Positional Momentum and Liquidity Management; A Bivariate Rank Approach

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Abstract

This paper develops a new positional momentum management strategy, which provides return and liquidity optimizing portfolios. The approach relies on the expected ranks of asset returns and trade volume changes predicted from a bivariate Vector Autoregressive (VAR) model. The new method is applied to a panel data of 1330 stocks traded on the NASDAQ between 1999 and 2016. The return-optimizing positional momentum strategy based on the bivariate ranks is shown to outperform the positional momentum strategy with predicted return ranks, the standard momentum strategy and the equally weighted portfolio. This suggests that volume ranks help improve the performance of a positional momentum portfolio. Moreover, a new positional liquidity management strategy is introduced. The positional liquidity portfolios are shown to produce even higher average and cumulative returns over time than the return-optimizing positional momentum portfolios.

Keywords: Positional Momentum Strategy, Gaussian Ranks, panel VAR Model, Positional Momentum Portfolio, Positional Liquidity Portfolio.

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1 Introduction

In the financial literature and practice, the mean-variance portfolio is usually considered as the benchmark for portfolio management. At each time t, the mean-variance allocation is derived by maximizing the expected future portfolio returns adjusted for conditional risk, which is measured by the volatility. The conditional expected return and volatility are usually computed from the lagged expected returns. It is known that this optimal strategy introduced by Markovitz (1952) can have poor performance in practice, especially when the number of assets in the portfolio is large. This is due to inaccurate estimation of the inverse of the volatility matrix that affects the estimated portfolio allocations and to highly erratic evolution of the mean-variance portfolio allocation, which requires high turnover and increases the associated trading costs. This explains why other portfolio managements can compete with the mean-variance management, some of which being naive, such as the equally weighted portfolio management, for example.

This paper is focused on the so-called momentum (or contrarian momentum) strategies. The basic momentum strategy consists in ranking the asset returns at time t and then building an equally weighted portfolio from the top alpha-percentile of all assets. The value of alpha is fixed to get the desired top percentile, such as the fifth top percentile, for example. The contrarian momentum strategy builds an equally weighted portfolio from the lower alpha percentile. These strategies, unlike the mean-variance approach, provide stable portfolio allocations over a given investment horizon, due to the discretization that underlies the rankings. They also lead to better ex-post realized Sharpe performance than the ex-post realized Sharpe performance of the mean-variance portfolio.

The aim of this paper is to extend the class of momentum strategies in two respects. First, I use jointly the ranks of asset returns and the ranks of trade volume changes. The motivation stems from the empirical evidence documented in financial literature which suggests that the trade volumes provide additional information and help predict the distribution of future returns. Conversely, the asset returns provide information and help predict the future trade volumes. We show that the momentum portfolios based on the bivariate ranks of return and trade volumes outperform the momentum portfolios based on the univariate return ranks only. Second, I modify the objective function of the investor under the momentum strategy. The investor considered in this paper is optimizing the future return by building the portfolio from assets with high future ranks of returns. The investor uses a positional criterion that takes into account the performance of other assets, instead of an absolute criterion such as the Sharpe ratio. Such a positional criterion replaces the momentum criterion based on the current observed ranks by the momentum strategy based on the expected future ranks. The approach involving the predicted future return ranks and called the positional momentum strategy has been proposed by Gagliardini, Gourieroux and Robin (2013). The search for returns with high future ranks (or high Sharpe performance) does not protect the investor from future high liquidity risk. In order to build a future return-optimal portfolio, the investor may end up with an illiquid portfolio. This paper introduces a new positional liquid portfolio that contains assets that display the highest (resp. lowest) changes in trade volumes. I find that the positional portfolios based on the expected bivariate ranks outperform the positional momentum portfolios of Gagliardini, Gourieroux and Robin (2013) as well as the standard momentum strategies [see. e.g. Jegadeesh and Titman (1993),(2000), Hellstrom (2000), Arena. et al., (2008))] and the equally weighted portfolio. Moreover, I introduce the positional liquid portfolio of stocks. The strategy that selects stocks with low expected volume changes lead to even higher average and cumulative portfolio gains than the return rank-based momentum and reversal strategies. The panel VAR model is used to represent the dynamics of bivariate ranks. The main advantages of panel VAR models are as follows (i) they have an ability to capture both dynamic and dynamic inter-dependencies; (ii) they can treat the links across units without any restrictions; (iii) they easily incorporate time variations in the coefficients and in the variance of the shocks, and (iv) they can account for considered for cross sectional dynamic heterogeneities. The panel VARs resemble standard VARs but, due to a cross sectional dimension, they are a much more powerful tool. In this paper, the panel VAR model is estimated on returns and trade volumes ranks of 1330 stocks traded on NASDAQ and used to build the positional momentum and liquidity portfolios. The paper is organized as follows: Section 2 provides the description and summary statistics of data on returns and trade volumes. Section 3 introduces the ranks, which at each point in time rank the securities according to their relative returns and trade volume changes cross-sectionally. Their transformation to Gaussian ranks is also explained. The panel VAR model of the bivariate Gaussian ranks and its estimation are discussed in Section 4. The optimal positional portfolios based on positional management are defined and examined in section 5. The proposed portfolios are compared to the equally weighted portfolio and the positional momentum and reversal portfolios. The conclusion of the portfolio management strategies based on the bivariate VAR model is given in Section 6. Additional results and proofs are provided in Appendices A and B.

2 Data Description

The panel data contains monthly returns and trade volumes of 1330 stocks traded on the NASDAQ market from October 1999 to October 2016. These stocks have been chosen with respect to the daily average of Turnover/Traded Value of all stocks in NASDAQ in 2015. The Turnover/Traded Value is defined as the total amount traded in the security's currency, which is calculated as the sum of the products of numbers of shares and their corresponding prices. Stocks which had the highest and the lowest 25% of Turnover/Traded Value have been selected. After dropping stocks with missing values during October 1999 and October 2016, we end up with 1330 stocks.

The trade volume of a security is defined as the total quantity of shares traded for a specified security multiplied by the closing price of that security. To get the return and the changes in trade volume, I calculate the log return and the log volume changes as follows:

$$\begin{aligned} r_{it} &= ln(\frac{P_{it}}{P_{it-1}}) \qquad t = 1,\cdots,T;\ i = 1,\cdots,n \\ tv_{it} &= ln(\frac{TV_{it}}{TV_{it-1}}) \qquad t = 1,\cdots,T;\ i = 1,\cdots,n \end{aligned} \tag{2.1}$$

where P_{it}, P_{it-1} are the price at time t and $t-1$, TV_{it}, TV_{it-1} are the traded volume at time t and $t-1$ and r_{it}, tv_{it} are the log return and log changes in trade volume for stock i at time t respectively. The panel contains $n = 1330$ stocks observed over $T = 214$ periods in time (month).

Figures 1 and 2 present the cross-sectional mean and variance of the returns (r_t) and trade volume changes (tv_t) over time. We see that, the mean returns and mean volume changes do not show any seasonality or trend over time. According to these figures, the mean and variance of trade volume changes are more volatile than those of return. From Oct,2000 up to Oct,2004, the mean return varies a lot, while it took a sharp downturn in July 2001 and May 2002. According to a report by the Cleveland Federal Reserve, this downturn can be viewed as part of a larger bear market or correction that began in 2000. The majority of the specialists believe that, this downturn could be a reversion to average stock market performance in a longer term context. From 1998 to 2000, the NASDAQ rose almost 85% while before that time it had the annual growth of 10% to 15%. So, after this bubble the index had only dropped to the same level it would have achieved if the annual growth rate followed during 1987-1995 had continued up to 2002. On September 16, 2008, the mean of returns reached its lowest value. The reason was the massive failures of financial institutions in the United States, due primarily to exposure to packaged sub-prime loans and credit default swaps issued to insure these loans and their issuers, which rapidly devolved into a global crisis. These financial failures resulted in a number of bank failures in Europe and sharp reductions in the value of stocks and commodities worldwide. Another major fall in stock market was Black Monday of 2011, which refers to August 8, 2011, when US and global stock markets crashed following the Friday night credit rating downgrade, by Standard and Poor's of the United States sovereign debt from AAA, or "risk free", to AA. After that in 2014 and 2016, the stock market had experienced the Bull Market. Retail investors, started to put money back in the market in 2013, allowing them to benefit from 2014 's advance.

The stock market in 2014 also reflected the significant shifts in the business world and wider society; for instance, a torrent of corporate mergers helped drive stocks higher. Although a near halving in the price of crude oil in 2014 was bad news for many energy companies, but the decline in the cost of fuel was a big boom for airlines. So as a result, most of airlines' stock rocketed in 2014. Another factor, which caused this upward trend, was the low bond yields while pushing investors into the stock market to earn returns that they would normally have earned from bonds.

In Figure 2 we observe that the variance of returns and the variance of volume changes are an average above 0.01. Although in most periods they fall within a bound of 0.01 to 0.02. There have been periods when variance was unusually high or unusually low, and often extreme periods in one direction are followed by oppositely extreme periods. From the beginning of 2000 the volatility of the return started to decrease gradually until 2004, when it reached a more steady pattern. In 2008, the volatility of returns surged to more than 0.07, which is fairly high by historical standards, yet not without precedent.

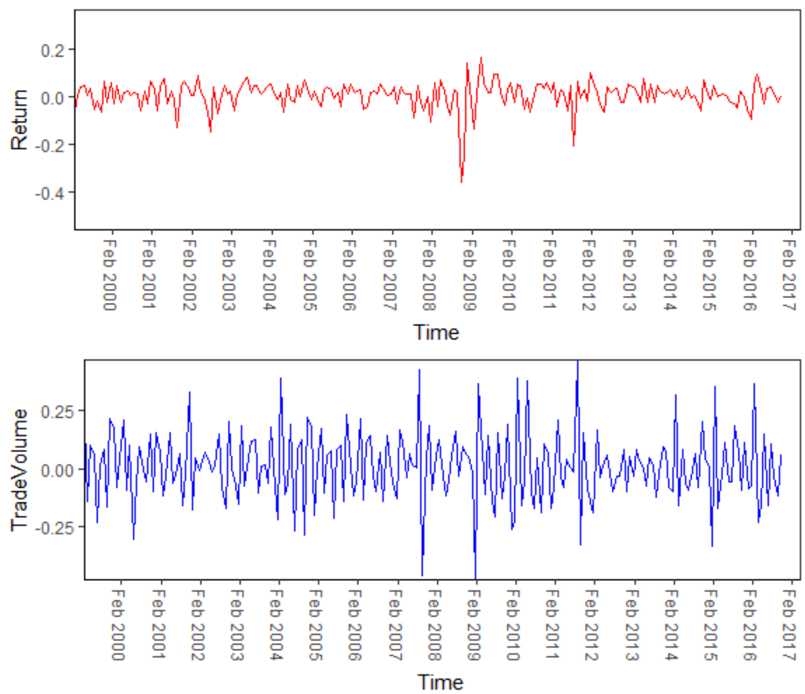

Figure 1: Cross-Sectional Mean Over Time

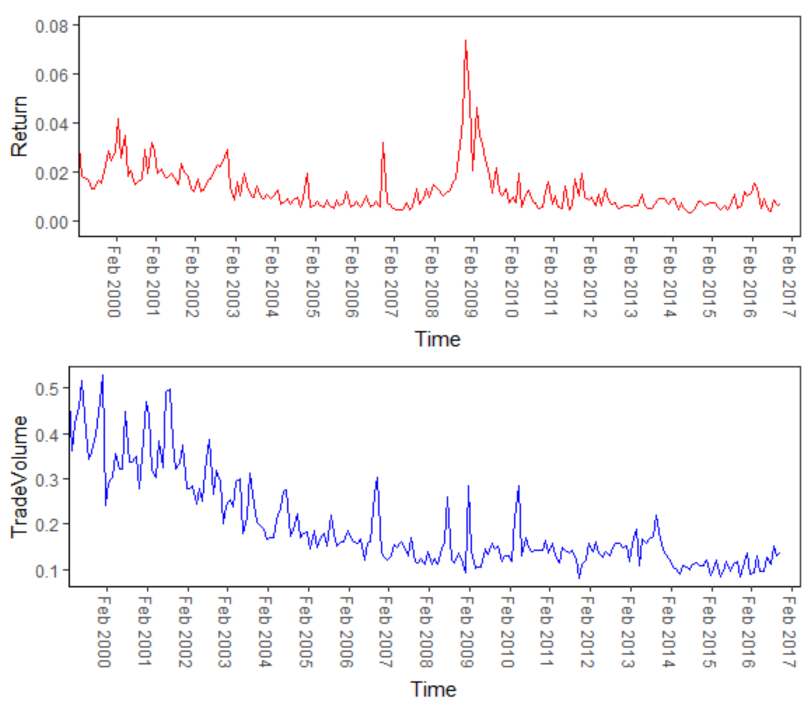

Figure 2: Cross-Sectional Variance Over Time

Soon after, the volatility settled back within the typical range the mid-range that acts as a holding pattern until volatility again breaks in either direction. The cross sectional variance of trade volume changes dropped from 0.4 in 1999 to less than 0.2 in 2004 and fluctuated between 0.1 and 0.2 until 2016. In some periods of time it had extreme positive movement which means that in those periods the trade have been differed largely compare to its mean.

2.1 Data Stationarity

To check the stationarity of returns and trade volume change series, the unit root test is applied while assuming that the series are cross-sectionally independent. In the panel unit root test literature, the null hypothesis is formally stated as H_0: "all of the series have one unit root". While the null hypothesis is common to all the panel unit root tests, the literature considers two different alternative hypotheses, H_1^a : "all of the series don't have unit root" and H_1^b : "at least one of the series has unit root". The alternative H_1^b has been criticized by some authors indicating that if H_0 is rejected we don't know which series have a unit root (Taylor and Sarno 1998). On the other hand, alternative H_1^a implicitly imposes a strong dynamic homogeneity restriction across the panel units (Levin et al. (2002), Im, Pesaran and Shin (2003)) while it may also has power in mixed situations where not all the series are stationary. In practice, those tests that consider the alternative H_1^a are less flexible and may be subject to the same criticism as those considering the alternative H_1^b. Given these two alternative hypotheses the panel unit root tests, can be obtained in two ways: Approach 1 is based on the t-ratio and approach 2 is based on the p-value. In the first case the alternative hypothesis is H_1^a and in the second one it is H_1^b (Maddala and Wu (1999) and Choi (2001)).

The tests based on the t-ratios are panel extensions of the standard Augmented Dickey-Fuller test (ADF) (Said and Dickey 1984). There are two ways of applying these tests for panel data, either by pooling the units before computing a pooled test statistic (Levin et al.(2002)), or averaging the individual test statistics in order to obtain a group-mean test (Im et al.(2003)). On the other hand, the p-value combination tests are based on the idea that the p-values from N independent ADF tests can easily be combined to obtain a test on the joint hypothesis concerning all the N units. The advantages of the p-value combination approach derive from its simplicity and flexibility in specifying a different model for each panel unit and the ease in allowing the use of unbalanced panels. In the following Table 1 provide the result of the stationarity tests for returns and trade volume changes.

Table 1: Stationary Test

Variables	*Levinlin*	*Ips*	*Madwu*	*Pm*	*Invnormal*	*Logit*
Returns	-565***	-539***	153322***	2065***	-377***	-1159***
TradeVolumeChanges	-473***	-561***	154848***	2086***	-380***	-1170***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

In Table 1, Column 1 and 2 show the outcomes of the tests based on t-ratio which was introduced by Levin and Lin (Lin and Chu (2002)) and Im et al.(2003). Columns 3 to 6 present the outcomes of tests based on p-value by Maddala and Wu (1999), the modified p-test proposed by Choi (2001), the inverse normal test by Choi (2001) and the logit test by Choi (2001), respectively. All of these tests indicate that the data of monthly returns and trade volume changes are stationary.

3 Ranks of Returns and Trade Volumes Changes

3.1 Ranks

The updating of the component of assets and their quantities in a portfolio is a key problem in portfolio management. One of the most commonly used approach is by ranking the stocks returns, and building the different momentum portfolios based on those ranks. The literature shows that a momentum strategy based on return ranks can outperform the mean-variance strategy based on returns. This is usually explained by the fact that returns are more volatile than their ranks and the momentum strategy is less sensitive to extreme volatility and more robust. Many articles show that, the rank of stock returns are more predictable than the individual returns. Hellstrom (2000), found that the ranks can be predicted with a linear model and his empirical results shows 63% hit rate for the sign of daily threshold-selected 1-day predictions.

There exist ex-post and ex-ante ranks. The ex-post ranks are obtained by ranking all asset returns at time t from the smallest one to the largest one and then dividing their position by the total number of observations. Equivalently the ex-post return rank of asset i can be derived by inverting the empirical cross-sectional (CS) cumulative distribution function (cdf) of the returns at date t. In this case the observed ex-post ranks have the discrete empirical uniform distribution on $(1/n, 2/n, ..., 1)$. In the ex-ante ranks, the empirical cross-sectional cdf is replaced by its theoretical distribution function, so the ex-ante ranks have a cross-sectional uniform distribution on the interval $[0, 1]$.

In other words, the ex-ante ranks are predicted ranks. Since the ranks are defined up to an increasing transformation, I choose the following transformation to build Gaussian ranks (see Gourieroux et al.(2013)). The Gaussian ranks are obtained from the corresponding uniform ranks by applying the quantile function of the standard Normal distribution. Standardizing the ranks ensures the cross-sectional Normal distribution of the rank variables. I consider two ex-post Gaussian ranks, one based on stock returns and the other based on their trade volume. These rank series are related to the returns and trade volume changes by the following equations:

$$u_{i,t} = \Phi^{-1}(\hat{F}_t^r(i,t)) \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{3.2}$$

$$v_{i,t} = \Phi^{-1}(\hat{F}_t^{tv}(i,t)) \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{3.3}$$

where $u_{i,t}$ is the Gaussian rank of return, $v_{i,t}$ is the Gaussian rank of trade volume changes, Φ^{-1} is the empirical cross-sectional cumulative distribution function (cdf) of the return and trade volume at date t and $\hat{F}_t^r, \hat{F}_t^{tv}$ are the cross-sectional empirical cumulative distribution functions of return and trade volume changes.

To compute the Gaussian ranks, first I ordered all returns and trade volume changes from the highest to the lowest for each month and assign them absolute ranks from 1 to 1330 (since my sample includes 1330 stocks which are trading in NASDAQ). Next, I divide these ranks by the total number of stocks which gives me the probability or the position of each stock in comparison to all observation in each month. That procedure provides the empirical cross-sectional cumulative distribution functions $\hat{F}_t^r, \hat{F}_t^{tv}$. Finally to transform these ranks to the Gaussian ranks, I find the equivalent quantile of the standard Normal distribution function for each probability. For instance, if asset i has return probability equal to $\hat{F}_t^r = 0.90$, it means that, there are 90% of assets in the sample, which have smaller or equal returns on time t, and other 10% of assets have larger returns. In another words, if asset i has rank 0.90, there is a probability equal to 90% that the return at time t of any other asset is smaller or equal to the return of asset i. For this particular stock, the corresponding Gaussian rank is $u_{it} = 1.28$, that is the 90% quantile of the standard normal distribution function.

Figures 3 and 4 display the Q-Q plots of the two transformed observed ranks in October 2016. The figures confirm the cross-sectional Gaussian distribution of return ranks and trade volume change ranks. We see that both ranks are cross-sectionally Normally distributed. Also, the Shapiro normality test for each month, indicate that both ranks are Normally distributed cross-sectionally at each period in time.

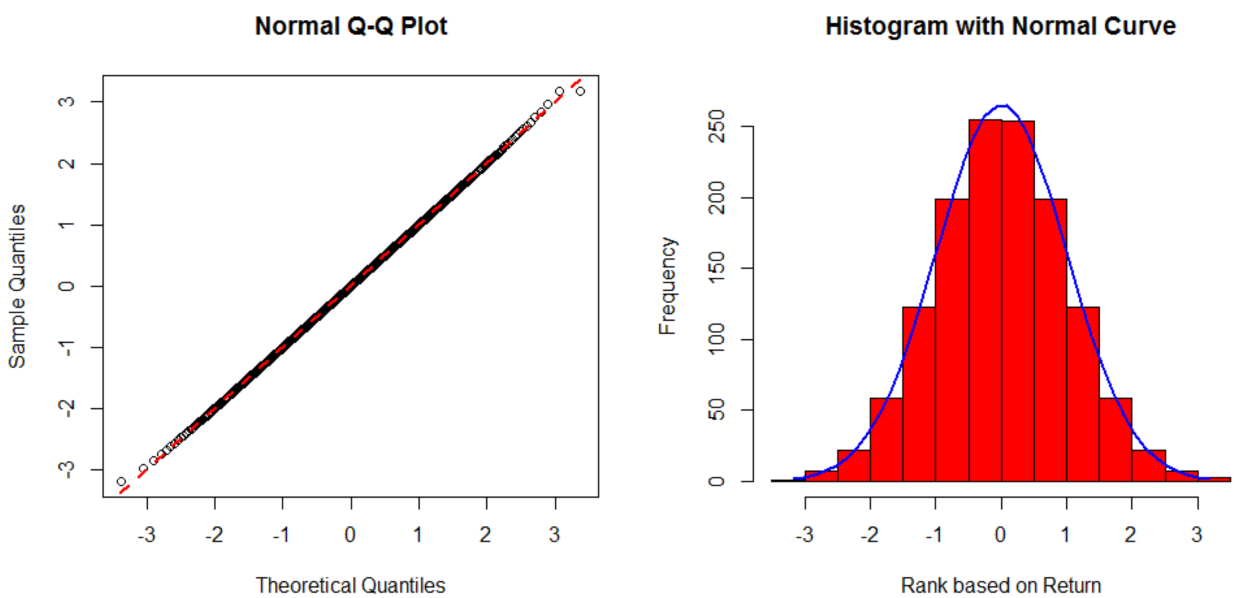

Figure 3: QQ Plot and Histogram of u in October 2016

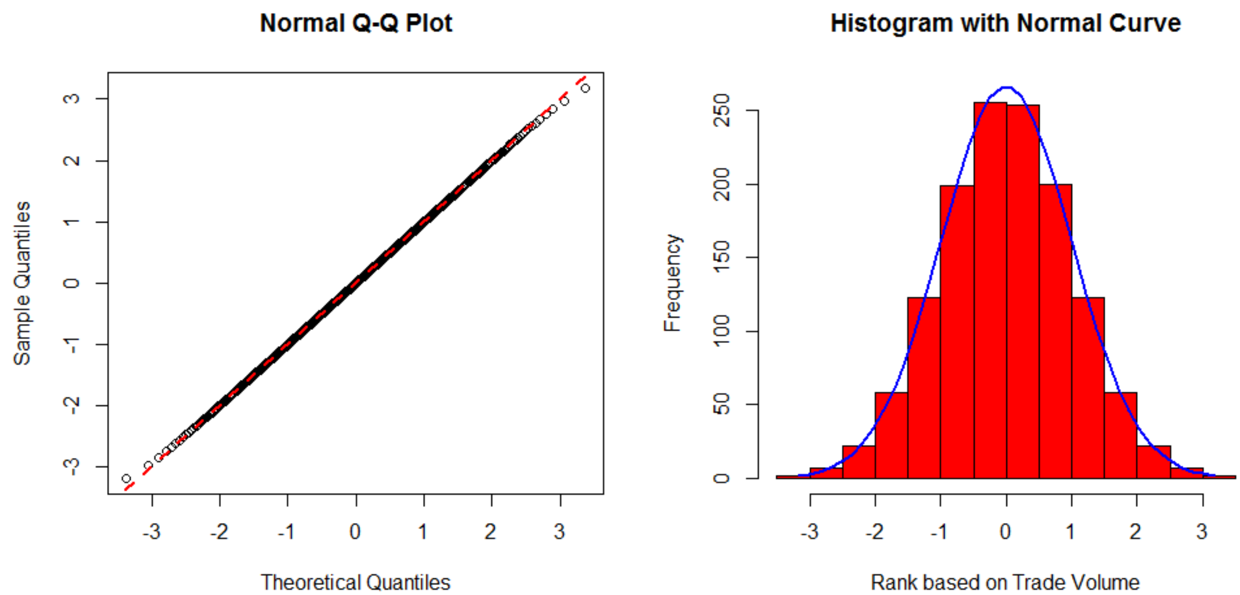

Figure 4: QQ Plot and Histogram of v in October 2016

3.2 Relation Between Return and Trade Volume changes Ranks

3.3 Contemporaneous Correlation

Let us study the relation between these two ranks, to see if the return and volumes rank series are correlated over time. Figure 5 shows the time series of the Contemporaneous correlation between these two ranks, which is computed from the sample of 1330 stocks at each time t. The cross-correlation is fluctuating between -0.2 and 0.2, in a way that in some period of time it is positive and in some periods of times it is negative. It reaches its highest value in 2001 (0.31) and it falls to its lowest value in 2014 (-0.19), while in average the correlation is 0.02. It means that on average, the ranks based on returns increases (decreases resp) as the ranks of trade volume changes increases (decreases resp). On average, their contemporaneous correlation coefficient is positive. Although the correlation coefficient is fluctuating over time, its non-zero average suggests that the past value of the ranks may be helpful to predict their future values.

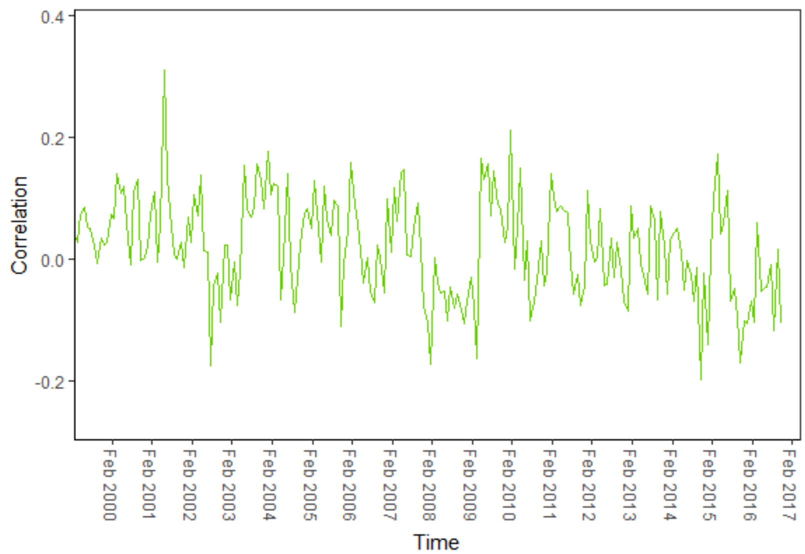

Figure 5: Cross-Correlation Between u and v

3.4 Dynamic Cross-Correlation

For further insights, let us illustrate the rank dynamics of the $S\&P500$ market index. Figure 6, shows the monthly returns and trade volume changes rank series of the $S\&P500$ during 2011-2016. We observe that, these two rank series of $S\&P500$ are fluctuating over time. In some periods of time they are moving in parallel while in other periods they are moving in opposite directions.

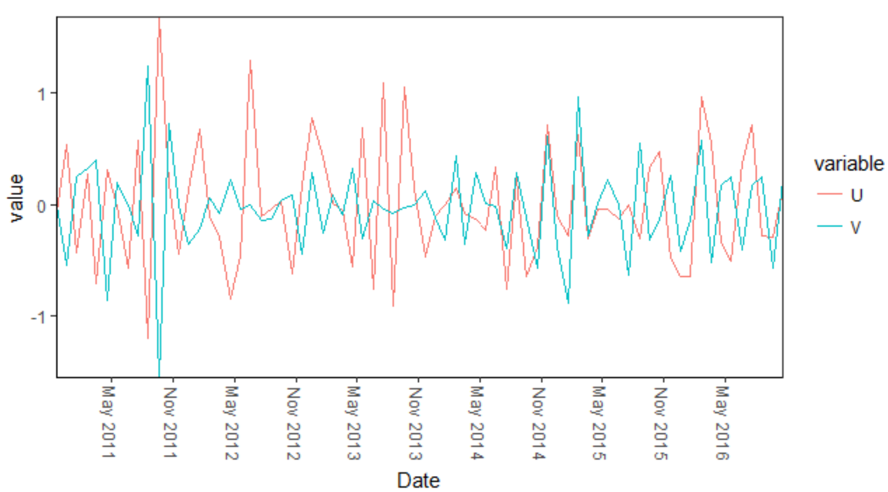

Figure 6: S&P.500 Ranks over Time

To illustrate the dynamic correlation between the two ranks series of the $S\&P500$, we plot the cross-correlation function (CCF). Figure 7, shows the CCF of the ranks of returns and trade volume changes of $S\&P500$ as a proxy of the market. The CCF shows negative contemporaneous correlation between these ranks for $S\&P500$. Also it shows significant correlations at the first and forth lags, which means that last period information may help us to predict the future ranks.

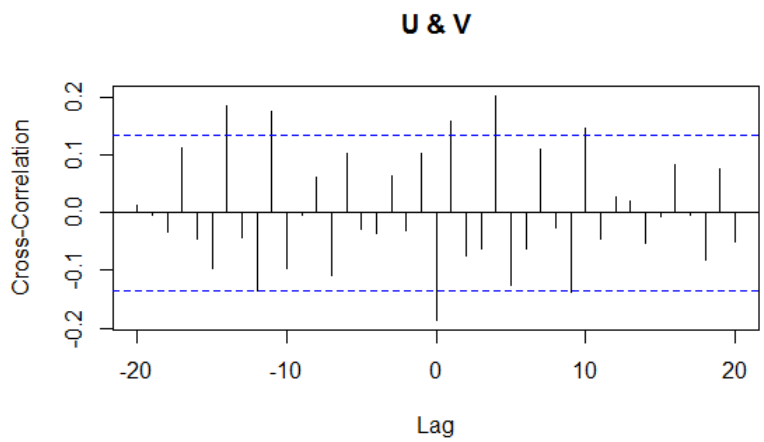

Figure 7: Cross-Correlation Function

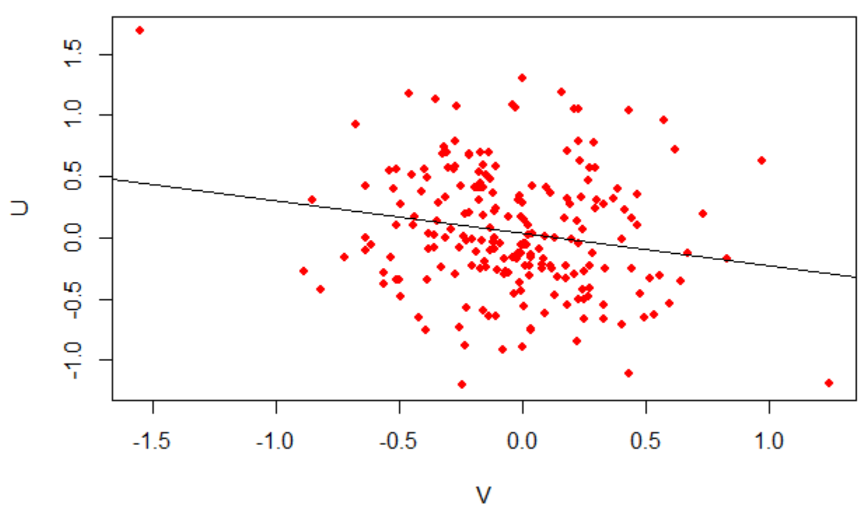

Figure 8: Regression Line for S&P.500

Figure 8, shows the linear regression of these two ranks for $S\&P500$. As observed in the CCF plot, the contemporaneous relationship between these two ranks for the market index $S\&P500$ is negative.

4 The Cross-Sectional Gaussian Ranks Model

In the positional portfolio strategy, we are interested to find the optimal portfolio based on the future position of all equities in the portfolio. The momentum positional portfolio, contains the optimal proportion of equities based on the future return ranks, while the liquid positional portfolio contains shares of each risky assets based on the future trade volume change ranks. To build a positional portfolio based on the future positions, we need to define a joint dynamic model of return and trade volume change ranks. This joint model, specifies the dynamics of the Gaussian ranks and the link between the asset returns and trade volume change ranks. According to the results presented in the previous section, the joint dynamic model of the two rank series is the Vector Autoregressive model of order one (VAR(1)).

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix} \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{4.4}$$

Where ρ_{11} and ρ_{12} are the correlation coefficients between u_{it}, and $u_{i,t-1}, v_{i,t-1}$, ρ_{21} and ρ_{22} are the correlation coefficients between v_{it} and $u_{i,t-1}, v_{i,t-1}$ and Σ represents the error variance matrix. We assume that in equation (4) the idiosyncratic disturbance terms $(e_{1it}, e_{2,it})$ are independent and identically (i.i.d) standard Normal distributed. We also assume the stationarity of the process of the Gaussian ranks and that the eigenvalues of $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ are of a modulus less than 1. I also assumed that, the Autoregressive coefficients are constant and do not change over time and across assets. To estimate the VAR(1) model, let us rewrite it as follow:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = R \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix} \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{4.5}$$

Where R presents a 2×2 matrix of coefficients of the VAR(1) model, $\Sigma^{1/2}$ shows the standard deviation of the error terms. These parameters will be estimated by the constrained maximum log likelihood with the following objective function:

$$logL = \sum_{i=1}^{N} \sum_{t=1}^{T} -log(2\pi) - \frac{1}{2} log(|\Sigma|) - \frac{1}{2} \Big(\begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix} \Big) \tag{4.6}$$

To model the cross-sectionally standard Normal distributed ranks, we need to impose a constraint on the error variance matrix Σ as follows:

$$\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} = R \begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} R' + \Sigma \tag{4.7}$$

This constraint ensures that the variance-covariance matrix of the vector $\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix}$ is $\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix}$. The diagonal values are the variance of u_{it} and v_{it} (which should be equal to 1 for the Gaussian variables) and η presents the contemporaneous correlation term between u_{it} and v_{it}. The VAR model parameters are estimated from the constrained log-likelihood function in equation 8 from the entire sample of 1330 stocks over the periods of 1999 to 2016. Table 2 shows the results of the constrained likelihood estimation.

Table 2: VAR(1) Model

$Coefficients$	$Values$
ρ_{11}	-2.43***
ρ_{12}	1.00***
ρ_{21}	-2.42***
ρ_{22}	-35.41***
η	3.90***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

According to the empirical results, all coefficients of the model are strongly significant. The returns' ranks are correlated negatively to their own past value, while they are correlated positively to the past value of trade volumes ranks. It means that if a stock return is ranked lower at time t or its trade volume is ranked higher than other stocks, we expect its return will have higher rank at time $t+1$. In addition, the ranks of trade volumes are correlated negatively to both past values of trade volumes and returns ranks. It tells us that, if at the current period, the return or trade volume of a stock is ranked lower, we will expect its trade volume has higher ranks in the next period. The contemporaneous correlation η between ranks of returns and ranks of trade volumes is positive. It means that if the rank of trade volume is high it positively affects the rank of return and makes it higher at the same period.

One important characteristic of a VAR process is its stability. When a VAR model is stable, it means that it generates stationary time series with time-invariant means, variances, and covariance structure, given sufficient starting values. In practice, the stability of an empirical VAR process can be analyzed by considering the companion form and calculating the eigenvalues of the coefficient matrix. The obtained eigenvalues for the VAR(1) model in equation (4) are -0.348 and -0.025. Since both eigenvalues are less than one, we can conclude that the system is stationary.

Once a VAR-model has been estimated, it is of essential interest to see whether the residuals obey the model's assumption. That is one should check for the absence of serial correlation and heteroscedasticity and see if the error process is normally distributed. The Durbin-Watson (DW) statistic is a test statistic used to detect the presence of autocorrelation at lag 1 in the residuals (prediction errors) from a regression analysis. The null hypothesis in DW test, indicates that the errors are serially uncorrelated and the alternative hypothesis is the existence of a first order autoregressive process in error terms. The result for DW test, did not reject the null hypothesis against the alternative, which says that there is no autocorrelation in error terms (Apendices A.1).

Given that the sampling period is long, one can be concerned about the stability of the parameters. To examine the fit of the model over the the long sampling period, I have computed the fitted values of the observed ranks. I also compute the time series of the coefficients, which are obtained by re-estimating the model (equation (4)) by rolling with the window of 109 months (9 years). Figures 9 and 10 show the time series of the estimated coefficients from rolling window of sampling. As we can see in these Figures, there is some variation in $\hat{\rho}_{11}$, which is more pronounced than that in $\hat{\rho}_{12}$, while both $\hat{\rho}_{21}$ and $\hat{\rho}_{22}$ are less fluctuating over time. $\hat{\rho}_{11}$ is fluctuating between -0.015 and -0.005, while $\hat{\rho}_{12}$ varies between $0-0.01$. On the other hand $\hat{\rho}_{21}$ fluctuates between -0.015 and -0.010

and $\hat{\rho}_{22}$ varies around -0.18. Given the slight variation of the parameters, I use the rolling estimation methods henceforth.

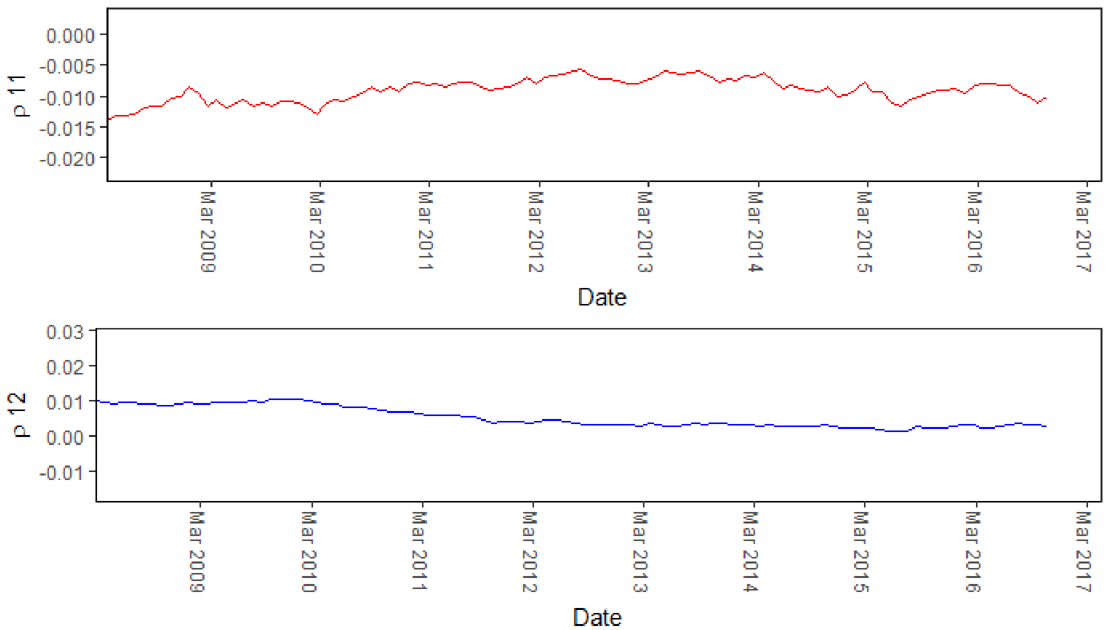

Figure 9: Time Series of ρ_{11}, ρ_{12}

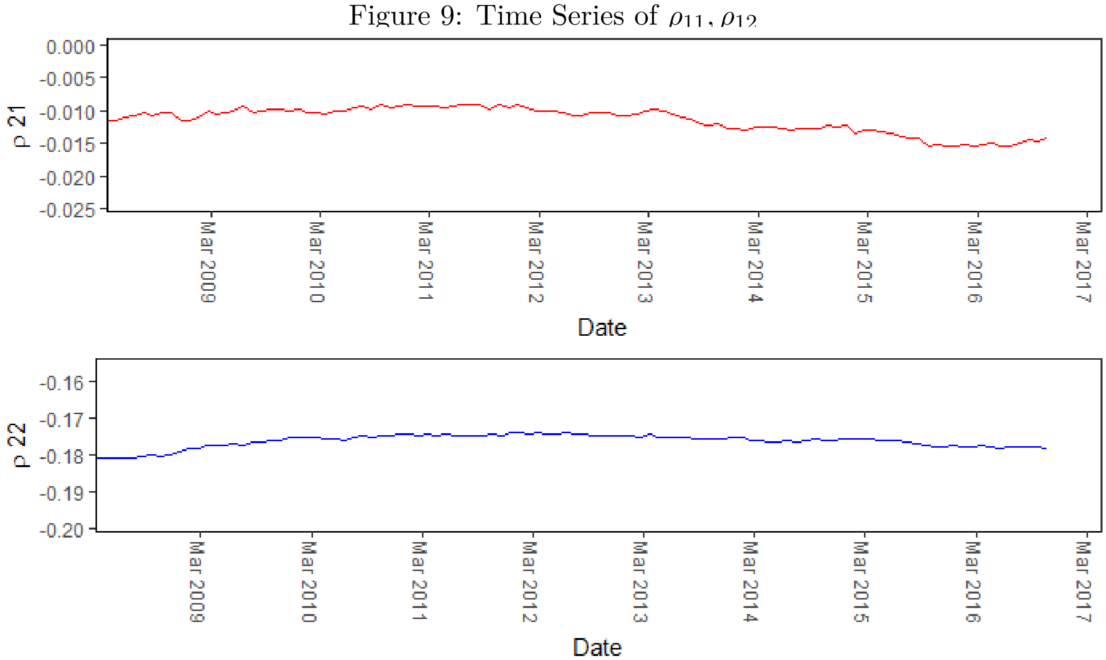

Figure 10: Time Series of ρ_{21}, ρ_{22}

4.1 Evaluating The Out of Sample Forecast Accuracy

To assess the model's forecasting performance we consider the out-of-sample forecasts. Let x_i denote the ith observation at time t fixed and $\hat{x_i}$ denote a forecast of x_i. The forecast error is defined as $e_i = x_i - \hat{x_i}$. The accuracy measures based on e_i are the commonly used scale-dependent measures are based on the absolute error and squared error vectors:

$$\begin{aligned} MeanAbsoluteError : MAE &= mean|e_i| \qquad && i = 1, \cdots, n \\ RootMeanSquaredError : RMSE &= \sqrt{mean(e_i)^2} \qquad && i = 1, \cdots, n \end{aligned} \tag{4.8}$$

To compare the forecasting power of the bivariate model of ranks (VAR) with the univariate model predicted on univariate rank, let us consider the following AR(1) models as the benchmarks for ranks of returns and ranks of trade volume changes, respectively:

$$u_{it} = \rho_{1t} u_{i,t-1} + \sqrt{1 - \rho_{1t}^2} \epsilon_{u,it} \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{4.9}$$

$$v_{it} = \rho_{2t} v_{i,t-1} + \sqrt{1 - \rho_{2t}^2} \epsilon_{v,it} \qquad t = 1, \cdots, T;\ i = 1, \cdots, n \tag{4.10}$$

where $u_{i,t}, u_{i,t-1}$ are the current and past values of return ranks and $v_{i,t}, v_{i,t-1}$ are the current and past values of trade volume change ranks. ρ_{1t}, ρ_{2t} are the autocorrelation coefficients between the current and past values of ranks, $\epsilon_{u,it}$ and $\epsilon_{v,it}$ are the error terms which are assumed to be i.i.d Normally distributed.

In the literature, there exist two methods for choosing the initial sample which are the recursive and Rolling methods. Under the recursive method, one observation is added to an initial data set at each time while under the Rolling method a fixed window of data set is applied to re-estimate the parameters each time. These methods provide two type of predictions: the dynamic and static predictions. The dynamic prediction is based on the fitted values from the previously estimated sub-sample, while the static prediction uses the true value of the data. In this section, the recursive method is applied to produce the dynamic and static forecasts and for computing the MEA and RMSE measures. A sample of 210 observation is used as the initial data set for both forecasting methods. Table 3 and 4 show the MAE and RMSE measures for last three months of the observed sample, i.e. from Aug,2016 to Oct,2016.

Table 3: MAE for Aug-Oct 2016

	$Date$	u_{it}		v_{it}	
		VAR	AR	VAR	AR
	$2016, 08$	0.795	0.796	0.743	0.758
$Dynamic$	$2016, 09$	0.797	0.797	0.813	0.799
	$2016, 10$	0.797	0.797	0.801	0.796
	$2016, 08$	0.795	0.796	0.743	0.758
$dynamic$	$2016, 09$	0.798	0.797	0.711	0.745
	$2016, 10$	0.796	0.797	0.732	0.752

Table 4: RMSE for Aug-Oct 2016

	$Date$	u_{it}		v_{it}	
		VAR	AR	VAR	AR
	$2016, 08$	0.996	0.997	0.935	0.949
$Dynamic$	$2016, 09$	0.998	0.998	1.020	1.00
	$2016, 10$	0.998	0.998	1.002	0.996
	$2016, 08$	0.996	0.997	0.935	0.949
$dynamic$	$2016, 09$	1.000	0.999	0.900	0.932
	$2016, 10$	0.997	0.998	0.929	0.946

Both the MAE and RMSE measures show that, in the first period, the dynamic and static prediction methods suggests the bivariate model of ranks (VAR) predicts future values of ranks more accurately than the (AR) model. In the second and third periods, the dynamic method-based bivariate and univariate models of ranks produce equal errors in predicting the ranks of return, while the bivariate VAR model displays higher error in predicting the ranks of trade volume changes. Over the second period, the static prediction method from the bivariate model of ranks (VAR) has less power in predicting the ranks of returns but more power in predicting the ranks of trade volume changes. Over the third period, the VAR model predict both ranks with less error than the AR model. According to both MAE and RMSE measures, the differences between the bivariate VAR model and the univariate AR model are very small; however, even this small prediction accuracy error can lead to a difference of a large amount in investment funds.

5 Optimal Positional Portfolio

As the return and volume ranks are Normally cross-sectionally distributed, at each time t the relationship between asset i returns and trade volume changes and their respective ranks can be defined by the following stochastic transformations:

$$r_{i,t} = \sigma_{r,t} u_{it} + \mu_{r,t} \qquad t = 1, \cdots, T \tag{5.11}$$

$$tv_{i,t} = \sigma_{tv,t} v_{it} + \mu_{tv,t} \qquad t = 1, \cdots, T \tag{5.12}$$

where $\mu_{rt}, \mu_{tv,t}$ are the cross-sectional means of returns and trade volumes and $\sigma_{rt}, \sigma_{tv,t}$, represent the cross-sectional standard deviations of the marginal Normal distributions of return and trade volume at time t. This transformation, implies that the cross-sectional marginal distributions of asset returns and trade volumes at date t are Gaussian as well ($N(\mu_{r,t}, \sigma_{r,t})$ and $N(\mu_{tv,t}, \sigma_{tv,t})$) respectively. Because $\mu_{j,t}$ and $\sigma_{j,t}$ where $j = \{r, tv\}$ vary over time, these parameters can be considered as dynamic macro factors which affect the ranks of all stocks in the same way [see Gourieroux et. al. (2013)]. In particular, this specification is compatible with the conditional heteroskedasticity of individual asset returns and trade volumes over time. As mentioned earlier, the function Φ^{-1} in equations (2) and (3), which maps the returns and trade volumes of all assets at time t into their Gaussian ranks, is the inverse of the Normal cumulative distribution function, which is also called the Quantile function. The Quantile function is denoted by Q_t as it is time varying and is defined below for the return and trade volumes:

$$Q_t(r_t) = \frac{r_t - \mu_{r,t}}{\sigma_{r,t}} \tag{5.13}$$

$$Q_t(tv_t) = \frac{tv_t - \mu_{tv,t}}{\sigma_{tv,t}} \tag{5.14}$$

Now let us consider a momentum and liquid portfolios which each contains both risky and risk free assets with the relative risky allocation vectors β_r' and β_{tv}' . The future return and trade volume of the risky part of these portfolios are given by:

$$\beta_r' r_{i,t+1} = \sigma_{r,t+1} \beta_r' u_{i,t+1} + \mu_{r,t+1} \tag{5.15}$$

$$\beta_{tv}' tv_{i,t+1} = \sigma_{tv,t+1} \beta_{tv}' v_{i,t+1} + \mu_{tv,t+1} \tag{5.16}$$

By substituting equations (15) and (16) in to (13) and (14) respectively, we obtain the future positions of the risky part of the portfolios as follow:

$$Q_{t+1}(\beta_r' r_{i,t+1}) = \frac{\sigma_{r,t+1}\beta_r' u_{i,t+1} + \mu_{r,t+1} - \mu_{r,t+1}}{\sigma_{r,t+1}} = \beta_r' u_{i,t+1} \tag{5.17}$$

$$Q_{t+1}(\beta_{tv}' tv_{i,t+1}) = \frac{\sigma_{tv,t+1}\beta_{tv}' v_{i,t+1} + \mu_{tv,t+1} - \mu_{tv,t+1}}{\sigma_{tv,t+1}} = \beta_{tv}' v_{i,t+1} \tag{5.18}$$

Equations (17) and (18) show that, the position of the future return and trade volume of the risky part of the momentum and liquid portfolios is a linear combination of the future Gaussian ranks of the individual risky assets $(u_{i,t+1}, v_{i,t+1})$, with weights equal to the relative risky allocations β_r' and β_{tv}'. This result is a consequence of the linearity of the transformed quantile function Q_{t+1} under the Normality assumption on the cross-sectional distributions, and holds for any dynamics of the ranks. In other word, the future positions of the return and trade volume of the risky part of the portfolio are equal to the shares of each asset in the portfolio multiplied by its future rank. Therefore in order to predict the future positions of returns and trade volumes, we can use their future Gaussian ranks weighted by their respective shares in each portfolio. More specifically, by considering the dynamics of the ranks introduced in section 3.2.(equation (4)), we obtain the future positions of the returns and trade volumes, based on their ranks as follow:

$$Q_{t+1}(\beta_r' r_{i,t+1}) = \rho_{11}\beta_r' u_{i,t} + \rho_{12}\beta_r' v_{i,t} + \beta_r' e_{1,it+1} \tag{5.19}$$

$$Q_{t+1}(\beta_{tv}' tv_{i,t+1}) = \rho_{21}\beta_{tv}' u_{i,t} + \rho_{22}\beta_{tv}' v_{i,t} + \beta_{tv}' e_{2,it+1} \tag{5.20}$$

These equations show that the future positions of the returns and the trade volumes can be easily computed from their past ranks of returns and trade volumes. To get the positional allocation [see Gourieroux et al. (2013)], let us assume a CARA (Constant Absolute Risk Aversion) utility function as $\mathscr{U}(\nu; A) = -exp(-A\nu)$, written on the Gaussian ranks of future returns and trade volumes of the risky part of the portfolio, with a risk aversion parameter $A > 0$. After substituting the quantile functions (equations (19) and (20)) in the utility function, the expected positional utility functions are as follows (see Appendix B.1):

$$-E[exp(-AQ_{t+1}(\beta_r r_{t+1})) \mid r_t, tv_t] = -exp\Big(-A\rho_{11}\beta_r' u_{i,t} - A\rho_{12}\beta_r' v_{i,t} + \frac{A^2}{2}\beta_r'\beta_r\sigma^2_{1,it+1}\Big) \tag{5.21}$$

$$-E[exp(-AQ_{t+1}(\beta_{tv} tv_{t+1})) \mid r_t, tv_t] = -exp\Big(-A\rho_{21}\beta_{tv}' u_{i,t} - A\rho_{22}\beta_{tv}' v_{i,t} + \frac{A^2}{2}\beta_{tv}'\beta_{tv} r\sigma^2_{2,it+1}\Big) \tag{5.22}$$

In each of the above equations (21) and (22), the expected positional utility is independent of the cross-sectional mean and standard deviation of returns and trade volumes (μ_{rt}, μ_{tvt} and $\sigma_{rt}, \sigma_{tvt}$) at time t and depends on the current position of asset i return and trade volume (u_{it} and v_{it}). The optimal positional portfolio allocations are derived by maximizing the terms inside the parentheses on the right hand sides of the equations with respect to vectors β_r, β_{tv} and subject to the budget constraint, which is $\beta_r e = 1, \beta_{tv} e = 1$ (where e is a vector of ones). By maximizing the expected utilities, the optimal relative positional allocation of the risky assets in the positional momentum and liquid portfolios are derived (proof is in Appendix B.2):

$$\beta_r^* = \frac{1}{ne} + \frac{1}{A}\frac{\rho_{11} + \rho_{12}}{(\sigma^2_{1,it+1})}(u_t - \overline{u_t}e) \tag{5.23}$$

$$\beta_{tv}^* = \frac{1}{ne} + \frac{1}{A}\frac{\rho_{12} + \rho_{22}}{(\sigma^2_{2,it+1})}(v_t - \overline{v_t}e) \tag{5.24}$$

In the above equations $\overline{u_t} = u_t'e/n$ and $\overline{v_t} = v_t'e/n$ are the cross-sectional averages of the Gaussian ranks at time t. When the number of assets (n) tends to infinity, these cross-sectional averages tend to 0, which is the mean of the standard Normal distribution. In that case, the optimal relative positional allocations $\beta_r'^*, \beta_{tv}'^*$ in expressions (23) and (24) are each a linear combination of two well-known portfolios. The first one is the equally weighted portfolio, with weight 1/n for each asset and the second portfolio is an arbitrage portfolio (zero-cost portfolio) with dynamic allocations proportional to the deviations of the current ranks from their cross-sectional averages. Since this arbitrage portfolio contains the vector of expected future ranks in deviation from their cross-sectional averages

$\{(\rho_{11}+\rho_{12})(u_t-\overline{u_t})\ ,\ (\rho_{21}+\rho_{22})(v_t-\overline{v_t})\}$, it can be interpreted as a momentum portfolio. Moreover, when the sign of the sum of persistence coefficients $\rho_{kk}+\rho_{kj}$ (where k,j=1,2) is positive, the arbitrage portfolio will be long in assets with large expected future ranks, and when the sum of persistence coefficients is negative then, it will be short in assets with small expected future ranks. This interpretation of the arbitrage part of the positional portfolio implies that, the optimal positional allocation vector in risky assets is formed from two portfolios, which are an equally weighted and a momentum portfolios. The weight of the arbitrage portfolio in the risky allocations β^* is positively correlated with the persistence of ranks coefficients ρ_{kj} and negatively correlated with the risk aversion coefficient A of the investor and the variances of the ranks. The optimal positional allocation deviates from the equally weighted portfolio by over-weighting the assets with larger (resp. smaller) current ranks, when the sum of persistence coefficients is positive (resp. negative).

The allocation vectors in equations (23) and (24) that determine the positional portfolio management depend on the choice of the positional utility function, the marginal distributions of ranks - which can be uniform or Gaussian - and on the positional universe of stocks which is used to compute the ranks. Moreover, the optimal allocations β^* of the positional fund manager are defined by considering functions Q_{t+1} as the exogenous functions. In this paper, the quantile function is used as an exogenous function.
In this paper the positional momentum portfolio is defined as follow;

Definition 1: The positional momentum portfolio based on the bivariate ranks of returns and volume changes is a portfolio of assets with high expected future return ranks conditional on the current and past ranks of their returns and volume changes.

As the liquidity ensures uninterrupted availability of funds to meet loan disbursements, debt servicing and other cash requirements, we extend this approach further and introduce a new positional liquid portfolio as follows;

Definition 2: The positional liquid portfolio based on the bivariate ranks of returns and volume changes is a portfolio of assets with high expected ranks of future trade volume changes conditional on the current and past ranks of stock returns and volume changes.

It has been observed that stocks reverse in returns at short monthly horizons (see e.g. Jegadesh (1990) and Avramov et. al. (2006)) likely due to overreaction of some investors to news (De Bondt and Thaler (1985)). Therefore, we also consider the positional reverse momentum portfolios and positional reverse liquid portfolios which contain stocks with low expected return ranks and low expected volume change ranks, respectively. The next Section illustrates the management of the above portfolios in practice and examines their performances.

5.1 Momentum and Liquid Portfolio Management Based on Bivariate Ranks

This section illustrates the positional momentum portfolio management and the liquid portfolio management based on the expected ranks of return and volume changes conditional on their past, and examines the comparative performance of the proposed portfolios. The basic momentum strategy consists in adjusting the portfolio by buying stocks or other securities with high observed past returns and selling stocks with poor observed past returns. The positional momentum strategy, introduced by Gourieroux et. al. (2013) extends that methodology by adjusting the portfolio so that at each time t it contains assets with high expected return ranks, which are forecasted out-of-sample from a univariate autoregressive model of return ranks.

The momentum positional portfolio based on the expected return ranks in Definition 1, is adjusted at each time t to contain stocks with high expected return ranks, conditional on the past return and volume change ranks. In our study, the expected ranks are forecasted out-of-sample from the bivariate VAR model (equation (4)) of return and volume change ranks at each time t. Section 5.2 examines the comparative performance of the positional portfolios. The positional liquid portfolio based on bivariate ranks in Definition 2, relies on the expected volume change ranks at time t. In practice, this portfolio is adjusted to contain assets with high forecasted volume change ranks from the bivariate VAR model of return and volume ranks (equation 4) at each time t. Section 5.3 examines the performance of the positional liquid portfolios.

In the positional strategy, it is important to define the investment universe which can be different than the positional universe. The investment universe is a set of assets potentially introduced in the portfolio, while the positional universe is a set of assets that has been used to define the ranks. For instance, for a fund manager, the investment universe may be a fraction of the stocks, whereas the positioning universe can be the set of all stocks which are trading in the stock market. This study, considers the investment universe equivalent to the positional universe, which contains 1330 stocks returns and trade volumes in the balanced panel from NASDAQ market from 1992 to 2016 .

5.2 The Positional Momentum Portfolio

This section examines the performance of the proposed VAR-based positional strategies. It also compares the proposed methodology the positional momentum strategy of Gourieroux et. al. (2013) and based on predicted return ranks from the AR model, which is estimated monthly using a rolling window of 9 years of data. The rolling estimation is used to produce all results in this sections. The following strategies are considered to compute portfolios with monthly adjustments of asset allocations over the sampling period 1999 to 2016:

1) **The Expectation-based Positional Momentum Strategy (EPMS)**, This strategy selects an equally weighted portfolios of stocks with the 5% highest expected return ranks in each month. The expected ranks are estimated from one-step ahead forecasts of current return ranks at time t from a) the bivariate VAR(1) model (equation (4.4)), b) the univariate AR(1) model (equations (4.9)).

2) **The Expectation-based Positional Reverse Strategy (EPRS)**, This strategy is similar to the EPMS except for including stocks the 5% lowest expected return ranks, which are estimated as one-step ahead forecasts of the current return ranks at time t from a) the bivariate VAR(1) model (equation (4.4)), b) the univariate AR(1) model (equations (4.9)).

3) **Equally weighted portfolio (EW)**, This portfolio is an Equally Weighted (EW) portfolio of all 1330 stocks. It is used in the performance study as the benchmark portfolio.

4) **The Perfect Foresight Positional Momentum Strategy (PMS)**, This is a theoretical strategy that selects an equally weighted portfolio including all stocks whose observed past returns are in the upper 5% quantile of the CS distribution respectively. The Gaussian ranks of return of these stocks are such that $u_{i,t} \geq 1.64$. This strategy is similar to the standard momentum strategy, but unlike that strategy, it relies on the past ranks of returns, which is assumed to be observed in the previous month, instead of the observed rank over a longer period in the past. This portfolio represents the upper bound of the positional momentum portfolios.

5) **The Perfect Foresight Positional Reversal Strategies (PRS)** , This theoretical strategy selects an equally weighted portfolio including all stocks with the observed

past return ranks in the lowest 5% quantile of the CS distribution. It represents the lower bound on the positional momentum portfolios performance.

Figure 11, shows the time series of monthly portfolio returns computed monthly according to the above strategies obtained from the bivariate model (VAR). We observe that the monthly returns on the EPMS portfolios are often higher than those on the EW portfolio. The monthly returns on the EPRS portfolio are sometimes slightly lower than those on the EW and sometimes they are higher. The graph also shows the returns on the theoretical PMS and PRS portfolios respectively.

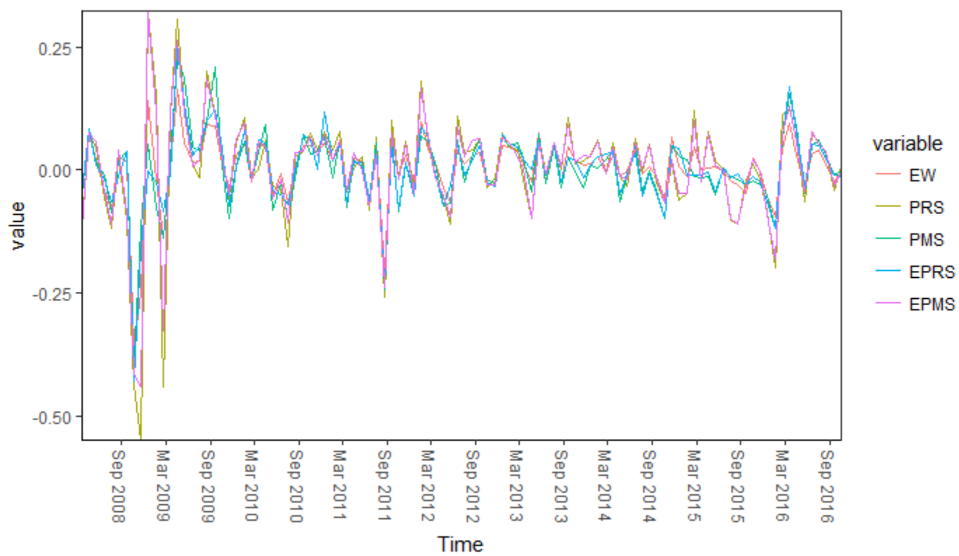

Figure 11: Monthly returns on positional momentum portfolios

Table 5 presents the statistics summarizing the monthly returns on the positional portfolios over the sampling period. On average, the VAR-based EPMS strategy provides the highest return and outperforms all the remaining strategies. The average return on the equally weighted portfolio (EW) is slightly lower than PRS while it is higher than PMS. This implies that the reversal portfolios based on the lower 5% past ranks of returns provides higher return than the PMS which is based on upper 5% past ranks of returns. In other word it says that the portfolio which is based on those stocks that had lower ranks in previous term, will provide more return than those which had higher ranks. Both the EPRS portfolios obtained from the AR and VAR models have lower average return than the EW portfolio, while both provide positive average returns. But, the reversal portfolios based on the expected ranks of returns and trade volumes (EPRS) obtained from the bivariate model (VAR) provide higher average returns than the one which is obtained from univariate model (AR). Second part of the table shows the average of the momentum portfolios based on excess returns. The highest average return belongs to EPMS obtained from bivariate VAR model, while EW and PRS both have same average return of 0.0021. Last line of the table provides the Sharp ratio of these momentum portfolios. As we can see the EPMS obtained from bivariate VAR model has the highest value in Sharp ratio, which means that it provides highest excess return for the extra volatility.

Table 5: Returns on Positional Portfolios

	EW	PMS	PRS	EPMS		EPRS	
				VAR	AR	VAR	AR
$GaussianRanks$							
$Mean$	0.0042	0.0020	0.0043	0.0064	0.0037	0.0034	0.0022
$Variance$	0.0046	0.0064	0.0144	0.0111	0.0119	0.0059	0.0055
$StandardDeviation$	0.0676	0.0798	0.1202	0.1054	0.1090	0.0770	0.0743
$Skew$	-2.1586	-1.0888	-1.654	-1.3467	-1.5703	-1.6173	-1.1998
$Kurt$	12.168	9.222	9.6130	8.444	9.6056	12.485	9.7066
$ExcessReturns$							
$Mean$	0.0021	-0.0001	0.0021	0.0042	0.0015	0.0013	0.0000
$Variance$	0.0045	0.0063	0.0144	0.0111	0.0119	0.0059	0.0055
$StandardDeviation$	0.0676	0.0798	0.1202	0.1054	0.1090	0.0770	0.0743
$Skew$	-2.1586	-1.0895	-1.6549	-1.3472	-1.5706	-1.6160	-1.1998
$Kurt$	12.168	9.224	9.6154	8.4475	9.6072	12.479	9.7070
$SharpRatio$	0.0416	0.0067	0.0240	0.0471	0.0206	0.0258	0.0103

Table 5 shows that, the positional momentum portfolios have higher average returns than all other portfolios. It also shows that the bivariate VAR-based EPMS and EPRS portfolios provide higher average return that the AR-based EPMS and EPRS. Also the highest Sharp ratio belongs to EPMS obtained from bivariate VAR model

Figures 12 and 13 show the cumulative portfolio returns over time for the equally weighted (EW), Positional Momentum (EPMS) and Positional Reversal (EPRS) portfolios, where the last two, involving the return ranks predicted from the bivariate VAR model (Figure 12) and the univariate AR model (Figure 13) respectively. In Figure 12, we observe that in early 2008, the cumulative returns on the EW and the VAR-based EPMS portfolios are close and higher than EPRS, while in late 2008 (crisis period) the EW outperforms the VAR-EPMS and VAR-EPRS portfolios.

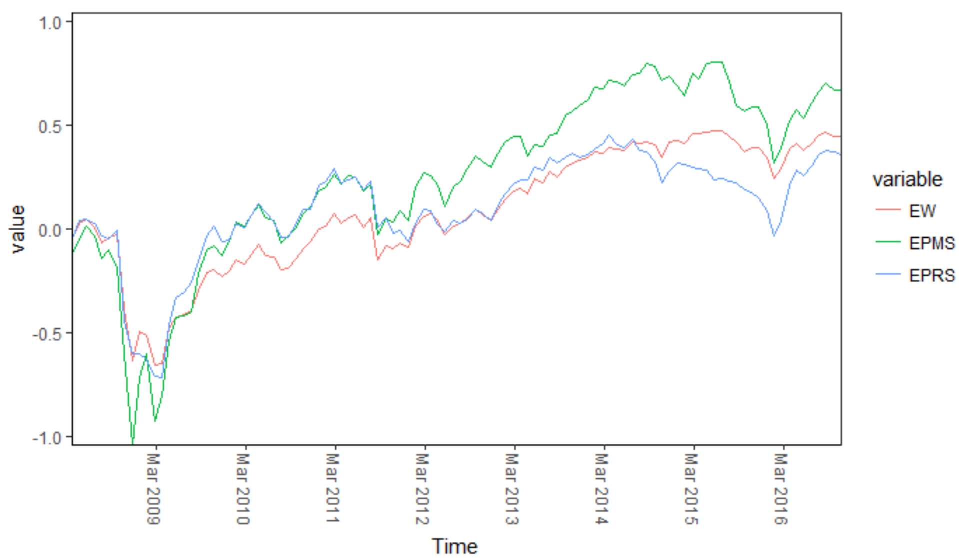

Figure 12: Monthly cumulative returns on positional portfolios - VAR(1) model

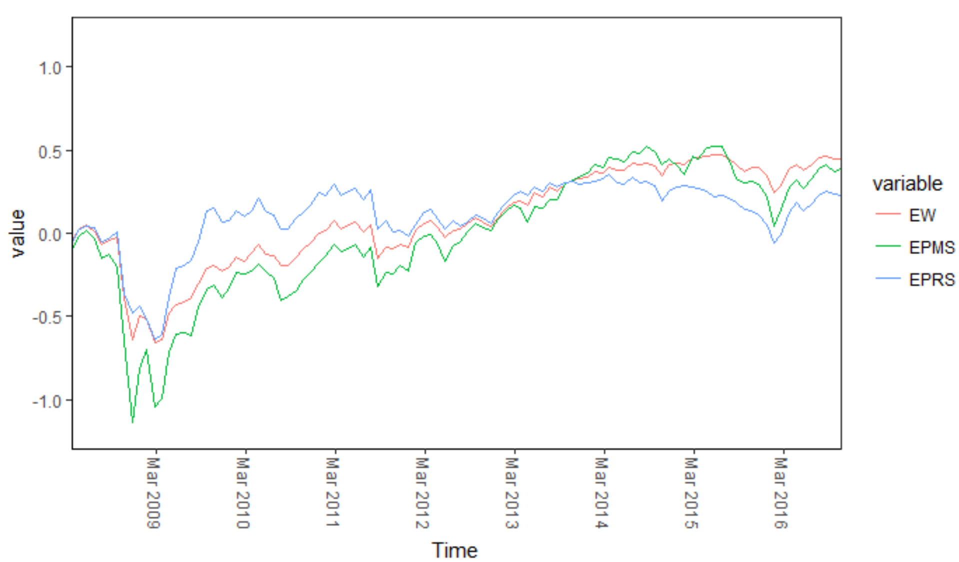

Figure 13: Monthly cumulative returns on positional portfolios- AR(1) model

Between 2009 and 2011, the cumulative returns on the EW and VAR-based EPMS portfolios are almost equal or greater than the cumulative returns on the VAR- based EPRS portfolio. After 2011, the VAR-based EPMS portfolio outperforms the equally weighted (EW) portfolio and EPRS. Also, between March 2012 and March 2014, all expectation based strategies, including even the VAR-based EPRS outperform the equally weighted portfolio EW. The next Figure 13, shows that in 2008 the AR-based EPMS portfolio provides the lowest cumulative return. From 2009 to 2011, the AR-based EPRS portfolio provides the highest cumulative return. From 2012 until the end of 2013, the cumulative returns of the EW and AR-based EPRS strategies are almost equal while the AR-EPMS provides the lowest cumulative return. Between March 2014 and July 2015, the cumulative return on the AR-based EPMS portfolio is often slightly higher than the EW return. However, after mid-2015, the EW portfolio outperforms the AR-based EPMS and EPRS portfolios.

Additional statistics in Table 6 show that the VAR-based EPMS is the best positional momentum portfolio which outperforms all other portfolios in terms of the cumulative return. The result show that even VAR-based EPRS provides higher average cumulative return than the AR-based EPRS and EW portfolios. Comparing the time series of cumulative returns of momentum portfolios also show that the positional portfolio based on expected ranks of return which was obtained from bivariate VAR model outperforms the market portfolio and the positional portfolio based on AR model.

Table 6: Cumulative Returns on Positional Momentum Portfolios

	EW	EPMS		EPRS	
		VAR	AR	VAR	AR
$Mean$	0.0906	0.2491	-0.0001	0.1028	0.1096
$Variance$	0.0045	0.0111	0.0118	0.0059	0.0055
$StandardDeviation$	0.0676	0.1054	0.1090	0.0770	0.0743

5.3 The Positional Liquid Portfolio

This section introduces the positional liquid portfolio management strategies that are defined below and named accordingly to the terminology introduced in the previous section, with the letter "L" for liquidity added to the acronyms:

1) **The Liquid Expectation-based Positional Momentum Strategy (LEPMS)**, This strategy selects an equally weighted portfolios of stocks with the 5% highest expected ranks of trade volume changes in each month. The expected ranks are estimated from one-step ahead forecasts of current trade volume ranks at time t obtained from a) the bivariate VAR(1) model (equation. (4.4)), b) the univariate AR(1) model (equations. (4.10)).

2) **The Liquid Expectation based Reverse Strategy (LEPRS)**, It selects the 5% lowest expected ranks of trade volume changes, which are estimated as one-step ahead forecasts of the current ranks at time t from a) the bivariate VAR(1) model (equation (4.4), b) the univariate AR(1) model (equations (4.10)) .

The LPMS and LPRS are the theoretical portfolios computed under the assumption that the past ranks of trade volume changes are observed. The LEW portfolio remains an equally weighted portfolio of all stocks, as before.

Figure 14 shows the time series of monthly trade volume changes of the above portfolios based on VAR model. The VAR-based LEPMS and LPRS displays higher changes in trade volume than the equally weighted portfolio (LEW), which itself shows higher changes in volume than the VAR-based LEPRS and LPMS consistently over the entire sampling period. We can see that the LEPMS which is based on expected highest 5% ranks of trade volume provides almost same changes in trade volume as LPRS which is based on lowest observed ranks of trade volume changes in previous term. It seems that those stocks which were ranked as lower trade volume changes in last period, now in current term will have higher trade volume changes. On the other hand the LEPRS is almost same as LPMS at most of the time. Which means that those stocks which had higher ranks in trade volume changes in last period, turn out to have higher trade volume changes in current period.

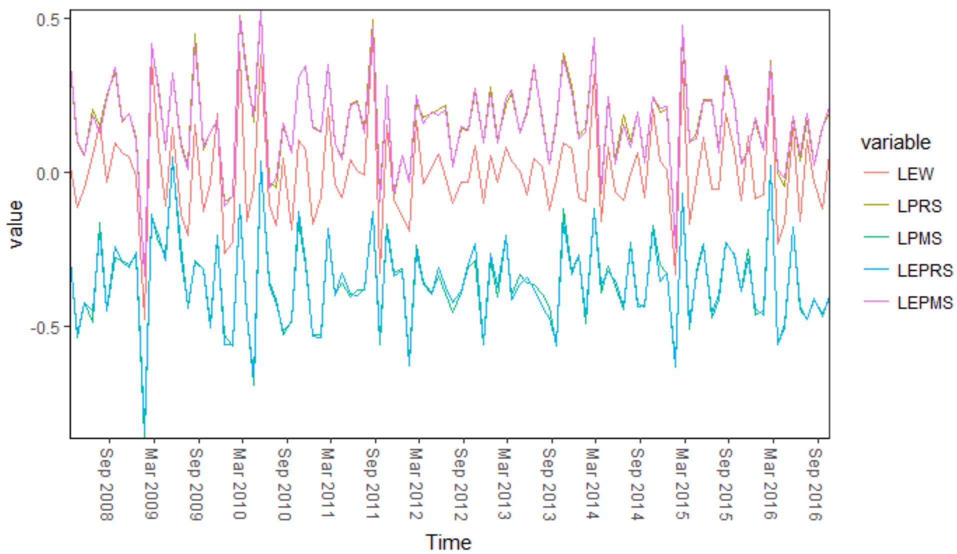

Figure 14: Monthly trade volume changes of positional liquid portfolios

Table 7 below shows the summary statistics concerning the liquidity of the positional liquid portfolios over the sampling period. The LPRS which is based on highest ranks of trade volume at time $(t-1)$, has highest average of trade volume changes at time (t). The VAR-based LEPMS portfolio displays the average trade volume change slightly lower than LPRS, followed by the AR-based LEPMS portfolio. Both these portfolios have higher average trade volume changes than the equally weighted EW portfolio.

Table 7: Trade volume changes of Positional Liquidity Portfolios

	LEW	$LPMS$	$LPRS$	LEPMS		LEPRS	
				VAR	AR	VAR	AR
$Mean$	-0.0017	-0.3563	0.1618	0.1611	0.1522	-0.3551	-0.2976
$Variance$	0.0253	0.0231	0.0225	0.0224	0.0219	0.0223	0.0226
$StandardDeviation$	0.1593	0.152	0.1500	0.1498	0.1480	0.1493	0.1504
$Skew$	0.3979	0.0521	-0.1233	-0.1389	-0.0880	0.1274	0.2018
$Kurt$	3.9871	3.7373	3.8129	3.7821	4.1474	3.6845	3.6073

Same as Figure 14, we can see that the results from Table 7 shows that, the LPRS which is base on the lowest 5% trade volume changes at time $(t-1)$ will have more trade volume changes at time (t). On average, among the expected positional strategies, the least change in trading volume characterizes the LPMS which is based on highest 5% observed trade volume ranks at time $(t-1)$.

Let us now compare these positional liquid portfolios in terms of their return. Table 8, shows the summary statistics for the returns on the positional liquid portfolios over the sampling period. According to these results, all expectation-based liquid portfolio strategies produce higher average returns than the market portfolio. The LPMS portfolio which is based on highest trade volume ranks at time $(t-1)$ has the highest average return. While the VAR-based LEPRS which is based on expected low trade volume ranks bits the market portfolio and all other portfolios. This result shows that the expected positional portfolio of stock with expected lower ranks of trade volume changes will provide highest average return.

Table 8: Returns on Positional Liquid Portfolios

	LEW	*LPMS*	*LPRS*	LEPMS		LEPRS	
				VAR	*AR*	*VAR*	*AR*
GaussianRanks							
Mean	0.0042	0.0101	0.0048	0.0045	0.0042	0.0085	0.0077
Variance	0.0046	0.0052	0.0047	0.0048	0.0048	0.0050	0.0051
StandardDeviation	0.0676	0.0723	0.06862	0.0692	0.0698	0.0710	0.0717
Skew	-2.1586	-1.0298	-2.1103	-2.1448	-2.2811	-1.0657	-1.4560
Kurt	12.168	6.1592	12.742	12.940	13.539	6.2801	8.2578
ExcessReturns							
Mean	0.0021	0.0079	0.0026	0.0023	0.0021	0.0064	0.0056
Variance	0.0045	0.0052	0.0047	0.0047	0.0048	0.0050	0.0051
StandardDeviation	0.0676	0.0722	0.0686	0.0692	0.0697	0.0710	0.0717
Skew	-2.1586	-1.0302	-2.1105	-2.1453	-2.2804	-1.0667	-1.4568
Kurt	12.168	6.1597	12.742	12.942	13.535	6.2826	8.2623
SharpRatio	0.0416	0.1198	0.0491	0.0442	0.0401	0.1004	0.0879

In terms of excess return, the VAR-based LEPRS outperform all other portfolio following the LPMS. The Sharp ratio of VAR-based LEPRS is also slightly lower than LPMS, but higher than AR-based portfolios and market portfolio. By comparing the results in Table 8 to those in Table 5 , we find that the highest average returns are obtained from the positional liquid VAR-based LEPRS strategy, followed by the VAR-based positional momentum EPMS strategy. Therefore, the positional liquid strategy seems to outperform the positional momentum strategy in terms of average portfolio returns.

To compare the cumulative returns on the positional liquid portfolios LEPMS and LEPRS over time, we compute the time series of their cumulative monthly returns. Figure 15 and 16 show the time series of the cumulative monthly returns for the LEW, LEPMS and LEPRS portfolios with the ranks predicted from the VAR and AR models respectively. Both Figures show that after 2008, the LEPRS strategy provides higher cumulative returns than the market portfolio. Between 2008 and 2014, the market portfolio has higher cumulative returns than the LEPMS portfolios, which become almost equal afterwards.

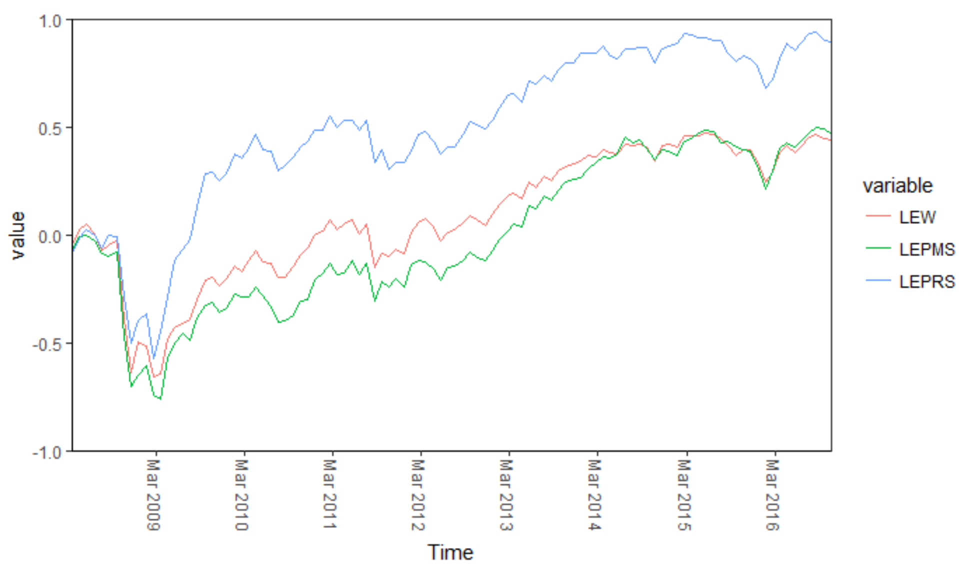

Figure 15: Monthly cumulative returns on positional liquidity portfolios - VAR(1) model

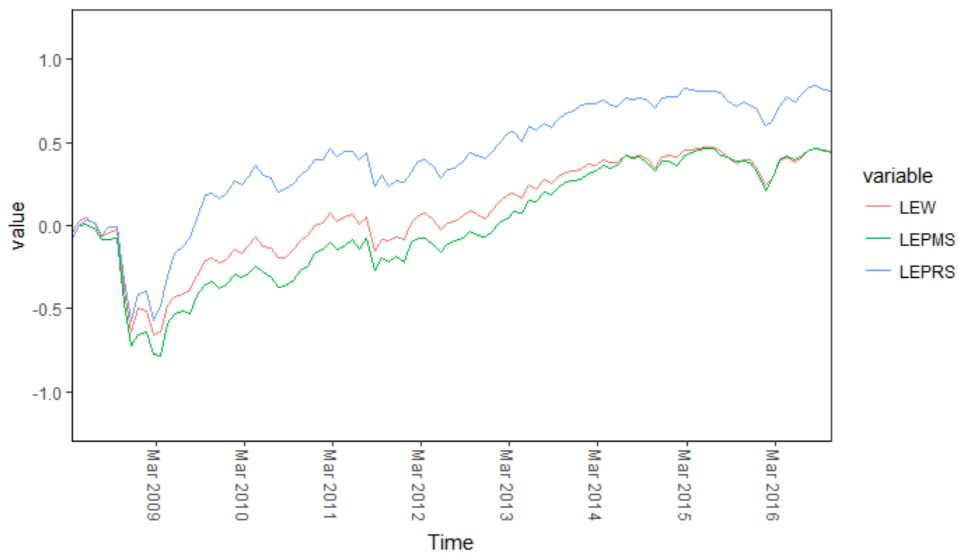

Figure 16: Monthly cumulative returns on positional liquidity portfolios - AR(1) model

To examine in detail the differences between the positional liquid portfolios, additional statistics on the cumulative returns on the LEPMS, LEPRS and EW portfolios are presented in Table 9. We see that all positional liquid portfolios provide positive average cumulative returns over the last the sampling period. We observe that the VAR-based LEPRS positional liquid portfolios outperform all other portfolios during the sampling period. The AR-based LEPMS hs higher cumulative return than VAR-base LEPMS but lower than the EW market portfolio. These findings suggest that the positional liquid portfolio of stocks with less expected volume changes provides higher cumulative returns than the positional liquid portfolio of stocks with high expected volume changes. Therefore, stocks with more stable trading volume yield higher average cumulative returns.

Table 9: Cumulative Returns on Positional Liquid Portfolios

	EW	LEPMS		LEPRS	
		VAR	AR	VAR	AR
$Mean$	0.0906	0.0000	0.0047	0.4939	0.4099
$Variance$	0.0856	0.1177	0.1152	0.1403	0.1229
$StandardDeviation$	0.2926	0.3431	0.3394	0.3746	0.3507

By comparing the results in Table 9 to those in Table 6, we conclude that the VAR-based positional liquid portfolio LEPRS outperforms the positional m momentum portfolio EPMS with high expected return ranks in terms of their cumulative returns.

6 Conclusion

This paper introduced a momentum and liquidity positional portfolio management strategies which are based on the expected bivariate ranks of returns and trade volume changes. The expected ranks are predicted from a conditionally Gaussian vector autoregressive model of order one VAR(1), which represents their joint dynamics. The ranks of returns and of trade volume changes are transformed to Gaussianity by the quantile function, i.e. the inverse of the cumulative Normal distribution function . Next, the predicted ranks are used to build the Expected Positional Momentum/Reversal portfolios (EPMS and EPRS) of stocks with high/low ranked expected returns. For portfolio liquidity management, I introduce the Liquidity Expected Positional Momentum and Reversal portfolios (LEPMS and LEPRS) of stocks with high and low ranked expected trade volume changes. These allocation strategies were applied to an investment universe consisting of 1330 stocks which are traded on the NASDAQ market between 1999-2016. The empirical results show that the VAR-based positional momentum EPMS portfolios outperform the equally weighted portfolio and the EPMS portfolios with return ranks predicted from a univariate AR model. This finding suggests that the liquidity ranks help predict the return ranks and improve the positional portfolio performance. The analysis of the positional liquid portfolios shows that average return on the VAR-based LEPRS portfolios is higher than on any positional momentum portfolios and the equally weighted portfolio. This interesting result shows that a positional portfolio of stocks with less volatile expected trading volumes outperforms a portfolio of stocks with more expected return. These finding are confirmed by examining the cumulative average return of the positional liquid portfolios as well. I fund that the LEPRS portoflio based on the predictions from the VAR model provides higher average cumulative returns than the equally weighted portfolio and the EPMS portfolio based on high expected returns ranks.

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Appendices

A Diagnostic Tests For Error Terms

A.1 Autocorrelation Test

In statistics, the Durbin-Watson (DW) statistic is a test statistic used to detect the presence of autocorrelation at lag 1 in the residuals (prediction errors) from a regression analysis. In DW test, the null hypothesis is that the errors are serially uncorrelated and the alternative is that they follow a first order autoregressive process. If e_t is the residual associated with the observation at time t, then the test statistic is:

$$d = \frac{\sum_{i=2}^{T}(e_t - e_{t-1})^2}{\sum_{i=1}^{T} e_t^2} \tag{A.25}$$

where T is the number of observations. Since d is approximately equal to $2(1-r)$, where r is the sample autocorrelation of the residuals, $d = 2$ indicates no autocorrelation. The obtained d for the VAR(1) model (equation(4)) are as follow:

$$\begin{aligned} d_{e_1} &= 1.99 \\ d_{e_2} &= 2.15 \end{aligned} \tag{A.26}$$

Since the obtained d's are almost equal to 2, we can conclude that there is no autocorrelation in error terms.

A.2 Normality Of Cross-Sectional Errors

The Shapiro Normality test for Cross-Sectional errors obtained from VAR(1) model in equation (4) confirmed that both errors are Cross-Sectionally Normally distributed. The following Figures show the Q-Q plot and the distribution function of the Cross-Sectional error terms in October,2016.

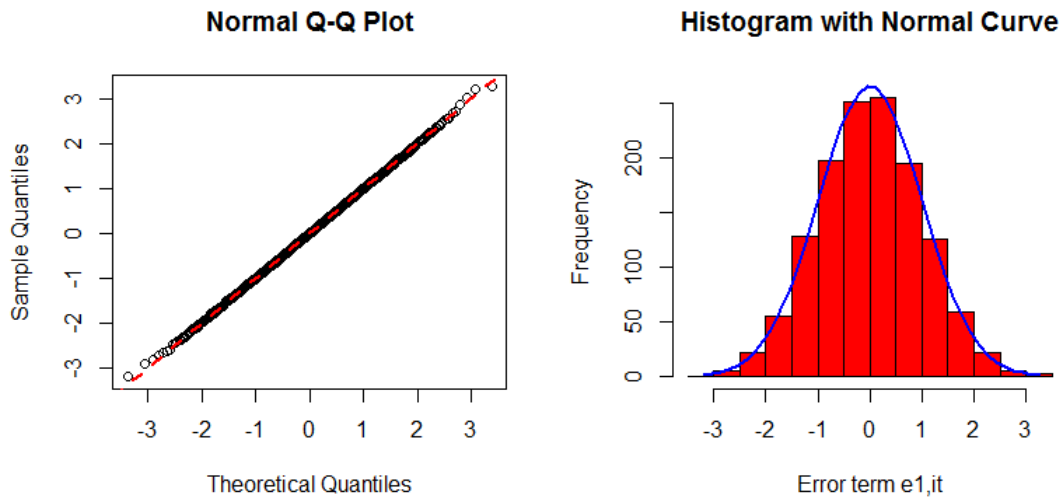

Figure 17: Cross-Sectional e_1 distribution for Oct,2016

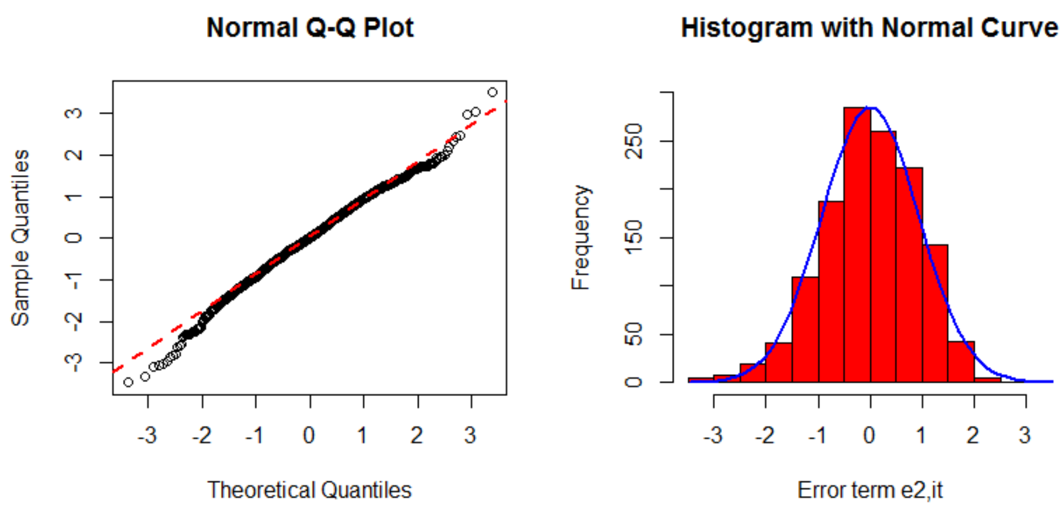

Figure 18: Cross-Sectional e_2 distribution for Oct,2016

B Optimal Positional Portfolio

B.1 Proof of Equation (21) and (22)

From the assumption of Gaussian CS distribution, the future position of the risky portfolio return and trade volume are:

$$\begin{aligned}
Q_{t+1}(\beta_r' r_{i,t+1}) &= \beta_r' u_{i,t+1} \\
&= \rho_{11}\beta_r' u_{i,t} + \rho_{12}\beta_r' v_{i,t} + \beta_r' e_{1,it} \\
Q_{t+1}(\beta_{tv}' tv_{i,t+1}) &= \beta_{tv}' v_{i,t+1} \\
&= \rho_{21}\beta_{tv}' u_{i,t} + \rho_{22}\beta_{tv}' v_{i,t} + \sum_{i=1}^{n} \beta_{tv}' e_{2,it}
\end{aligned} \tag{B.27}$$

Then, by using that the $e_{1,it}, e_{2,it}$ are independent Gaussian white noise processes, the expected positional utility is:

$$\begin{aligned}
&- E[exp(-AQ_{t+1}(\beta_r r_{t+1})) \mid r_t, tv_t] \\
&= -E\{E[exp(-AQ_{t+1}(\beta_r r_{t+1})) \mid r_t, tv_t] \mid r_t, tv_t\} \\
&= -E\Big[exp\Big(-A\rho_{11}\beta_r' u_{i,t} - A\rho_{12}\beta_r' v_{i,t} + \frac{1}{2}A^2\beta_r'^2 e_{1,it}\Big) \mid r_t, tv_t\Big]
\end{aligned} \tag{B.28}$$

$$\begin{aligned}
&- E[exp(-AQ_{t+1}(\beta_{tv} tv_{t+1})) \mid r_t, tv_t] \\
&= -E\{E[exp(-AQ_{t+1}(\beta_{tv} tv_{t+1})) \mid r_t, tv_t] \mid r_t, tv_t\} \\
&= -E\Big[exp\Big(-A\rho_{21}\beta_{tv}' u_{i,t} - A\rho_{22}\beta_{tv}' v_{i,t} + \frac{1}{2}A^2\beta_{tv}'^2 e_{2,it}\Big) \mid r_t, tv_t\Big]
\end{aligned} \tag{B.29}$$

B.2 Proof of Equation (23) and (24)

The Lagrangian function for the maximization of the expected positional utility w.r.t. the portfolio allocation β subject to the constraint $\beta' e = 1$ is:

$$L = -E\Big[exp\Big(-A\rho_{11}\beta_r' u_{i,t} - A\rho_{12}\beta_r' v_{i,t} + \frac{1}{2}A^2\beta_r'^2 e_{1,it}\Big) \mid r_t, tv_t\Big] + \lambda(\beta_r' e - 1) \tag{B.30}$$

$$L = -E\Big[exp\Big(-A\rho_{21}\beta_{tv}' u_{i,t} - A\rho_{22}\beta_{tv}'^2 v_{i,t} + \frac{1}{2}A^2\beta_{tv}' e_{2,it}\Big) \mid r_t, tv_t\Big] + \lambda(\beta_{tv}' e - 1) \tag{B.31}$$

where λ is the Lagrange multiplier. The first-order condition for β_r, β_{tv} are:

$$AE\Big[(\rho_{11}\beta_r' u_{i,t} + \rho_{12}\beta_r' v_{i,t} - A\beta_r')exp\Big(-A\rho_{11}\beta_r' u_{i,t} - A\rho_{12}\beta_r' v_{i,t} + \frac{1}{2}A^2\beta_r'^2 e_{1,it}\Big) \mid r_t, tv_t\Big] + \lambda = 0 \tag{B.32}$$

$$AE\Big[(\rho_{21}\beta'_{tv}u_{i,t}+\rho_{22}\beta'_{tv}v_{i,t}-A\beta'_{tv})exp\Big(-A\rho_{21}\beta'_{tv}u_{i,t}-A\rho_{22}\beta'_{tv}v_{i,t}+\frac{1}{2}A^2\beta'^2_{tv}e_{2,it}\Big) \mid r_t, tv_t\Big]+\lambda = 0 \tag{B.33}$$

We deduce that the solution $\beta = \beta^*$ satisfies the implicit equation:

$$\beta^*_r = \frac{1}{ne} + \frac{1}{A}\frac{\rho_{11}+\rho_{12}}{(\sigma^2_{1,it+1})}(u_t - \overline{u_t}e) \tag{B.34}$$

$$\beta^*_{tv} = \frac{1}{ne} + \frac{1}{A}\frac{\rho_{12}+\rho_{22}}{(\sigma^2_{2,it+1})}(v_t - \overline{v_t}e) \tag{B.35}$$