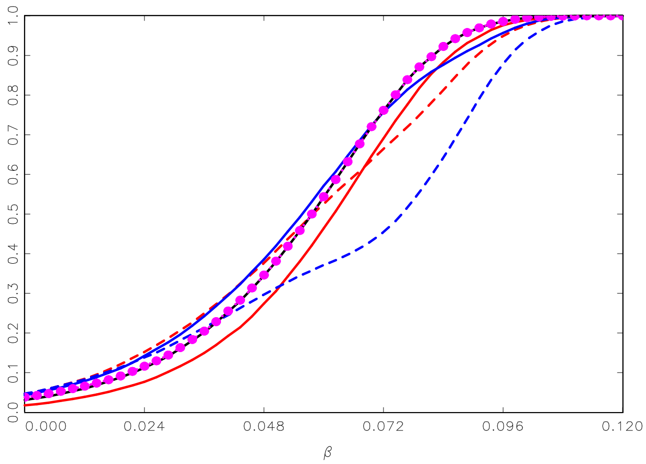
(a) $\kappa = 0.0$



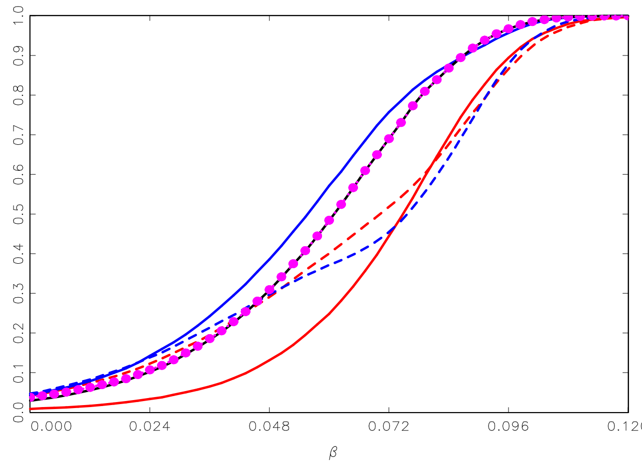
(b) $\kappa = 0.1$



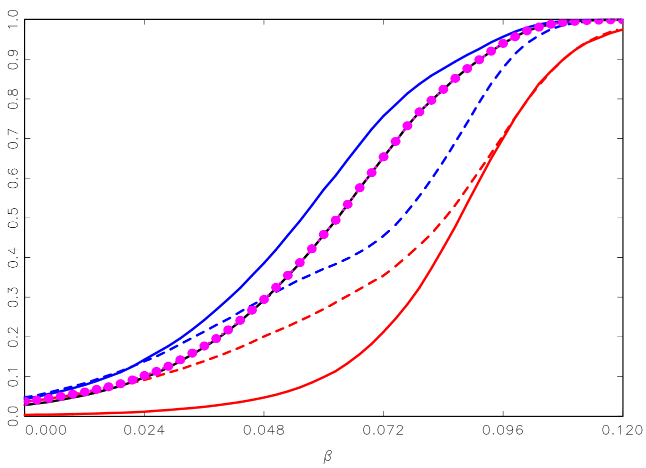
(c) $\kappa = 0.2$



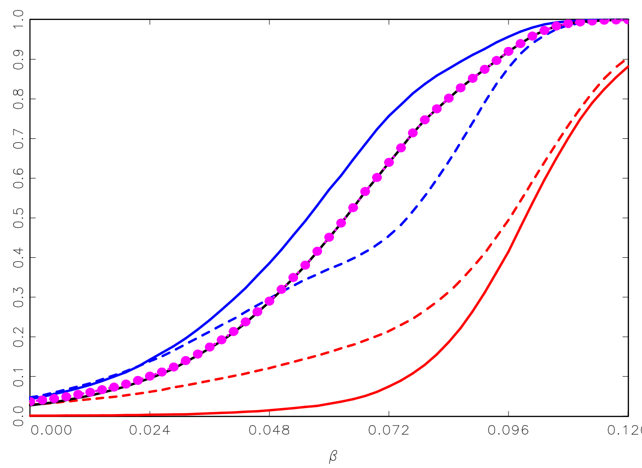
(d) $\kappa = 0.3$



(e) $\kappa = 0.4$

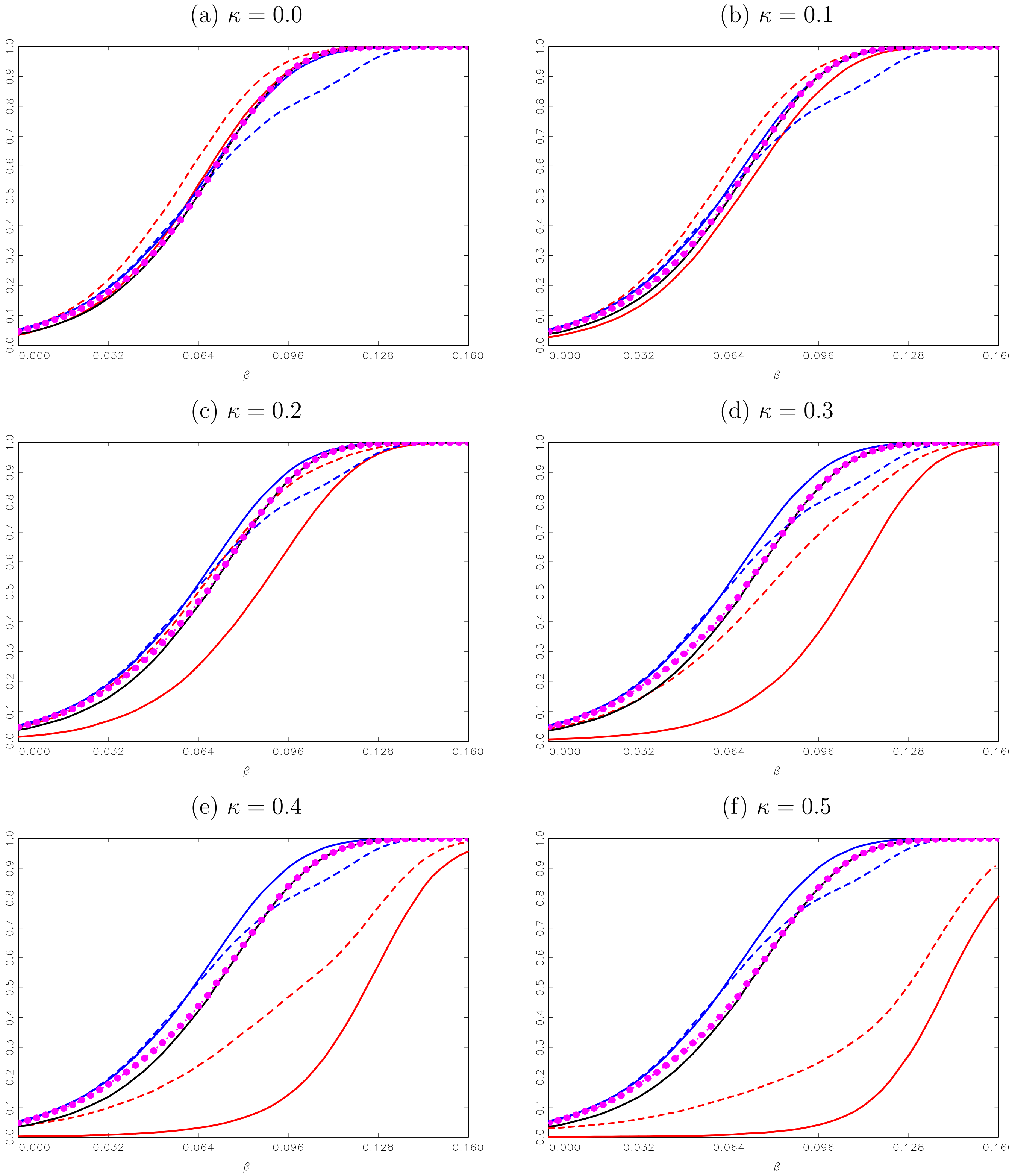


(f) $\kappa = 0.5$



t_μ^{OLS}: - - , Q_μ^{GLS}: —— , t_τ^{OLS}: - - , Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · ● ·

Figure S.11: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -30$



t_μ^{OLS}: - - , Q_μ^{GLS}: —— , t_τ^{OLS}: - - , Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.12: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -40$

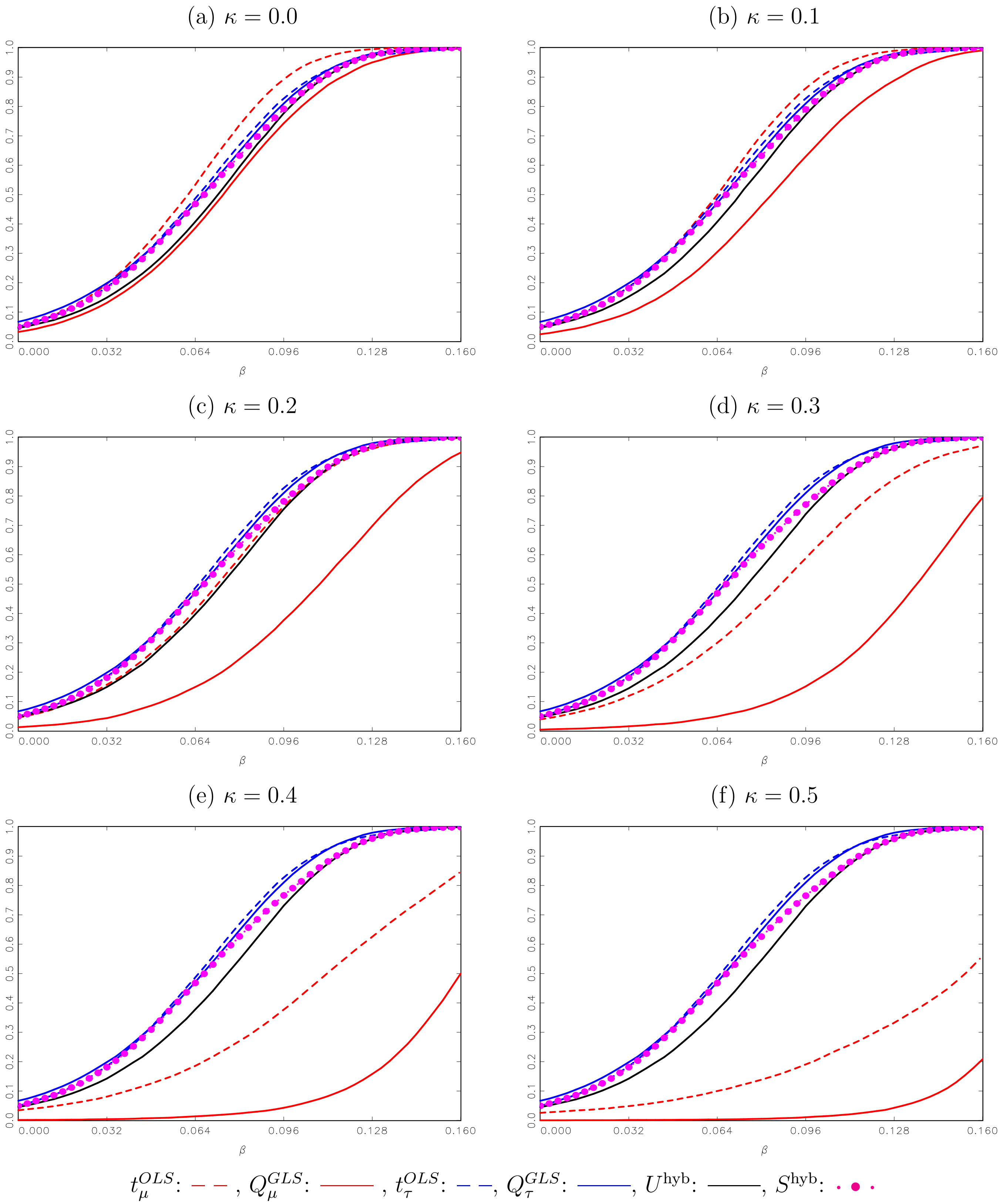


Figure S.13: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -50$

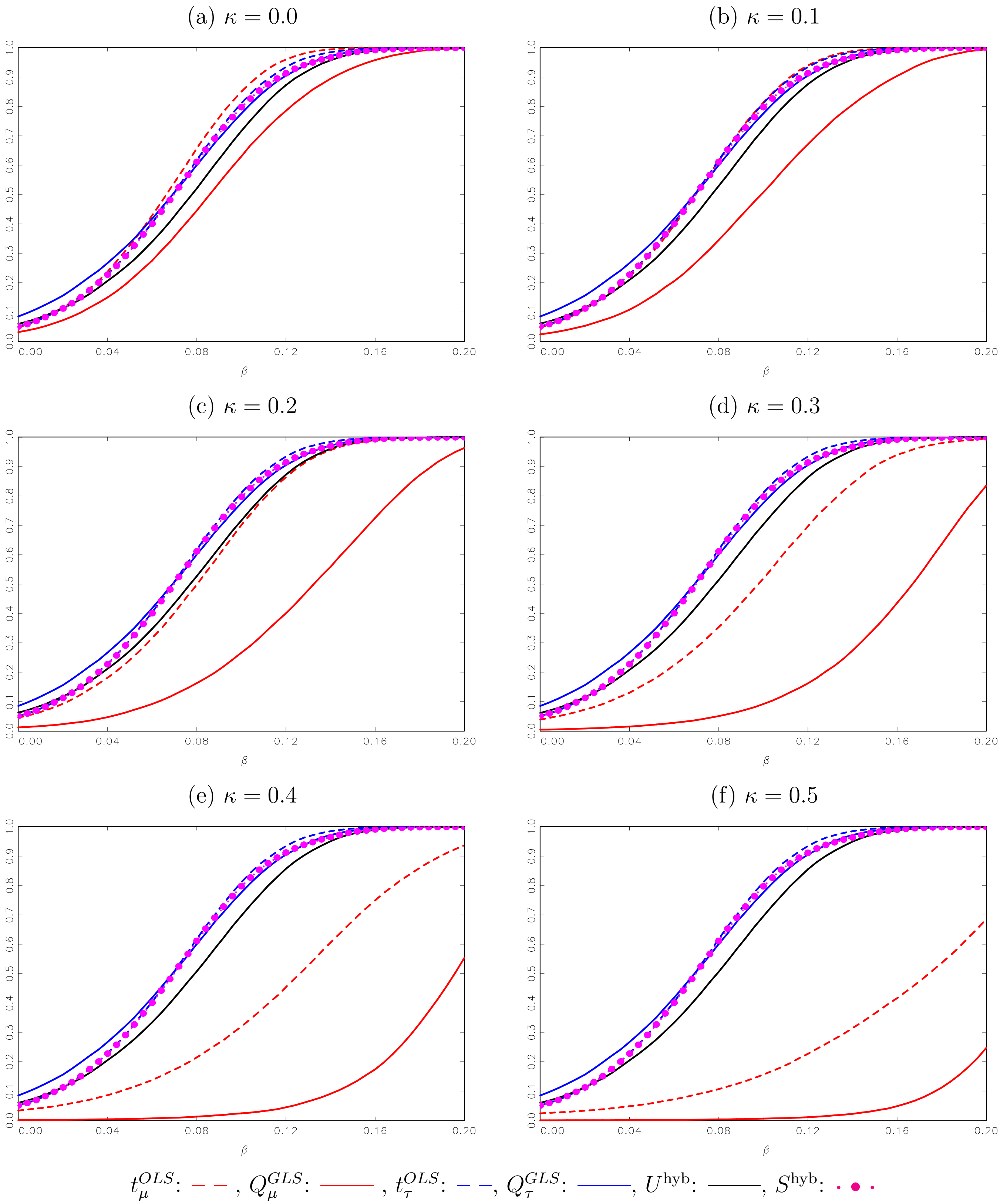
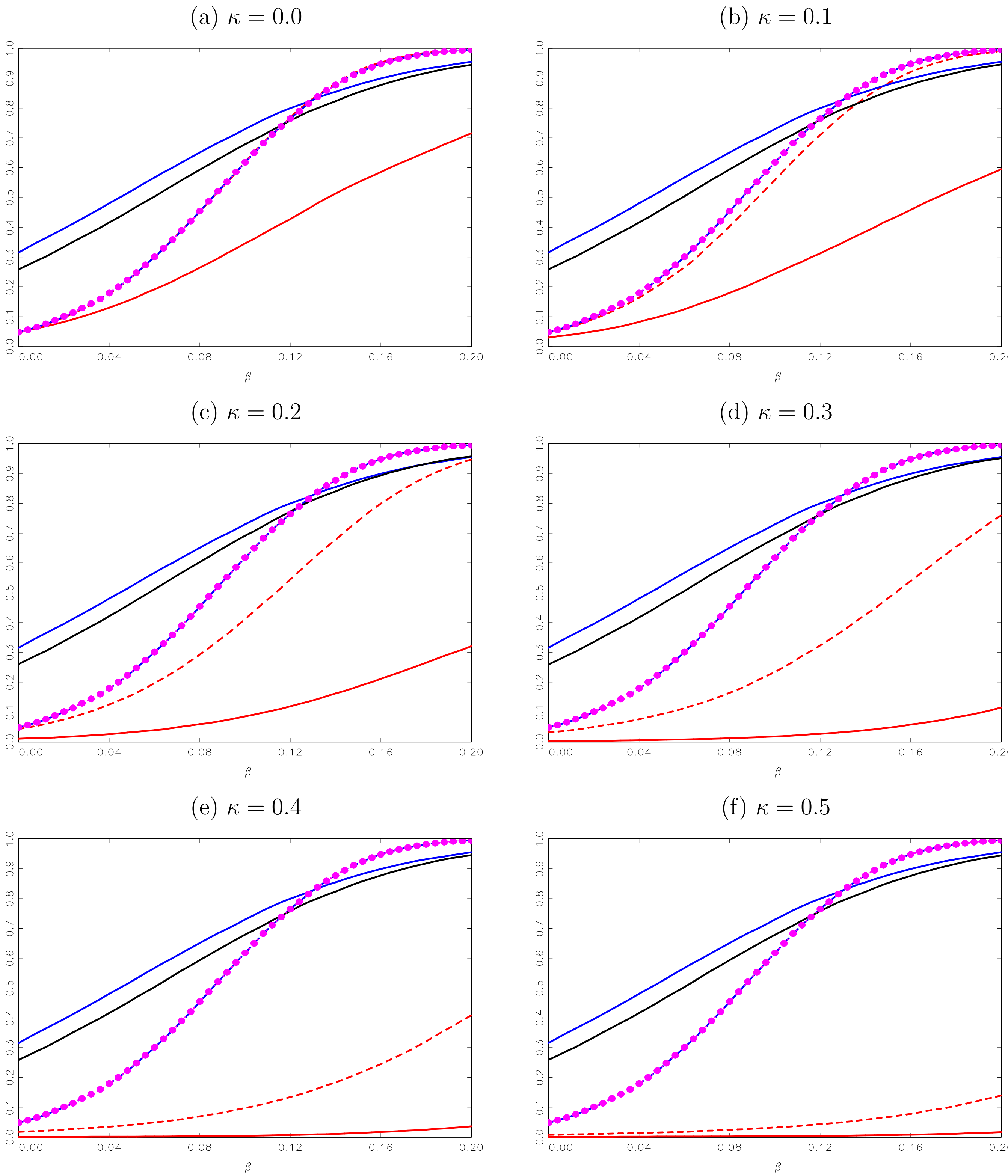
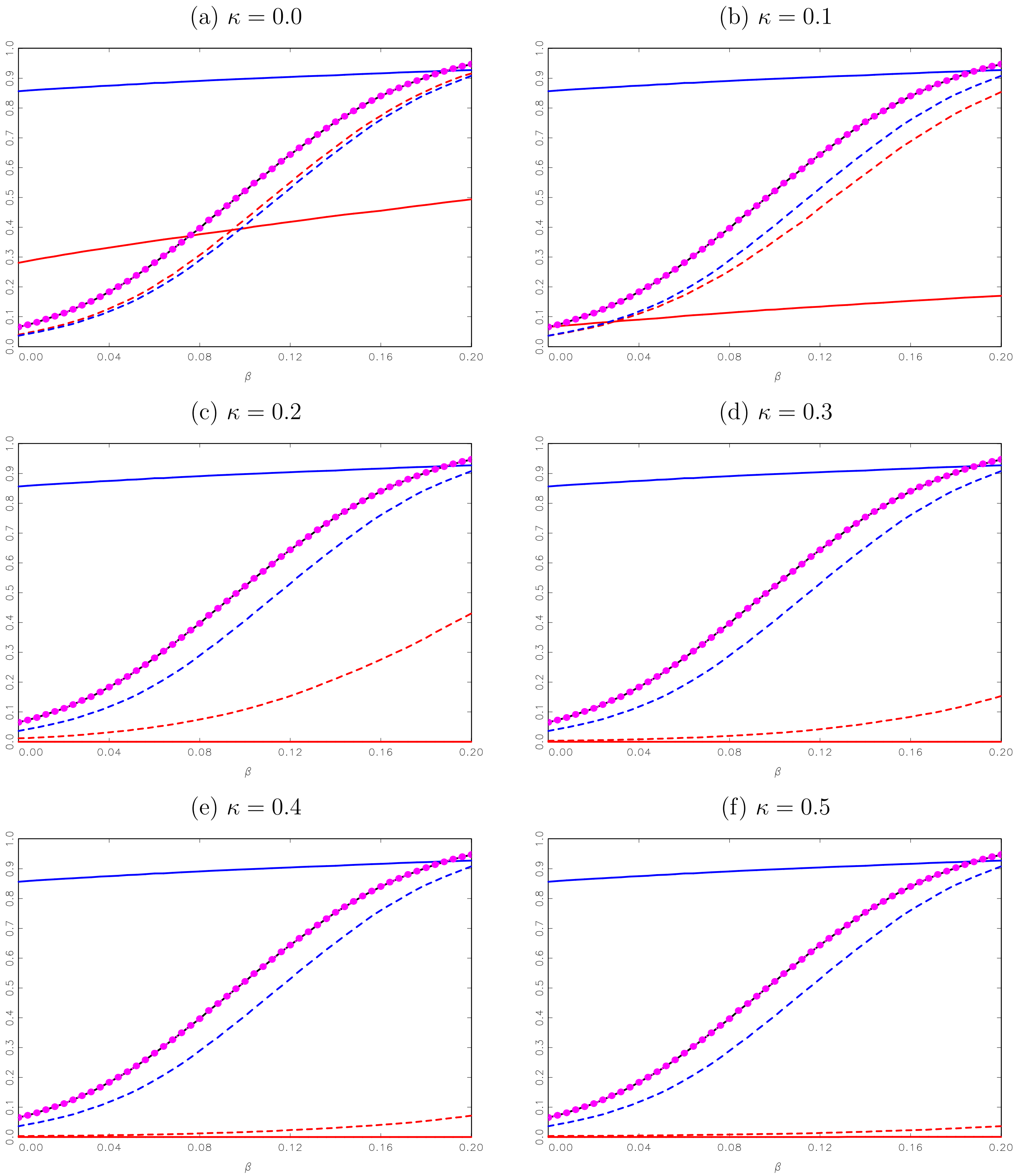


Figure S.14: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -100$



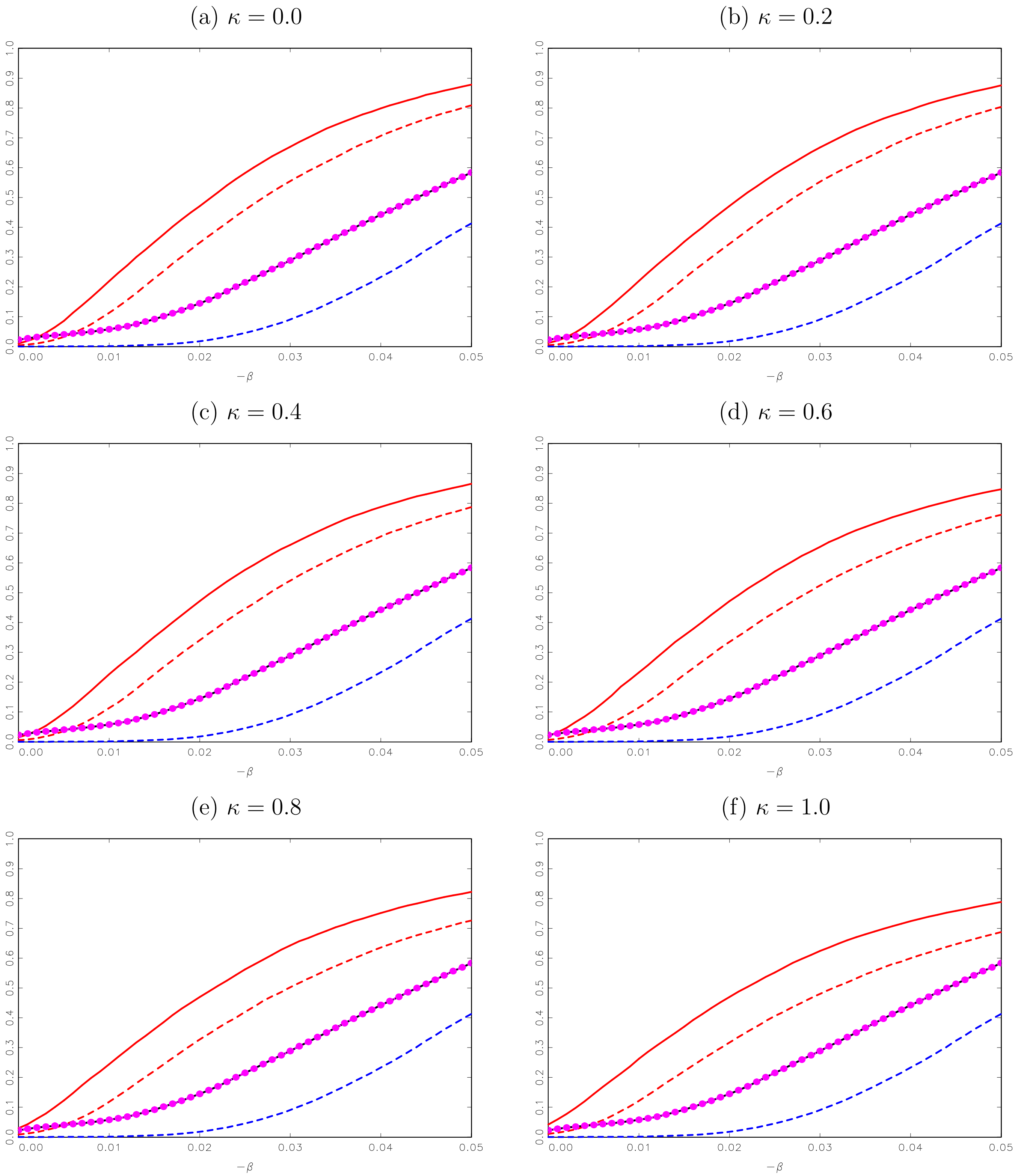
t_μ^{OLS}: – –, Q_μ^{GLS}: ——, t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.15: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = -250$



t_{μ}^{OLS}: – –, Q_{μ}^{GLS}: ——, t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · ● ·

Figure S.16: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = 0$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.17: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -2$

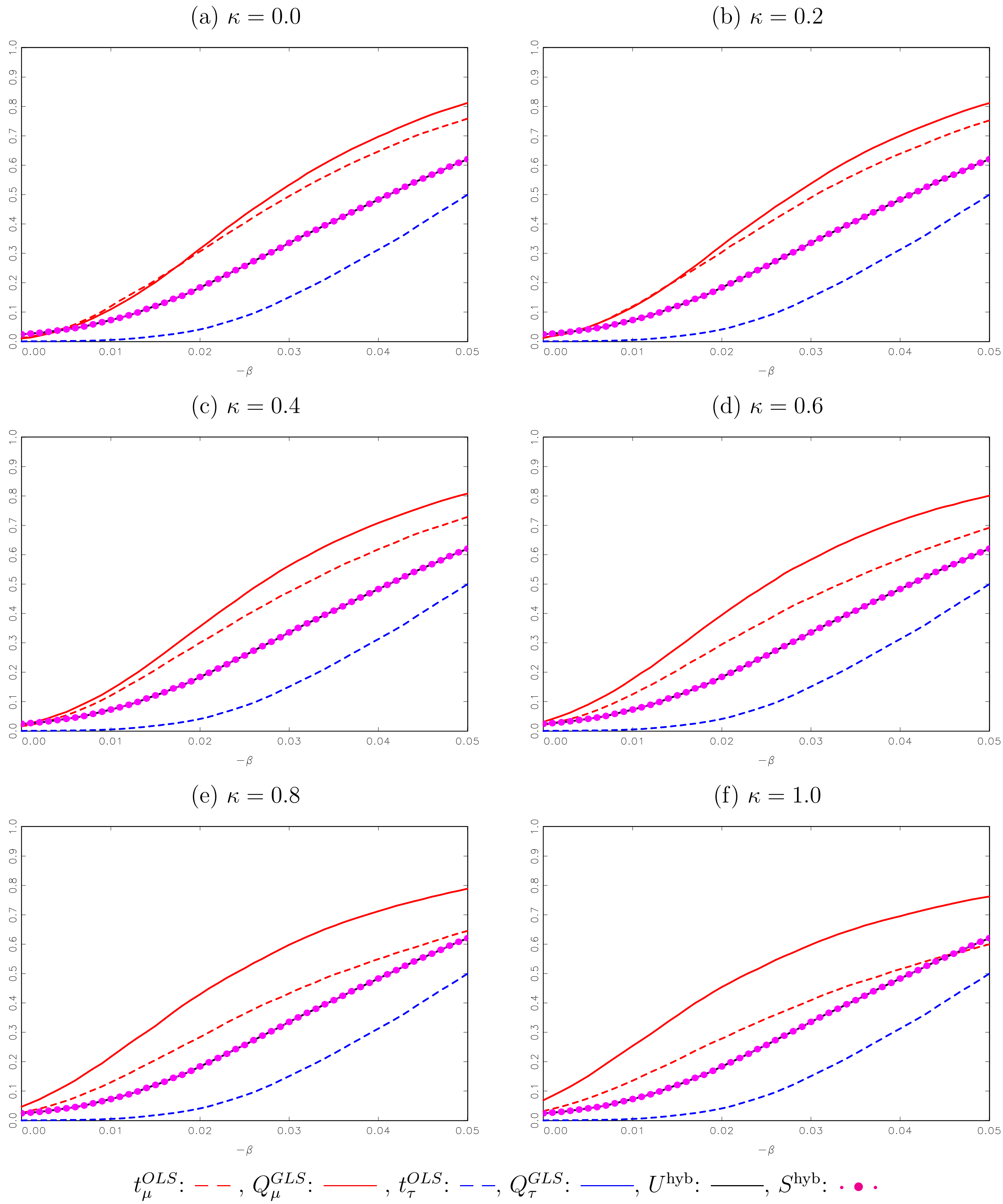
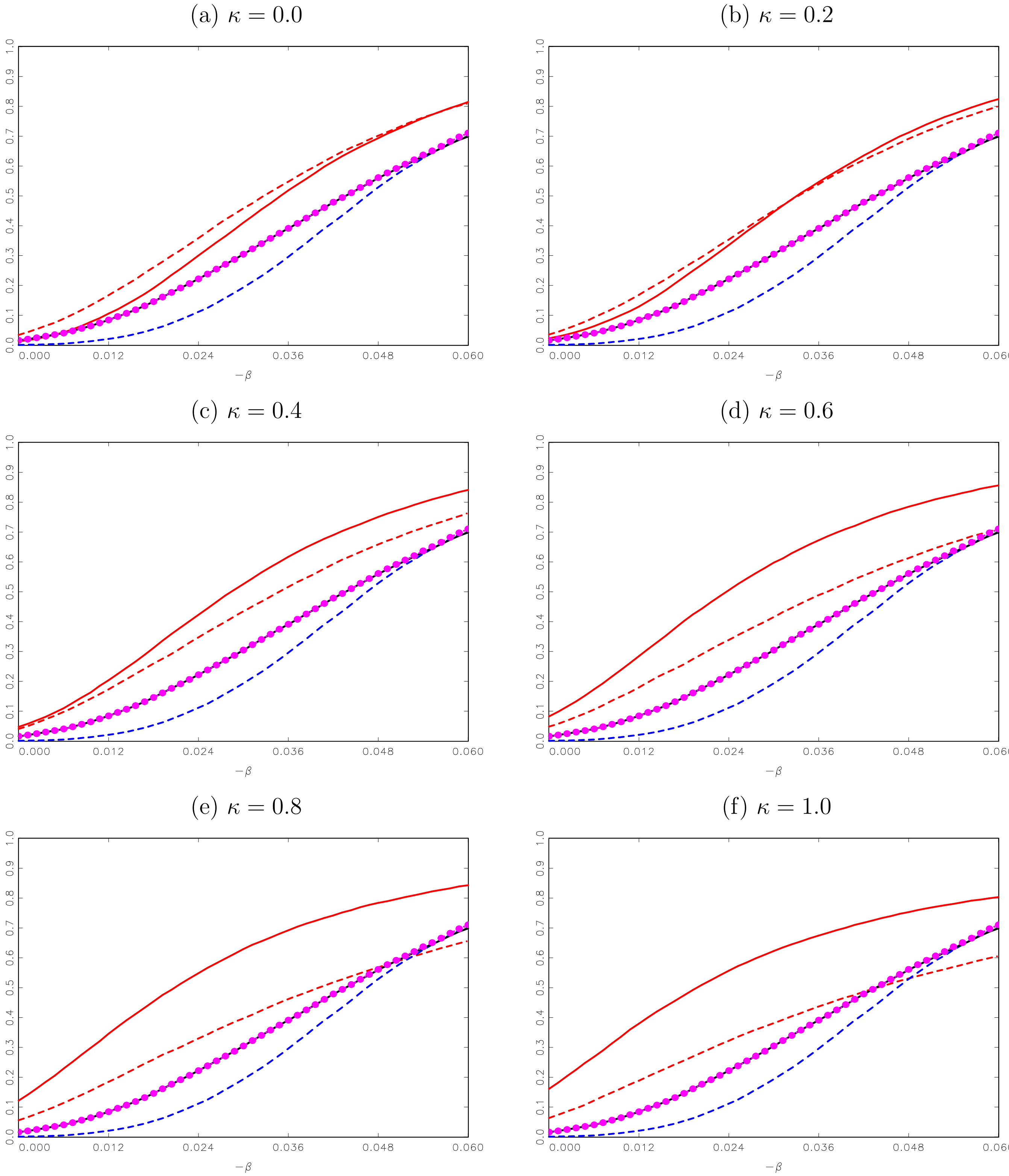


Figure S.18: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -5$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.19: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -10$

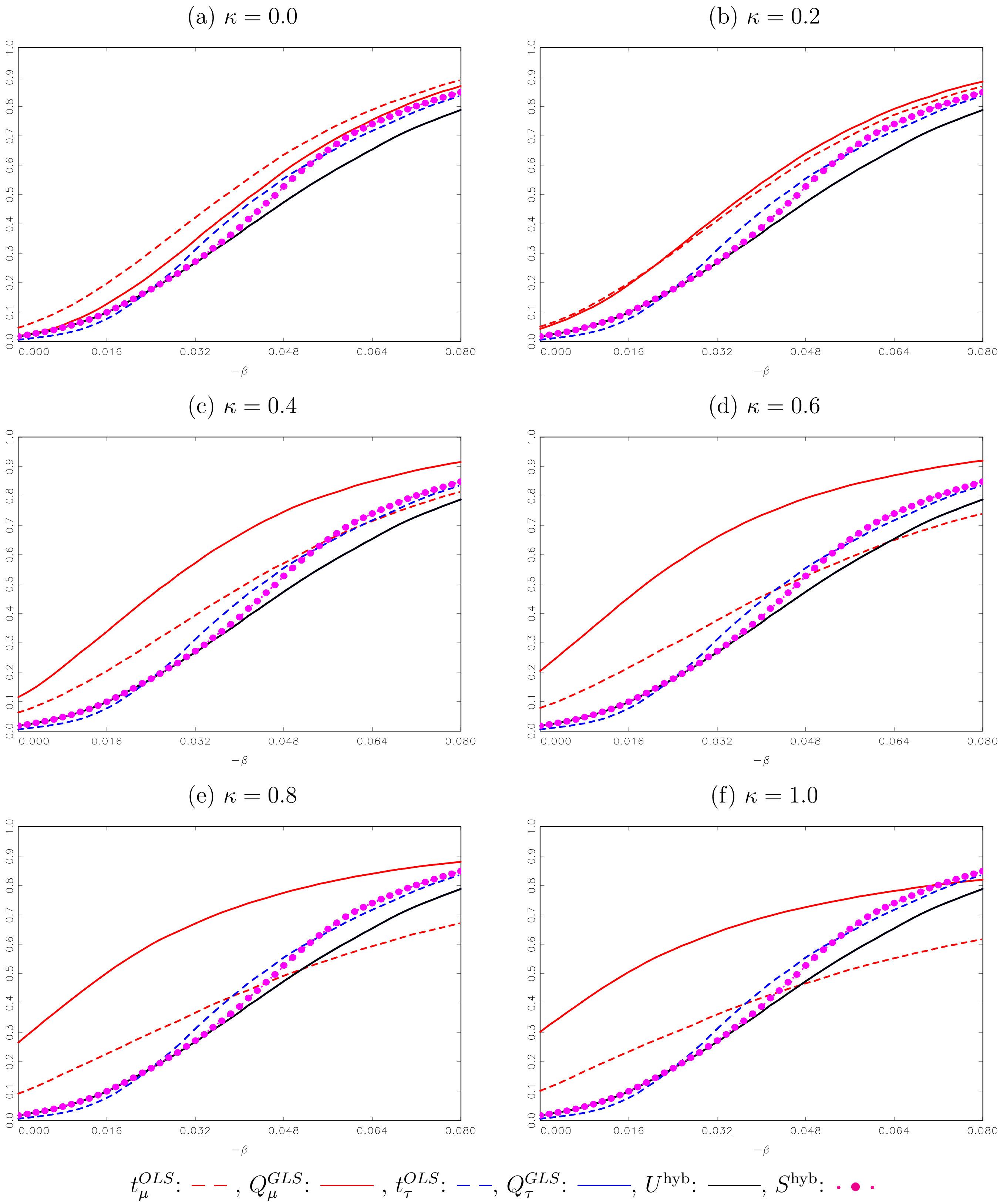
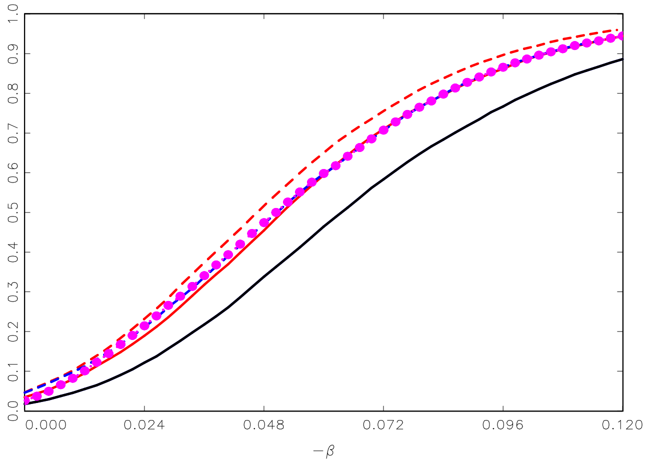
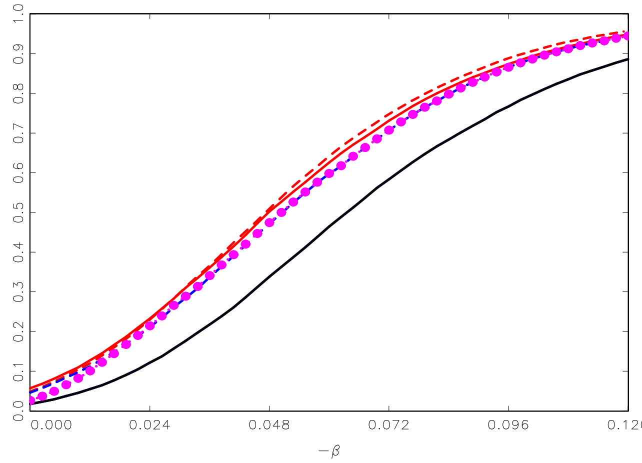


Figure S.20: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -20$

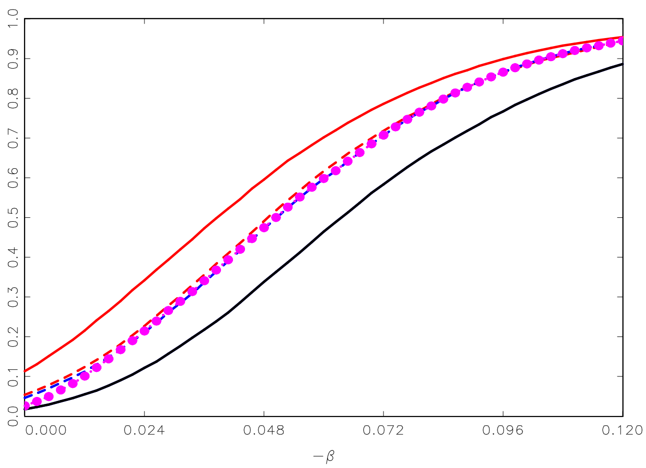
(a) $\kappa = 0.0$



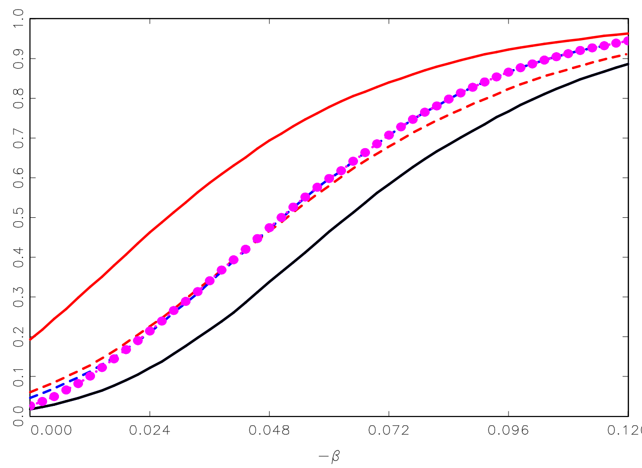
(b) $\kappa = 0.1$



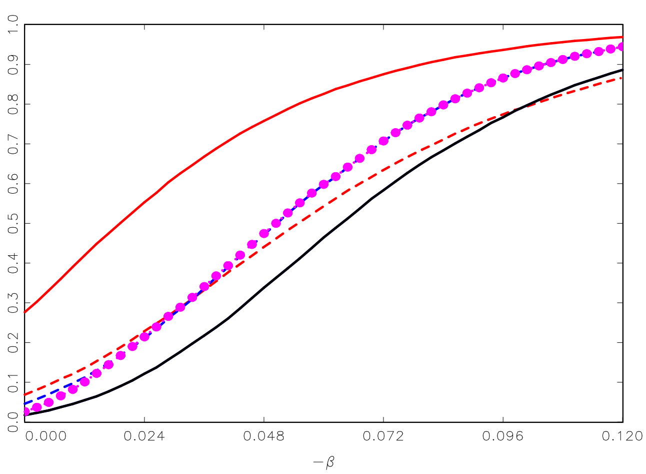
(c) $\kappa = 0.2$



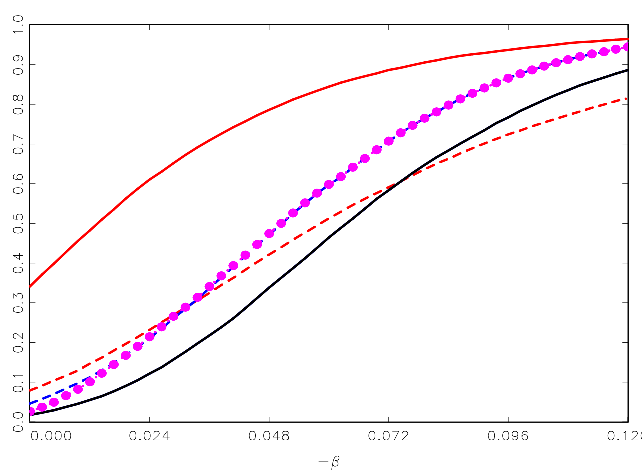
(d) $\kappa = 0.3$



(e) $\kappa = 0.4$

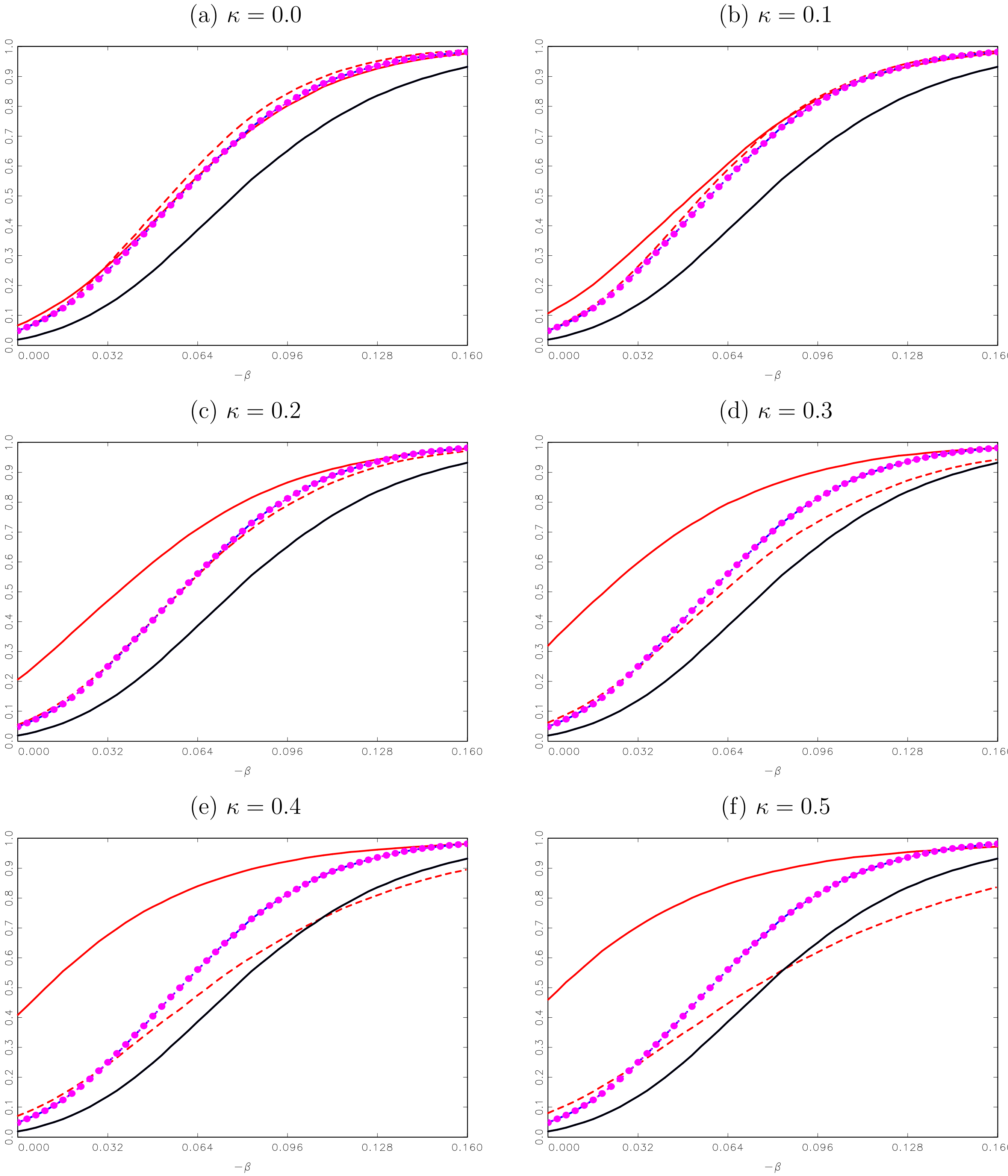


(f) $\kappa = 0.5$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – – , Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: • • •

Figure S.21: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -30$



t_μ^{OLS}: – –, Q_μ^{GLS}: ——, t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.22: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -40$

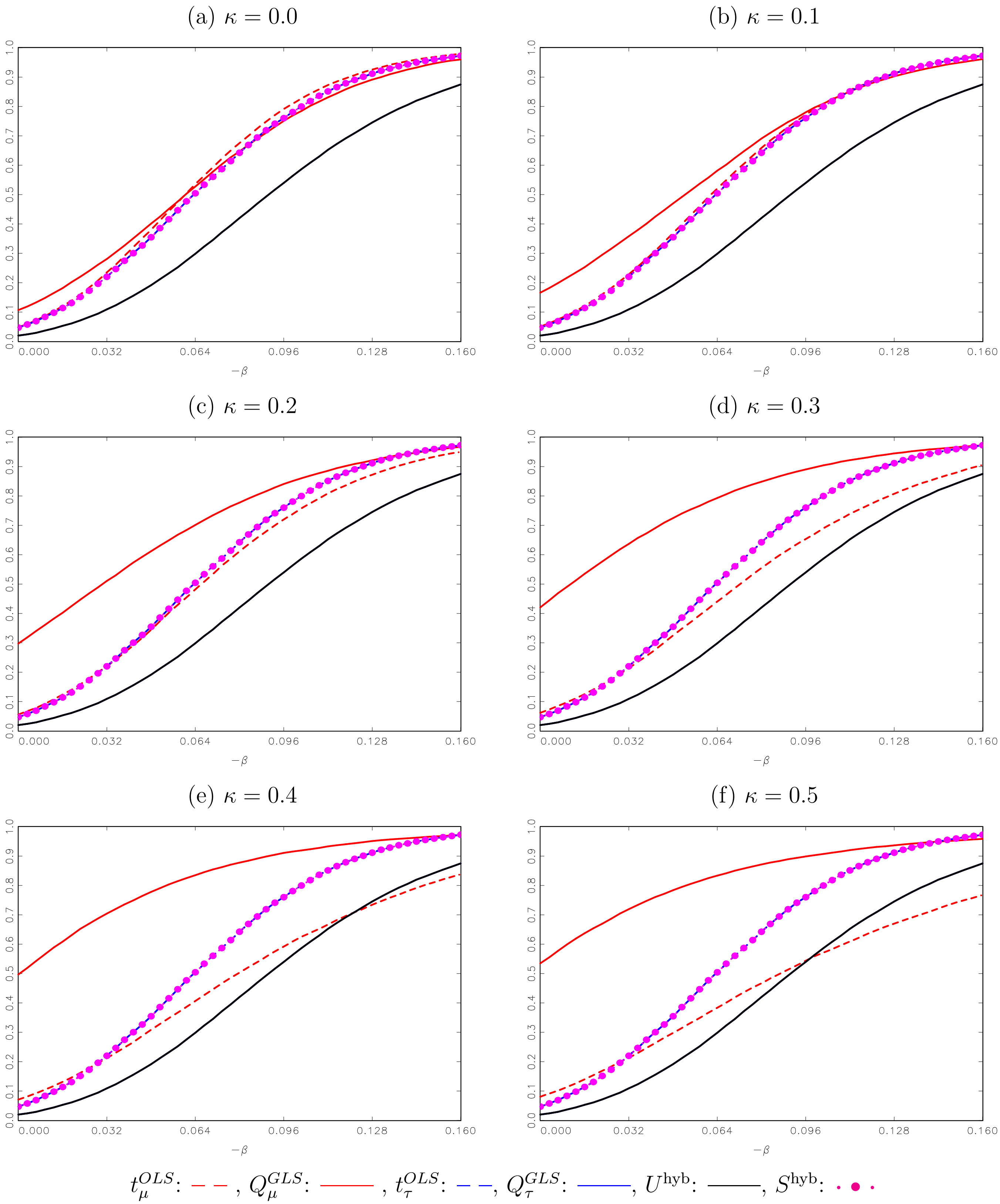
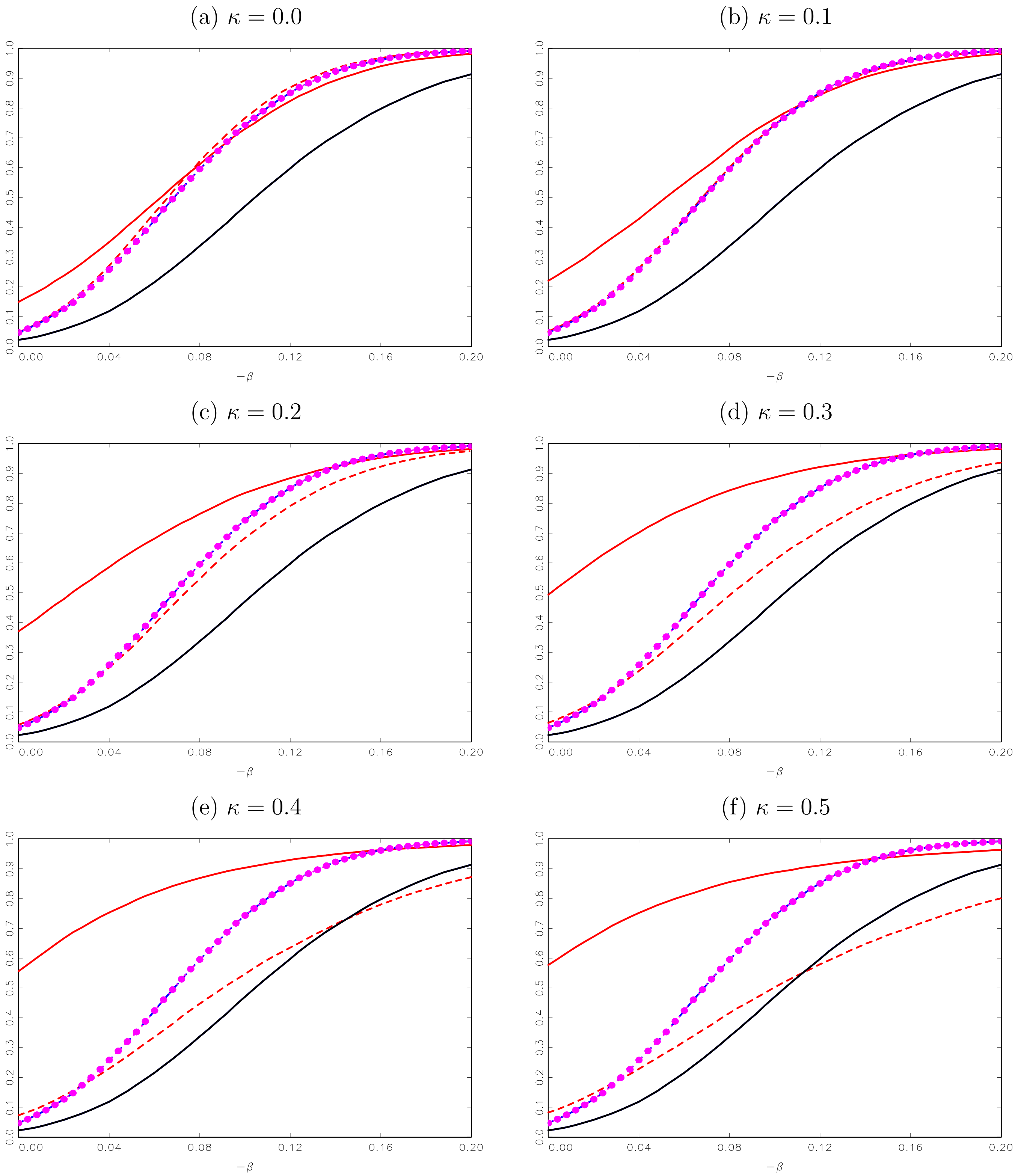


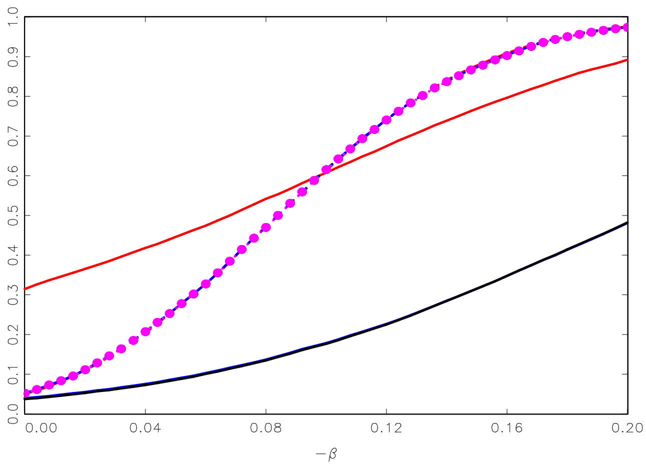
Figure S.23: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -50$



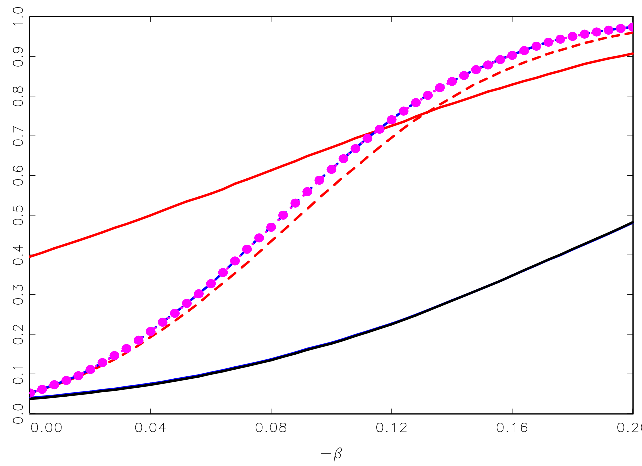
t_μ^{OLS}: – –, Q_μ^{GLS}: —, t_τ^{OLS}: – –, Q_τ^{GLS}: —, U^{hyb}: —, S^{hyb}: • • •

Figure S.24: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -100$

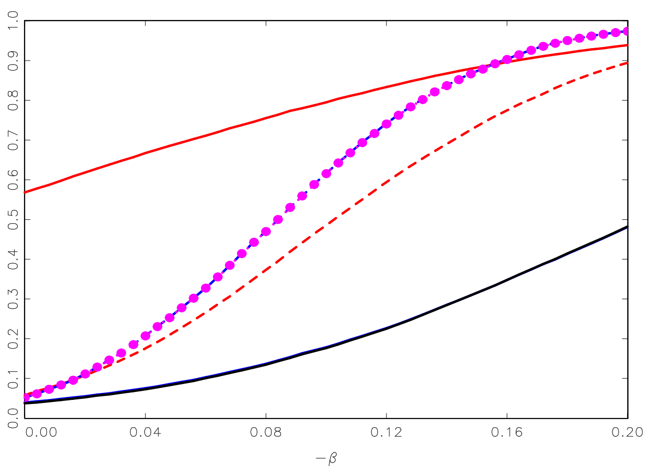
(a) $\kappa = 0.0$



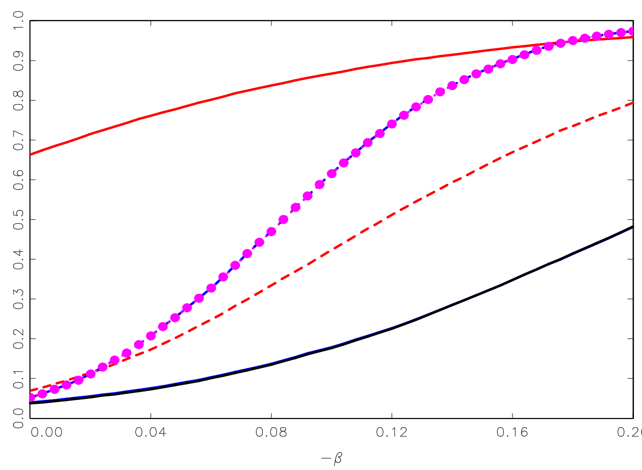
(b) $\kappa = 0.1$



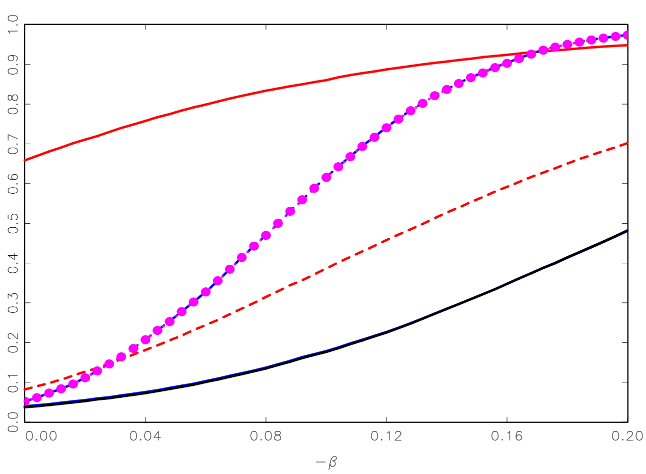
(c) $\kappa = 0.2$



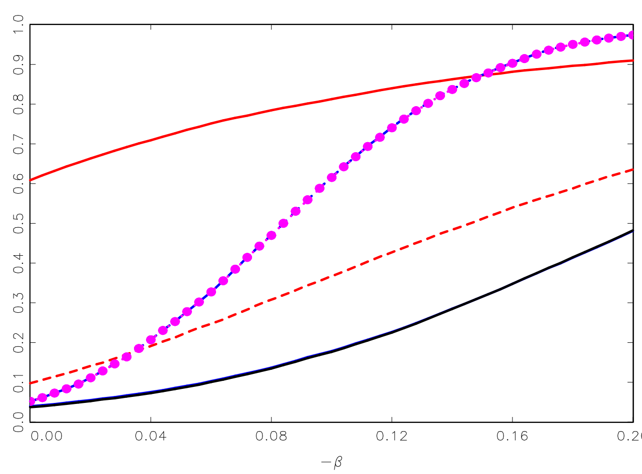
(d) $\kappa = 0.3$



(e) $\kappa = 0.4$



(f) $\kappa = 0.5$



t_μ^{OLS}: – –, Q_μ^{GLS}: ——, t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: · • ·

Figure S.25: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = -250$

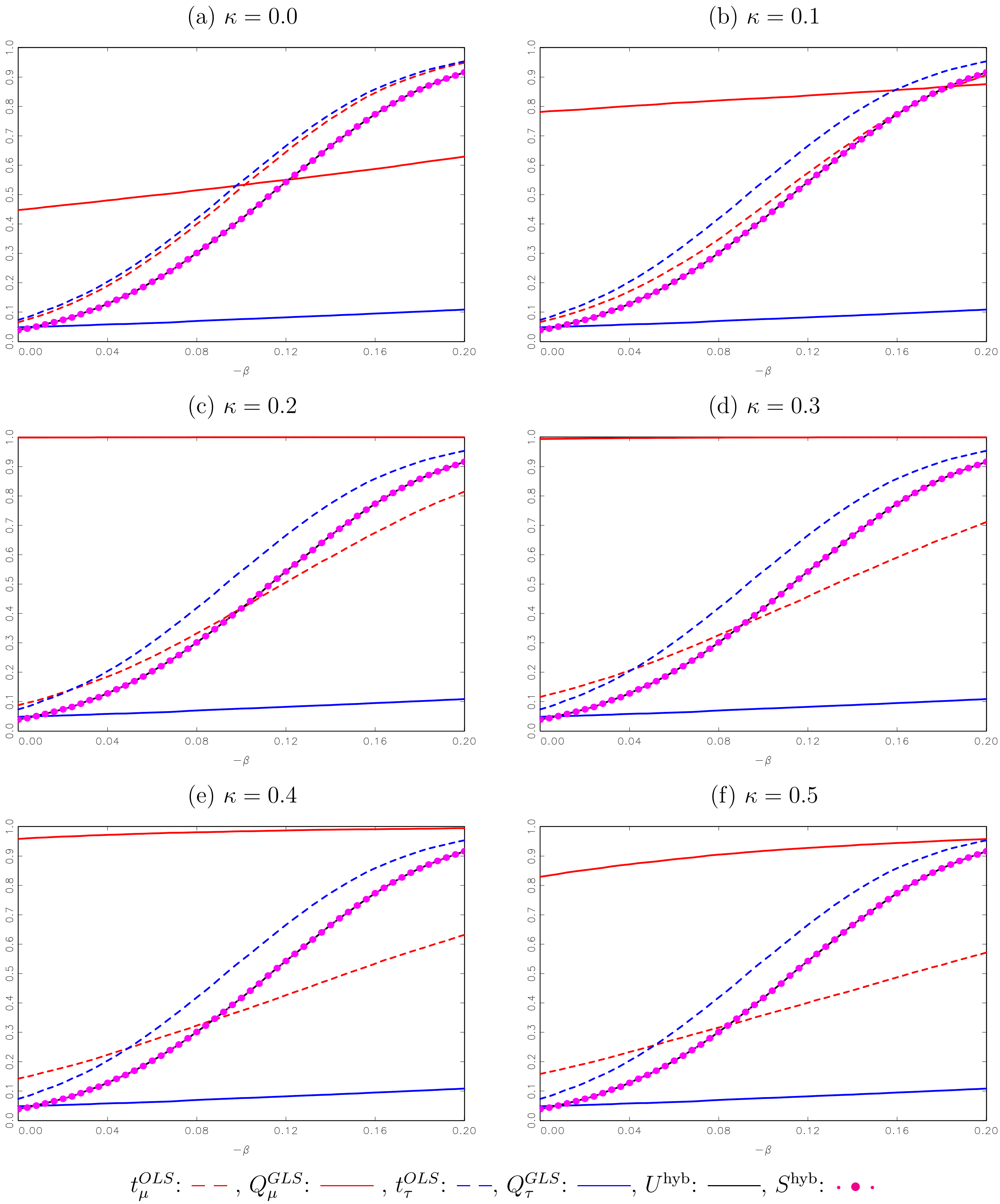


Figure S.26: Finite Sample Power of Right Tailed Tests - $\delta = -0.95$, $c = 2$

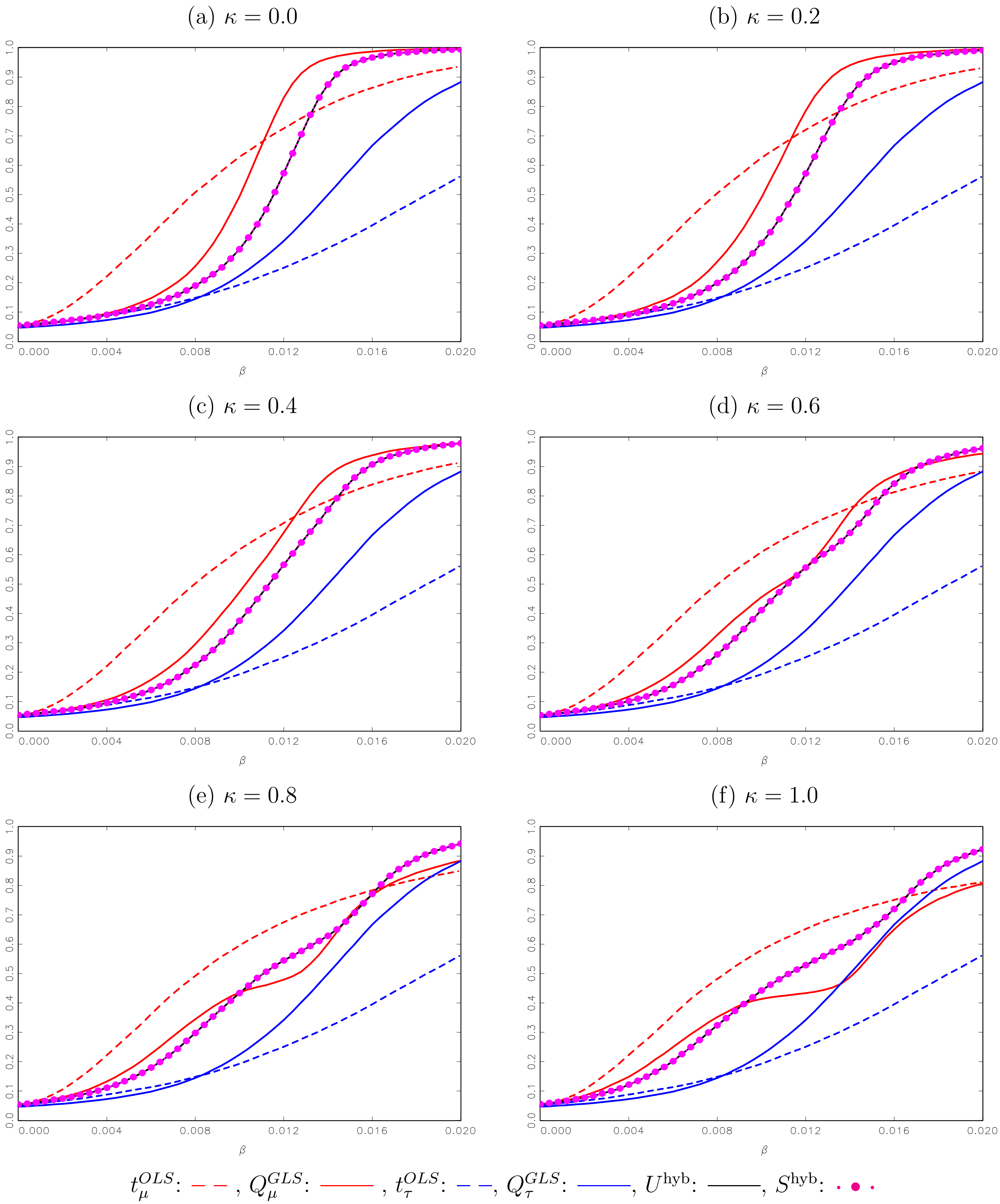


Figure S.27: Finite Sample Power of Left Tailed Tests - $\delta = -0.95$, $c = 2$