Statistical inferences for price staleness*

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Disclaimer

Abstract

Asset prices recorded at a high frequency can be more stale than implied by the semimartingale hypothesis. The staleness emerges due to illiquidity and materializes in a form of zero returns. We propose a new general framework formalizing this phenomenon. A limit theory for Multi-Idle-Time (an economic indicator for price staleness) and related quantities is provided. This allows measuring the *level* and *volatility* of staleness of asset price adjustment and conducting non-parametric specification tests. We consider two different hypotheses. First, whether the extent of staleness is constant or time-varying. Second, whether its dynamics can be described by a Brownian semimartingale. The empirical application on NYSE stocks provides the evidence that the level of stock price staleness is typically time-varying and can be described with adequate realism by an $(0, 1)$-valued Brownian semimartingale.

Keywords: staleness, idle time, liquidity, zero returns, stable convergence

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1 Introduction

Traditional modelling in continuous time entail that the price of an asset, traded in a frictionless market, evolves as a semimartingale. Bandi et al. (2017) provide the evidence that real asset prices do not update as frequently as expected under the semimartingale assumptions. Indeed, while in the standard jump-diffusion setting high-frequency returns should exceed an appropriately-defined shrinking threshold with large probability, in real data often the converse is true. The main cause of this large number of (extremely) small returns is the lack of price updates at a high-frequency. In particular, Bandi et al. (2018) provide empirical evidence on the fact that zero returns, henceforth *zeros*, are driven by volume, magnitude of execution cost, and only immaterially by institutional effects, such as price discreteness. Thus, motivated by the fact that zeros seem to be a genuine economic phenomenon twisted to, e.g., cost of trading, price formation mechanism, extent of asymmetric information, in this paper we focus on the statistical inference of zeros.

In this work, we develop an inference theory on the occurrence of zero returns. Since ubiquitous jump-diffusion models in continuous time does not implied an occurrence of zeros compatible with that observed in real data, we frame our inferential theory under a frictional dynamics for asset prices, one which captures the lack of prices adjustment in high-frequency data. As a starting point, we assume the existence of an efficient price process Y, which we define as the asset price that would have been observed if the market was perfectly liquid. In presence of illiquidity frictions (such as trading costs or asymmetric information) the trading activity is inhibited, whence the random occurrence of periods in which the observed price process stays constant[1] a situation that in this paper we address, following the nomenclature of Bandi et al. (2018), as "price staleness". The higher the "magnitude" of these frictions, the more probable and the more persistent the staleness of the observed price. We model this frictional price dynamics following the formalism introduced by Bandi et al. (2017) and Bandi et al. (2018). Hence, on top of the existence of the latent efficient price process Y, we assume that the logarithmic price process, $(X_t)_{t\geq 0}$, sampled in any partition $0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1$ of the unit time interval $[0, 1]$ (for example one trading day), is driven by the recursive equation:

$$X_{t_{j,n}} = Y_{t_{j,n}} (1 - \mathbb{B}_{j,n}) + X_{t_{j-1,n}} \mathbb{B}_{j,n}, \quad j = 1, \ldots, n, \tag{1}$$

[1] Here we are implicitly assuming a previous-tick interpolation scheme that attributes to each instant of the sampling partition the last available observation, hence a period without trading activity is straightforwardly translated into a stale price.

with the initial condition $X_0 = Y_0$, where $Y_{t_{j,n}}$ is the efficient price sampled in the j-th element of the partition and where $(\mathbb{B}_{j,n})_{j=1,\dots,n}$ is a triangular array of Bernoulli random variates such that, for some (random) $p_\infty \in (0,1)$,

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{B}_{j,n} \xrightarrow{p} p_\infty,$$

as $n \to \infty$. The recursive equation (1) implies that, at each instant $t_{j,n}$, the observed price $X_{j\Delta_n}$ may either coincide with the latent efficient price ($\mathbb{B}_{j,n} = 0$) or not update and stay constant ($\mathbb{B}_{j,n} = 1$), thus leading to a stale price. Developing a statistical inference on price staleness translates in studying the statistical properties of the triangular array $(\mathbb{B}_{j,n})_{j=1,\dots,n}$.

The inclusion of price staleness in the data generating process results to be pivotal from both an economic and an econometric point of view. Bandi et al. (2017) provide a model based on micro-structural theories of price formation (Kyle, 1985; Hasbrouck and Ho, 1987; Glosten and Milgrom, 1985) where insurgence of zero returns is triggered by the joint effect of asymmetric information, transaction costs and delays in the incorporation of the information flow into the assets' prices. Kolokolov and Renò (2017), instead, support the inclusion of price staleness in the data generating process from an econometric perspective, showing that neglecting price staleness leads to severe distortions on the widely used power- and multi-power estimators (Woerner, 2006; Barndorff-Nielsen et al., 2006; Barndorff-Nielsen and Shephard, 2004; Lee and Mykland, 2008; Caporin et al., 2014). Importantly, it is shown that both detection of jumps and estimation of the jump activity index are jeopardized even by a moderate levels of staleness.

The main contribution of our paper consists in developing an inferential theory for the triangular array $(\mathbb{B}_{j,n})_{j=1,\dots,n}$, governing the intra-day dynamics of price staleness. As follows from the previous empirical literature on zero returns (Lesmond et al., 1999; Bekaert et al., 2007; Naes et al., 2011; Bandi et al., 2018), the characterization of the array $(\mathbb{B}_{j,n})_{j=1,\dots,n}$ is tantamount to the characterization of the intra-day dynamics of illiquidity or, more precisely, to the dimension of illiquidity captured by price staleness. We answer to the following questions: 1) Does illiquidity varies stochastically during the day? 2) If yes, which kind of stochastic process is more suitable to describe its dynamics and is it possible to define and measure its volatility? To this purpose, we introduce a very general econometric framework to model a triangular array of possibly dependent Bernoulli random variables. The main idea is to represent the probability of observing a zero as a (latent) continuous-time process $(p_t)_{t\in[0,1]}$ taking values in $(0,1)$. We provide a set of novel results.

Our first result is to show that the intraday fraction of zeros, dubbed as *idle* *time* in Bandi et al. (2017), is a consistent estimator of the integrated probability of price staleness. Then, under the assumption that the process $(p_t)_{t\in[0,1]}$ evolves as a Brownian semimartingale, we derive a (stable) Central Limit Theorem (henceforth CLT) for *idle time*. In order to set-up a feasible confidence interval, we introduce a new economic indicator, named *(m)-multi-idle-time*, and we derive its limiting properties by using a standard infill asymptotic design. Next, we introduce the notion of *local idle-time*, an estimator of the instantaneous stochastic probability of price staleness. This quantity permits us to construct estimates of general integrated function of probability of staleness and to conduct a fine-tuning analysis on the dynamical properties of zeros. Precisely, using the developed limit theory, we construct 1) a non-parametric test to distinguish between a constant and a time-varying p_t, 2) a non-parametric test, which, having established that p_t varies stochastically, allows to assess whether a Brownian semimartingale type dynamics is suitable to describe p_t.

Using 250 NYSE-listed stocks, we show that the assumption of the constancy of instantaneous probability of stale prices is fairly rejected. Simultaneously, for the large majority of our sample, a Brownian semimartingale specification of the instantaneous probability of zero returns can not be rejected in favour a more persistent alternative. This result paves the way to a new research topic: the consistent estimation of volatility of illiquidity. Under the assumption that p_t is a Brownian semimartingale, we provide an estimator of the integrated (over, say, one day of trading) volatility of p_t. Our estimation theory is the analogue for illiquidity of the estimation of volatility of volatility of financial prices (see, e.g., Barndorff-Nielsen and Shephard, 2002a,b; Vetter et al., 2015).

The plan of the paper is as follows. Section 2 introduces the setting. In particular, we give assumptions on both the triangular array of Bernoulli random variates and on the probability of staleness. Section 3 contains the limit results. Section 5 shows the finite sample accuracy of our asymptotic theory through a Monte Carlo exercise, whereas Section 6 presents the empirical plausibility of our assumptions. Section 7 concludes. All technical proofs are confined to the Appendix.

2 The settings

We work on a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P}\right)$, one which supports all stochastic elements defined below. The structure of the filtration $(\mathcal{F}_t)_{t\geq 0}$ is quite technical and it is reported in the Appendix A.1. The value of a generic stochastic process X at a point $t_{j,n}$ of any partition $0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1$ of the time interval $[0,1]$ will be denoted with $X_{t_{j,n}}$ or, to avoid excessive subscripts, simply with $X_{j,n}$. In what follows, for simplicity, we will always assume that the partition is equispaced, hence we put $t_{j,n} = j/n$ with $j = 0, ..., n$ and the distance between two consecutive points is given by $\Delta_n = 1/n$. We assume that the Bernoulli random variables representing price staleness as described in Equation (1) have the following structure.

Assumption 1. *There exists a (latent) continuous-time stochastic process $(p_t)_{t\in[0,1]}$ taking values in $(0,1)$. Let $(\mathbb{B}_{j,n})_{j=1,\dots,n}$ be the triangular array of $\mathcal{F}_{t_{j,n}}$-measurable Bernoulli random variables defined as*

$$\mathbb{B}_{j,n} \doteq \mathbb{I}_{\left\{U_{t_{j,n}} \le p_{t_{j,n}}\right\}}, \quad j = 0, \dots, n, \tag{2}$$

where $\mathbb{I}_{\{.\}}$ is the indicator function, $(U_t)_{t\in[0,1]}$ is a collection of Uniform random variables independent of p_t and satisfying $U_t \perp U_{t'}$, $\forall t \neq t'$.

In other words, we assume that the process $(p_t)_{t\in[0,1]}$ is responsible, at any sampling frequency, for the occurrence of the event $\{\mathbb{B}_{j,n} = 1\}$, in the sense that $\mathbb{P}\left[\mathbb{B}_{j,n} = 1\right] = \mathbb{E}\left[p_{t_{j,n}}\right]$.

Note that Assumption 1 preserves the *compatibility relationship* (cfr. Aït-Sahalia and Jacod, 2014, Pag. 211) over different sampling frequencies. Formally, this property guarantees that if $t_{j,n} = j/n$ and $t_{j',n} = j'/n$ are two equally spaced partitions of $[0,1]$, with $j = 1, \dots, n$ and $j' = 1, \dots, n'$, then $\mathbb{B}_{j,n} = \mathbb{B}_{j',n'}$ whenever $j/n = j'/n'$.

Assumption 1 allows for different specifications of $(\mathbb{B}_{j,n})_{j=1,\dots,n}$. For instance, if $p_t = p^F$ for all $t \in [0,1]$, then the Bernoulli variates are i.i.d with probability of staleness given by p^F. Nonetheless, this case is very restrictive. A more sophisticated one is obtained when $(p_t)_{t\in[0,1]}$ is described by a Brownian semimartingale, as in the Example 1 below. In this case, indeed, the Bernoulli variates can be autocorrelated.

Example 1. *Let $F : \mathbb{R} \to (0,1)$ be a smooth function and $(Z_t)_{t\in[0,1]}$ be a Brownian semimartingale described by the following SDE:*

$$Z_t = Z_0 + \int_0^t a_u \, du + \int_0^t \sigma_u^{(p)} \, dW_u,$$

where W_t is a $\mathcal{F}$-Brownian motion and the processes a_t and $\sigma_t^{(p)}$ are cádlág and $\mathcal{F}$-adapted. Then, set $p_t = F(Z_t)$ for each $t \in [0,1]$. The latter is a well-defined process taking values in $(0,1)$. By Itô lemma, $(p_t)_{t\in[0,1]}$ is itself a Brownian semimartingale of the form:

$$p_t = p_0 + \int_0^t \left(a_u \frac{\partial F}{\partial Z} + \left(\sigma_u^{(p)}\right)^2 \frac{1}{2} \frac{\partial^2 F}{\partial Z^2} \right) dt + \int_0^t \left(\sigma_u^{(p)} \frac{\partial F}{\partial Z} \right) dW_u.$$

To gain intuition, in Figure 1 we report simulated stale stock prices. In the top panel, we generate stale prices under the more general situation of $(p_t)_{t\in[0,1]}$ semimartingale, whereas in the bottom one stale prices are generated under the i.i.d. assumption for the Bernoulli variates. Although the number of zeros (signaled by a red cross) is the same, the two graphs look rather different. In the semimartingale case we see that there is some clustering of lack of price adjustments. On the other hand, in the i.i.d. situation stale prices are uniformly distributed over the trading day. However, in the former case, zeros are *nearly* independent within each cluster. This because Assumption 2 implies that the covariance between two consecutive Bernoulli random variates is of the same probability order of the Brownian motion (i.e. $\mathbb{C}\text{ov}\left[\mathbb{B}_{j,n}, \mathbb{B}_{j+1,n}\right] = O_p(\Delta_n^{1/2})$. See Remark 4 in the Appendix). Thus, for sufficiently large n, consecutive zero returns are approximately uncorrelated.

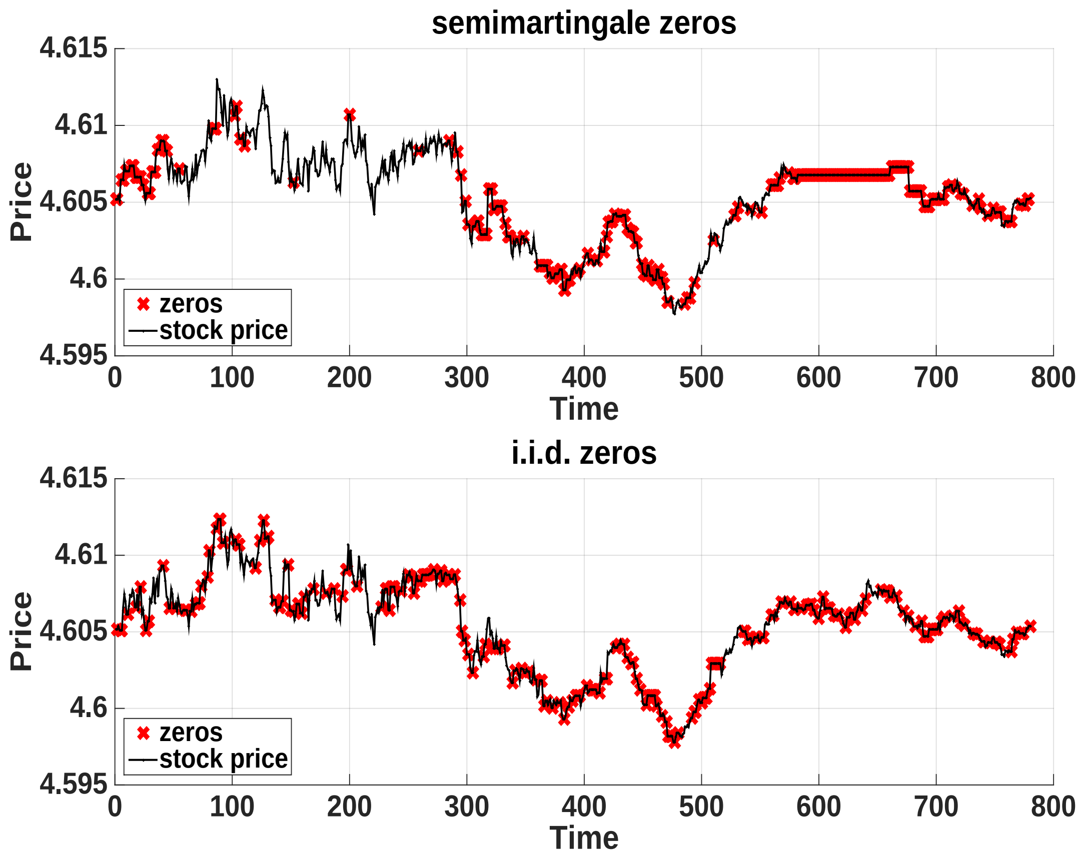

Figure 1: We report example of stale stock price where zero returns are signaled by a red circle. The probability of observing a zero return either follows a semimartingale dynamics (upper panel) or it is equal to a constant (lower panel). In both cases, the number of zeros is the same.

In what follows, we will not assume any particular parametric specification for the process p_t, but instead, we want to make some kind of (non-parametric) inference about the regularity of its trajectories. Thus, we generalize Example 1 and we make the following general assumption.

Assumption 2. *The process* $(p_t)_{t\in[0,1]}$ *is described by the following SDE:*

$$p_t = \int_0^t \mu_s \, ds + \int_0^t \nu_s \, dW_s, \tag{3}$$

where W_t *is a standard Brownian motion, and* μ_t *and* ν_t *are cádlág processes, such that* $\forall t$, $p_t \in (0,1)$ *almost surely.*

Under Assumption 2, the average probability of price staleness is proxied by $\int_0^1 p_t \, dt$, whereas its intraday variability is captured by the integral $\int_0^1 \nu_t^2 \, dt$.

In this paper, we use the scaling property of the autocorrelation function of zeros for testing the adequacy of the Brownian semimartingale assumption (Assumption 2). As an alternative, we consider a specification of the process p_t, which permits a slower (w.r.t. the semimartingale case) vanishing correlation between two consecutive Bernoulli random variates even for large n. One possibility consists in describing the process p_t through a rough dynamic, e.g. p_t is generated by a fractional process with Hurst parameter $H < 1/2$. However, we will turn later on this specification (Section 4.2).

3 Asymptotic results

We begin with the derivation of limiting results for the sum of intraday zeros, which coincides with the notion of idle time introduced in Bandi et al. (2017). First, we prove that idle time converges in probability to the integrated probability of price staleness. Then, under Assumption 2, we derive a (stable) CLT for this quantity (Theorem 3.1). Next, we investigate the problem of estimation of integrated quantities of the form $\int_0^1 f(p_s) \, ds$ (for a suitable test function $f(\cdot)$), which are useful for setting-up a feasible confidence interval for idle time and for the specification analysis of the dynamics of the process p_t based on zeros[2]. For this purpose, we introduce the notions of *m-multi-idle time*, *k-staggered multi-idle time local idle time* and we establish the corresponding limit theory (Theorem 3.2 and Theorem 3.3). Finally, we construct an estimator of the quadratic variation of the process $(p_t)_{t\in[0,1]}$ under Assumption 2.

[2]Hereafter, for sake of brevity, we will write only "specification analysis" when referring to this last statement.

3.1 Idle, multi-idle and staggered-idle time

Following Bandi et al. (2018)[3] we (formally) define, for any frequency of observation n, idle time as the average number of zeros within a trading day:

$$\mathrm{IT}_n = \frac{1}{n}\sum_{j=1}^{n} \mathbb{B}_{j,n}.$$

Despite of its simplicity, IT_n encompasses an important economic information since it constitutes an illiquidity proxy retrieved from the high-frequency data. However, we decide do not expand further and we refer to the original work(s) for an exhaustive description of the economic meaning of IT_n. Instead, we focus on the mathematical meaning of IT_n. The limiting properties of IT_n are summarized by the following theorem.

Theorem 3.1. *Assume that Assumption 1 holds. Then, as $n \to \infty$,*

$$\mathrm{IT}_n \overset{u.c.p}{\longrightarrow} \int_0^1 p_s\, ds.$$

If both Assumptions 1 and 2 hold, as $n \to \infty$,

$$\sqrt{n}\left(\mathrm{IT}_n - \int_0^1 p_s\, ds\right) \overset{stably}{\Longrightarrow} \mathcal{MN}\left(0, \int_0^1 p_s\,(1-p_s)\, ds\right), \tag{4}$$

where $\mathcal{MN}(0, V^2)$ denotes the mixed-normal distribution with a stochastic variance V^2.

Proof. See Appendix A.2. □

Theorem 3.1 implies that IT_n is a consistent estimator of the integrated probability of price staleness over a trading day under very general assumptions on the dynamics of the process p_t. If p_t is a Brownian semimartingale (Assumption 2), IT_n admits a stable CLT. In case of constant probability of price staleness, e.g. $p_t = p_0, \forall t \in [0,1]$, the asymptotic variance is simply equal to $p_0(1-p_0)$, which coincide with the variance of a Bernoulli random variable with mean p_0.

Under Assumption 2 (hereafter, both Assumptions 1 and 2 are tacitly assumed if not explicitly stated), a feasible confidence interval for IT_n can be readily constructed provided a consistent estimator of $\int_0^1 p_s^2\, ds$. We consider a slightly more general problem of estimating $\int_0^1 (p_s)^m\, ds$, for some integer $m \geq 2$. For this pur-

[3] The concept of *idle time* appeared for the first time in Bandi et al. (2017) as the average number of price adjustments below a suitably defined threshold ξ_n.

pose, we define the $(m-)$*multi-idle time* as:

$$\mathrm{MIT}_n^{(m)} \stackrel{\text{def}}{=} \frac{1}{n-m} \sum_{j=1}^{n-m} \prod_{q=0}^{m} \mathbb{B}_{j+q,n}.$$

Intuitively, multi-idle time counts the number of runs of zeros of length m. For example, fix $m \geq 2$ and $j \in \{1, \ldots, n-m\}$. Then, if all of the m consecutive price adjustment are zero, the product $\prod_{q=0}^{m} \mathbb{B}_{j+q,n}$ is equal to one and contribute to the summation. If at least one of these m price adjustment is different from zero, the product $\prod_{q=0}^{m} \mathbb{B}_{j+q,n}$ is equal to zero, and does not contribute to $\mathrm{MIT}_n^{(m)}$. Hence, in i.i.d. case $\mathrm{MIT}_n^{(m)}$ naturally estimates the joint probability of m consecutive zeros. In general case, we have the following theorem.

Theorem 3.2. *Assume Assumptions 1 and 2 hold. Then, as $n \to \infty$,*

$$\mathrm{MIT}_n^{(m)} \xrightarrow{u.c.p} \int_0^1 (p_s)^m \, ds.$$

Moreover, as $n \to \infty$

$$\sqrt{n} \begin{bmatrix} \mathrm{IT}_n - \int_0^1 p_s \, ds \\ \mathrm{MIT}_n^{(m)} - \int_0^1 (p_s)^m \, ds \end{bmatrix} \stackrel{stably}{\Longrightarrow} \mathcal{MN}\left(0, \Sigma_{\mathrm{MIT}}\right)$$

where $\mathcal{MN}(0, \Sigma_{\mathrm{MIT}})$ denotes the mixed-normal distribution with covariance matrix Σ:

$$\Sigma_{\mathrm{MIT}} = \begin{bmatrix} \int_0^1 p_s \left(1 - p_s\right) ds & \int_0^1 m \, p_s^m \left(1 - p_s\right) ds \\ \int_0^1 m \, p_s^m \left(1 - p_s\right) ds & \int_0^1 p_s^m \, \frac{p_s^m \, (2\,m+1) - p_s^{m+1} \, (2\,m-1) - (1+p_s)}{1-p_s} \, ds \end{bmatrix}.$$

Proof. See Appendix A.2. □

A consistent estimator of the matrix Σ_{MIT} can be obtained through a suitable combination of $\mathrm{MIT}_n^{(m)}$. Note that $\mathrm{MIT}_n^{(2)}$ corresponds to the first order auto-covariance of zeros. Under Assumption 2, the difference between the first and higher order auto-covariances of zeros becomes negligible as n increases, because of the scaling properties mentioned above ($\mathbb{C}\mathrm{ov}\left[\mathbb{B}_{j,n}, \mathbb{B}_{j+1,n}\right] = O_p(\Delta_n^{1/2})$). Hence, integrated squared probability of staleness, $\int_0^1 (p_s)^2 \, ds$, can be estimated not only by $\mathrm{MIT}_n^{(2)}$, but also by the empirical auto-covariance of some (finite) order $k \geq 2$. In order to generalize the empirical auto-covariance to the k-th order, we introduce the notion of $(k-)$*staggered multi-idle time*, defined as:

$$\mathrm{SIT}_n^{(k)} \stackrel{\text{def}}{=} \frac{1}{n-k} \sum_{j=1}^{n-k} \mathbb{B}_{j,n} \mathbb{B}_{j+k,n}.$$

The limiting properties of staggered multi-idle time are summarized by the following theorem.

Theorem 3.3. *Assume that Assumptions 1 and 2 hold. Then, For any finite $k \geq 1$, as $n \to \infty$,*

$$\mathrm{SIT}_n^{(k)} \xrightarrow{u.c.p} \int_0^1 (p_s)^2 \, ds.$$

Moreover, as $n \to \infty$,

$$\sqrt{n} \begin{bmatrix} \mathrm{SIT}_n^{(1)} - \int_0^1 (p_s)^2 \, ds \\ \mathrm{SIT}_n^{(k)} - \int_0^1 (p_s)^2 \, ds \end{bmatrix} \overset{stably}{\Longrightarrow} \mathcal{MN}\left(0, \Sigma_{\mathrm{SIT}}\right),$$

where $\mathcal{MN}\left(0, \Sigma_{\mathrm{SIT}}\right)$ denotes the mixed-normal distribution with covariance matrix Σ_{SIT}:

$$\Sigma_{\mathrm{SIT}} = \begin{bmatrix} \int_0^1 \left(p_s^2 + 2p_s^3 - 3p_s^4\right) ds & \int_0^1 4p_s^3 \left(1 - p_s\right) ds \\ \int_0^1 4p_s^3 \left(1 - p_s\right) ds & \int_0^1 \left(p_s^2 + 2p_s^3 - 3p_s^4\right) ds \end{bmatrix}.$$

Proof. See Appendix A.2. □

Theorem 3.3 shows that, under Assumption 2, the limiting value of $\mathrm{SIT}_n^{(k)}$ is independent of k. This allows to test the reliability of Assumption 2 on real data by comparing the statistics $\mathrm{SIT}_n^{(k)}$ for different values of k. We will return to this point in Section 4, dedicated to the specification analysis.

3.2 Local estimation of probability of staleness

In this section we consider the problem of estimating functionals of the probability of staleness of the form

$$U(f) = \int_0^1 f(p_s) \, ds,$$

for a (relatively) general test function $f(\cdot)$. To this purpose, we introduce the notion of *local idle time* and develop corresponding limit theory under Assumption 2. Precisely, let k_n be a sequence of integer numbers satisfying $k_n \to \infty, \quad k_n \Delta_n \to 0$. Local idle time is defined as:

$$\widehat{p}_i(k_n) = \frac{1}{k_n} \sum_{j=0}^{k_n - 1} \mathbb{B}_{i+j,n}, \quad i \in \{1, \ldots, n - k_n\}. \tag{5}$$

For any given $i \in \{1, \ldots, n-k_n\}$, $\widehat{p}_i\left(k_n\right)$ is a consistent estimator of $p_{i,n}$ (as follows from the proof of the Theorem 3.4 below). Consequently, $U\left(f\right)$ can be estimated by the Riemann sum of local idle times as:

$$U\left(\Delta_n, f\right)^n = \Delta_n \sum_{i=1}^{n-k_n+1} f\left(\widehat{p}_i\left(k_n\right)\right).$$

Theorem 3.4. *Let $f\left(\cdot\right)$ be a locally bounded function and assume that Assumptions 1 and 2 hold. Then, as $n \to \infty$,*

$$U\left(\Delta_n, f\right)^n \stackrel{u.c.p.}{\longrightarrow} \int_0^1 f\left(p_s\right) ds.$$

Proof. See Appendix A.3 □

The idea of estimating the functionals $U(f)$ using $U\left(\Delta_n, f\right)^n$ mimics the idea of estimating volatility functionals developed by Jacod and Rosenbaum (2013, 2015). As for the case of estimation of volatility functional, $U\left(\Delta_n, f\right)^n$ admits a stable CLT with $\mathcal{F}$-conditionally Gaussian limit, which is, however, not centered. If $k_n \sim \theta/\sqrt{\Delta_n}$, for some constant θ, the $\mathcal{F}$-conditional mean of the limit consists of several bias terms depending on end-effects, the second derivative of f and the quadratic variation of p_t. If k_n diverges slower than $1/\sqrt{\Delta_n}$, the $\mathcal{F}$-conditional mean of the limit depends only on the second derivative of f, while the other bias terms are immaterial.

In order to obtain a CLT with a conditionally centered Gaussian limit, $U\left(\Delta_n, f\right)^n$ ought to be bias-corrected. The biases depending on end-effects and on the second derivative of f can be easily estimated. On the other hand, the bias term depending on the quadratic variation of p_t is more complicated to estimate (in particular, the convergence rate of the estimator is slower, (see Jacod and Rosenbaum, 2015)). Hence, in order to eliminate the latter we focus on the case with k_n converging to infinity slower than $1/\sqrt{\Delta_n}$. In such a case, the bias-corrected version of $U\left(\Delta_n, f\right)^n$ takes the following form:

$$U'\left(\Delta_n, f\right)^n = \Delta_n \sum_{i=1}^{n-k_n+1} \left(f\left(\widehat{p}_i\left(k_n\right)\right) - \frac{1}{2\,k_n} f''\left(\widehat{p}_i\left(k_n\right)\right) \widehat{p}_i\left(k_n\right)\left(1-\widehat{p}_i\left(k_n\right)\right)\right).$$

Then, we have the following CLT

Theorem 3.5. *As $n \to \infty$, let k_n a sequence of integers such that $k_n^2 \Delta_n \to 0$ and $k_n^3 \Delta_n \to \infty$. Besides, let f a test function satisfying the following conditions*

$$\left|f^{(j)}\left(p\right)\right| \leq K\left(1+|p|^{m-j}\right), \quad j = 0, 1.$$

for suitable positive constants K and m and assume that Assumptions 1 and 2 hold. Then, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}}\left(U'\left(\Delta_n, f\right)^n - \int_0^1 f\left(p_s\right)\, ds\right) \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma),$$

where $\mathcal{MN}(0,\Sigma)$ denotes the mixed-normal distribution with covariance matrix

$$\Sigma = \int_0^1 f'\left(p_s\right)^2 p_s(1-p_s)\, ds.$$

Proof. See Appendix A.3 □

In order to improve the performance of $U'\left(\Delta_n, f\right)^n$ in any finite sample, we adjust $U'\left(\Delta_n, f\right)^n$ by an asymptotically negligible correction for the end-effect as follows:

$$U''\left(\Delta_n, f\right)^n = \frac{\left(n-k_n+1\right)^{-1}}{\Delta_n} U'\left(\Delta_n, f\right)^n.$$

The adjusted version, $U''\left(\Delta_n, f\right)^n$, is used for the estimation of $U(f)$ in the rest of the paper. In the simulation study, we compare the performance of $U''\left(\Delta_n, f\right)^n$ and $\text{MIT}_n^{(m)}$ for estimating integrated powers of p_t.

3.3 On the estimation of the volatility of staleness

In this section, under the assumption that p_t evolves as a Brownian semimartingale (Assumption 2), we investigate the possibility of non-parametrically estimating the quadratic variation of p_t, i.e. $\int_0^1 \nu_s^2\, ds$, which represents the integrated (intraday) volatility of staleness. As explained in the introduction, price staleness constitutes an illiquidity measure. Thus, $\int_0^1 \nu_s^2\, ds$ is readily interpreted as the integrated as the volatility of liquidity. Measuring the volatility of liquidity is of relevant economic importance. For instance, as pointed out by Persaud (2003) “there is also broad belief among users of financial liquidity – traders, investors and central bankers – that the principal challenge is not the average level of financial liquidity... but its variability and uncertainty...”. On the other hand, the problem is volumetric and deserve special attention, which is worth for a separate paper. In this section we provide a first step by deriving a consistent estimator of the quadratic variation of p_t.

If p_t were observed, its quadratic variation would be consistently estimated by the realized variance $\sum_{i=1}^n \left(p_{i,n} - p_{i-1,n}\right)^2$. However, the increments of p_t are not observable, hence a proxy of them, constructed using local idle time[4], is used instead. Replacing the increments of p_t with their estimates induces a bias in measuring the quadratic variation. Theorem 3.6 below shows that the (properly rescaled) squared increments of local idle time converges in probability to the sum between of the volatility of staleness and the asymptotic variance of the idle time. The latter bias term can be estimated and corrected by straightforward application of Theorem 3.4.

Theorem 3.6. *Let $k_n = \theta \lfloor \sqrt{n} \rfloor$ be a sequence of integers, for some constant $\theta > 0$. Besides, assume that Assumptions 1 and 2 hold. Then, as $n \to \infty$,*

$$k_n^{-1} \sum_{i=1}^{n-2k_n+1} \left(\widehat{p}_{i+k_n}(k_n) - \widehat{p}_i(k_n)\right)^2 \xrightarrow{p} \frac{2}{3} \int_0^1 \nu_s^2 \, ds + \frac{2}{\theta^2} \int_0^1 p_s \left(1 - p_s\right) \, ds,$$

where $\widehat{p}_i(k_n)$ is the local idle time as in Eq. (5).

Proof. See Appendix A.4. □

Then, by combining Theorem 3.6 with Theorem 3.4, a consistent estimator of $\int_0^1 \nu_s^2 \, ds$, i.e. of the integrated volatility of the process p_t can be defined as

$$\mathrm{VIL}_n = \frac{3}{2} \left(k_n^{-1} \sum_{i=1}^{n-2k_n+1} \left(\widehat{p}_{i+k_n}(k_n) - \widehat{p}_i(k_n)\right)^2 - \frac{2}{\theta^2} U''(\Delta_n, f)^n \right), \tag{6}$$

where $f(x) = x(1 - x)$. Since IT_n can be used as a measure of illiquidity, the acronym VIL_n stands for "volatility of illiquidity" at frequency n.

Unfortunately, VIL_n is not non-negative by construction. It can take a negative value if the integrated volatility of staleness is small relative to the variance of $\frac{2}{\theta^2} U''(\Delta_n, f)^n$. This situation is especially likely when the volatility of staleness is close to zero. Hence, in order to avoid negative estimates in practice, we use the following modified estimator:

$$\mathrm{VIL}'_n = \max \left\{ \mathrm{VIL}_n, 0 \right\}. \tag{7}$$

At the costs of additional assumptions regarding the dynamics of ν_t, it is possible to derive a CLT for VIL_n, similarly to what is done for the volatility of volatility (Vetter et al., 2015). However, deriving a CLT and further investigation of the volatility of liquidity is left for further research.

[4]Note that, in contrast to assumptions in Theorem 3.5, in order to estimate the volatility of staleness we have to take $k_n \sim \theta / \sqrt{\Delta_n}$

4 Statistical tests

We now turn to the construction of statistical tests for investigating the dynamical properties of p_t. First, we consider testing for the constancy of a path of p_t over a given time interval. Second, in order to examine if the semimartingale Assumption 2 reflects the properties of the financial data, the we test for the smoothness of a path of p_t.

4.1 Testing for constant probability of staleness

For some $m \geq 2$, define the following two complementary subsets of Ω:

$$\Omega^0 = \left(\omega \in \Omega \,\middle|\, \int_0^1 (p_t(\omega))^m \, dt = \left(\int_0^1 p_t(\omega)\, dt \right)^m \right),$$
$$\Omega^1 = \left(\omega \in \Omega \,\middle|\, \int_0^1 (p_t(\omega))^m \, dt \neq \left(\int_0^1 p_t(\omega)\, dt \right)^m \right).$$

Then, testing for constancy of p_t amounts to distinguish the two complementary subsets of Ω based on the observed sample of Bernoulli random variables. In other words, testing for constant probability of staleness is equivalent to testing the following two hypothesis:

$$\mathcal{H}_{-1}: \ \left(\mathbb{B}_{i,n}(\omega)\right)_{i=1,\dots,n} \in \Omega^0 \quad \text{v.s.} \quad \overline{\mathcal{H}}_{-1}: \ \left(\mathbb{B}_{i,n}(\omega)\right)_{i=1,\dots,n} \in \Omega^1.$$

Indeed, if the trajectory $p_t(\omega)$ is constant over $[0,1]$, the observed sample $\left(\mathbb{B}_{i,n}(\omega)\right)_{i=1,\dots,n}$ belongs to Ω_0. On the contrary, the equality characterizing the set Ω_0 does not hold provided that $p_t(\omega)$ is time-varying.

By Theorems 3.1 and 3.2 (which, in particular, provide the stable convergence of IT_n and $\mathrm{MIT}_n^{(m)}$ on Ω^0) and delta method, the test statistics is naturally defined as:

$$\Psi_{n,m} \stackrel{\text{def}}{=} \frac{\sqrt{n}\left(\mathrm{MIT}_n^{(m)} - (\mathrm{IT}_n)^m\right)}{\sqrt{\frac{(\mathrm{IT}_n)^{2m+1}(m^2+2m-1)-(\mathrm{IT}_n)^{2m}(2m^2+2m+1)+(\mathrm{IT}_n)^{m+1}+(\mathrm{IT}_n)^m}{\mathrm{IT}_n-1}}}. \tag{8}$$

The asymptotic behaviour of the $\Psi_{n,m}$ statistics is described the following corollary.

Corollary 1. *Assume Assumptions of Theorem 3.2 hold. As $n \to \infty$*

$$\begin{cases} \Psi_{n,m} \stackrel{stably}{\Longrightarrow} \mathrm{N}(0,1) & on\ \Omega^0, \\ \Psi_{n,m} \stackrel{p}{\longrightarrow} +\infty & on\ \Omega^1. \end{cases}$$

Proof. See Appendix A.2. $\square$

On Ω^0, $\Psi_{n,m}$ converges (stably) to a zero-mean normal distribution with unit variance. On Ω^1, i.e. when p_t is not constant on the whole interval $[0, 1]$, it diverges as the number of observations n increases.

4.2 Testing for smoothness of the probability of staleness

In this section we propose a test for the Brownian semimartingale specification of p_t (Assumption 2). It implies that $\mathbb{C}\mathrm{ov}\left[\mathbb{B}_{j,n}, \mathbb{B}_{j+1,n}\right] = O_p\left(\Delta_n^{1/2}\right)$, which guarantees that the difference between the first and higher order auto-covariances of zeros becomes negligible as n increases. Hence, Assumption 2 can be tested by comparing the first and higher order auto-covariances of zeros, captured by staggered multi-idle times, $\mathrm{SIT}_n^{(k)}$, with different k's.

An alternative to Assumption 2 should postulate a different scaling property for the auto-covariance of zeros. This can be achieved if, for example, p_t follow a process with rough sample paths. Instead of specifying a particular process describing the dynamics of p_t for formulating an alternative hypothesis, we consider the following high-level alternative to the Assumption 2:

Assumption 3. *The process p_t has Riemann integrable paths and $\left|\mathbb{E}_t\left[|p_{t+\Delta_n} - p_t|\right]\right| = K\Delta_n^q + o_p(\Delta_n^{q+\epsilon})$ pointwise on Ω, for some $0 < q < \frac{1}{2}$, $K, \epsilon > 0$.*

Then, the testing problem can be formulated as:

$$\mathcal{H}_0 : \text{Assumption 2 holds} \quad \text{v.s.} \quad \mathcal{H}_1 : \text{Assumption 3 holds.}$$

The test is defined as:

$$\Phi_{n,k} \stackrel{\text{def}}{=} \frac{\sqrt{n}\left(\mathrm{SIT}_n^{(1)} - \mathrm{SIT}_n^{(k)}\right)}{\sqrt{2 \cdot \Delta_n \sum_{j=1}^{n-k_n}\left(\left(\widehat{p}_j(k_n)\right)^2 - 2\left(\widehat{p}_j(k_n)\right)^3 + \left(\widehat{p}_j(k_n)\right)^4\right)}}. \tag{9}$$

The asymptotic behaviour of $\Phi_{n,k}$ is described in the following corollary.

Corollary 2. *Assume Assumptions of Theorem 3.3 hold. As $n \to \infty$*

$$\begin{cases} \Phi_{n,k} \xrightarrow{\text{stably}} \mathrm{N}(0,1) & \text{under } \mathcal{H}_0, \\ |\Phi_{n,k}| \xrightarrow{p} +\infty & \text{under } \mathcal{H}_1. \end{cases}$$

Proof. See Appendix A.2. □

If p_t is a Brownian semimartingale, $\Phi_{n,k}$ converges (stably) to a zero-mean normal distribution with unit variance. Instead, if p_t is a rough process, the test statistic diverges as the number of observations increases.

5 Monte Carlo

5.1 Simulation settings

In absence of finite-sample distortions, the implementation of the asymptotic theory developed in Section 3 and Section 4 would require the adoption of the highest frequency available for the data: the larger the frequency the closer the random quantities to their limits (either in probability or stably in law). Nevertheless, price discreteness may affect these limits, producing unwanted spurious effects. More precisely, in presence of rounding, there could be some extra zero returns not generated by "genuine" flatness. In this section we explore the finite sample contaminations of the asymptotic theory by means of Monte Carlo simulations. In particular we want to asses the sizes and the powers of the two tests $\Psi_{n,m}$ and $\Phi_{n,k}$ defined, respectively, in (8) and (9). For this purpose we generate a large artificial dataset of efficient price paths contaminated by flatness and rounded at one cent (as imposed by the actual settings of electronic financial markets). We simulate, for each replication, a trading day of 6.5 hours on a time-gird of one second, for a total of $6.5 \times 60 \times 60$ steps. First of all, we create the path of an efficient log-price process $Y_t = \log(P_t)$ driven by a one-factor stochastic volatility model, whose dynamics is described by the SDE:

$$\begin{aligned} d\log\sigma_t^2 &= \left(\alpha - \beta\,\log\sigma_t^2\right)\,dt + \eta\,dW_{\sigma,t}, \\ dY_t &= \mu\,dt + c_\sigma\,\sigma_t\,dW_{Y,t}, \end{aligned} \tag{10}$$

where $W_{\sigma,t}$ and $W_{Y,t}$ are two Brownian motions with $\mathrm{corr}\,(dW_{\sigma,t}, dW_{Y,t}) = \rho\,dt$. We adopt the values for the parameters α, β, η, μ and ρ estimated by Andersen et al. (2002) on S&P500. The volatility factor c_σ can be tuned to generate different scenarios. We impose $c_\sigma = 2$ that corresponds to, roughly, a daily volatility of 1%. Numerical integration of the SDE in (10) is performed on a one-second time grid via a standard Euler scheme and with the initial conditions $Y_0 = \log(P_0)$, with $P_0 = 100$, and $\log\sigma_0^2 = \alpha/\beta$. Once simulated, the efficient prices are sampled every thirty seconds. This sub-sampling produces, for each replication, the efficient log-prices $Y_{j,n}$ with $j = 1, ..., n$ and $n = 780$. Then, on the time grid of thirty seconds, we construct the flatness-contaminated price process $X_{j,n}$ following the recursive equation:

$$\begin{cases} X_{0,n} = Y_{0,n} = \log(P_0) \\ X_{j,n} = (1 - \mathbb{B}_{j,n})\;Y_{j,n} + \mathbb{B}_{j,n}\,X_{j-1,n}, \end{cases} \tag{11}$$

where $\mathbb{B}_{j,n}$ are Bernoulli random variables specified in one of the three ways described below. Finally, the flat-prices $\exp(X_{j,n})$ are rounded at one cent. The rounding is the only very reason that prevents to take the highest frequency available.

For each path of the efficient price process Y we consider three different specifications of triangular arrays $\mathbb{B}_{j,n}$.

Constant probability of staleness. In this specification the $\mathbb{B}_{j,n}$'s are i.i.d. Bernoulli random variables with constant expected value, $\mathbb{E}[\mathbb{B}_{j,n}] = p_F$ for all j. We put[5] $p_F = 0.5$.

Semimartingale-type probability of staleness. This specification corresponds to Assumption 2. First, at each replication, we generate a path of a latent stochastic process u with the following (discrete-time) integration scheme:

$$\begin{cases} u_{0,n} = F^{-1}(p_F) \\ u_{j,n} = u_{j-1,n} + \left(F^{-1}(p_F) - u_{j-1,n}\right)\Delta_n + \sigma_u\, \varepsilon_{j,n} \sqrt{\Delta_n}, \end{cases} \tag{12}$$

with $j = 1, ..., n$, $\Delta_n = 1/n$, $n = 780$, $p_F = 0.5$ and where $F^{-1}(x)$ is the inverse of the cumulative distribution function of a standard Gaussian variable. The $\varepsilon_{j,n}$'s are i.i.d. standard Gaussian shocks while σ_u is a tuning-parameter that we set to $\sigma_u = 1.5$. Then, a path of the stochastic probability p_t, defined in equation (2) of Assumption 1, is generated as:

$$p_{j,n} = \int_{-\infty}^{u_{j,n}} \frac{1}{\sqrt{2\,\pi}}\, e^{-z^2/2}\, dz = F(u_{j,n}).$$

Note that since, by construction, u is a mean-reverting around $F^{-1}(p_F)$, then p_t is mean-reverting around p_F. Hence, on average, the probability of zeros is similar to the value used in the constant probability case.

Rough probability of staleness. This specification corresponds to the alternative for the semimartingale-type behaviour of the probability of staleness. Instead of simulating rough probability paths explicitly, we adopt the following scheme, approximating the dependence of Bernoulli random variables for a fixed frequency. First, we generate two sequences of i.i.d. Bernoulli random variables, $\mathbb{B}^F_{j,n}$ and $\mathbb{B}^R_{j,n}$, with $j = 1, ..., n$ and $n = 780$ as for the other cases considered. These two sequences are characterized by two different expected values, i.e. we put $p_F \doteq \mathbb{E}\left[\mathbb{B}^F_{j,n}\right] = 0.5\%$

[5] With this numerical choice we are assuming that, at the frequency of 30 seconds, fifty percent of the log-returns are zeros. This corresponds to a moderately high level of illiquidity for the asset.

and $p_R \doteq \mathbb{E}\left[\mathbb{B}^R_{j,n}\right] = 0.2\%$. Then, they are assembled together via the recursive equation:

$$\begin{cases} \mathbb{B}_{1,n} = \mathbb{B}^F_{1,n} \\ \mathbb{B}_{j+1,n} = \mathbb{B}^F_{j+1,n}\left(1 - \mathbb{B}^R_{j+1,n}\right) + \mathbb{B}^R_{j+1,n}\mathbb{B}_{j,n}, \quad j \geq 1. \end{cases}$$

Hence, the Bernoulli random variables $\mathbb{B}_{j,n}$'s mimic the persistency, which could be implied by a rough probability process.

Figure 2 shows an example of a path of p_t and the corresponding path of the stale price process generated by the model (12). It illustrates the flexibility of our semimartingale model in controlling the occurrence of zeros via the realization of the process p_t. For instance, in the example, the probability of flat trading becomes very small after the middle of the trading day. The number of observed zeros declines accordingly. In particular, the price is stale only in the first part of the day.

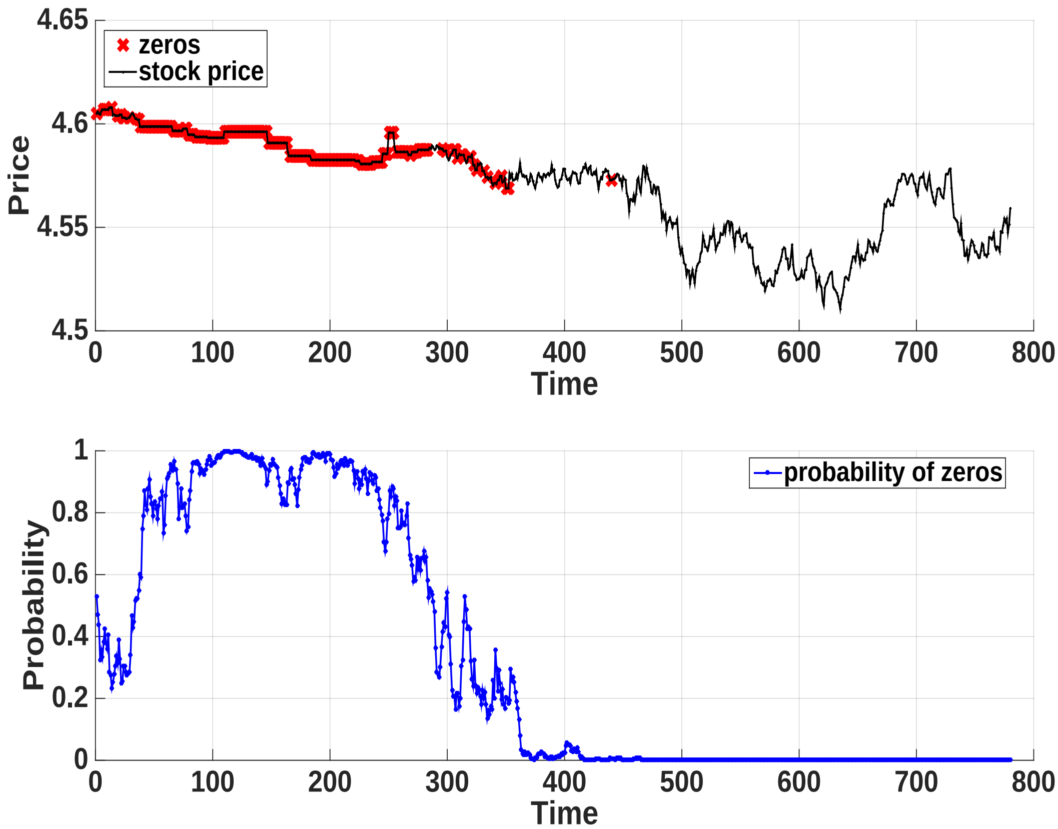

Figure 2: Stale stock price and the probability of observing zero returns generated by models 10 and (12) respectively. Zeros are indicated with red circles.

5.2 Estimation of p_t functionals

Here we illustrate estimation of integrated functionals of the instantaneous probability of zeros, under the Assumption 2. We focus on estimating the most relevant functional, i.e. $\int_0^1 p_s(1-p_s)\,ds$, which represents the asymptotic variance of idle time. It can be estimated either as a difference of idle and multi-idle times, $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$, or by integrated local idle time, $U''(\Delta_n, f)^n$, with $f(x) = x - x^2$.

Figure 3 shows the two estimates of $\int_0^1 p_s(1-p_s)\,ds$ for different levels of the true value. It indicates that both estimators are remarkably precise. However, the variance of $U''(\Delta_n, f)^n$ (computed using block size $k_n = 13$) is considerably smaller than the variance of $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$. The later result is expected, since $U''(\Delta_n, f)^n$ constitutes a localized maximum likelihood estimator.

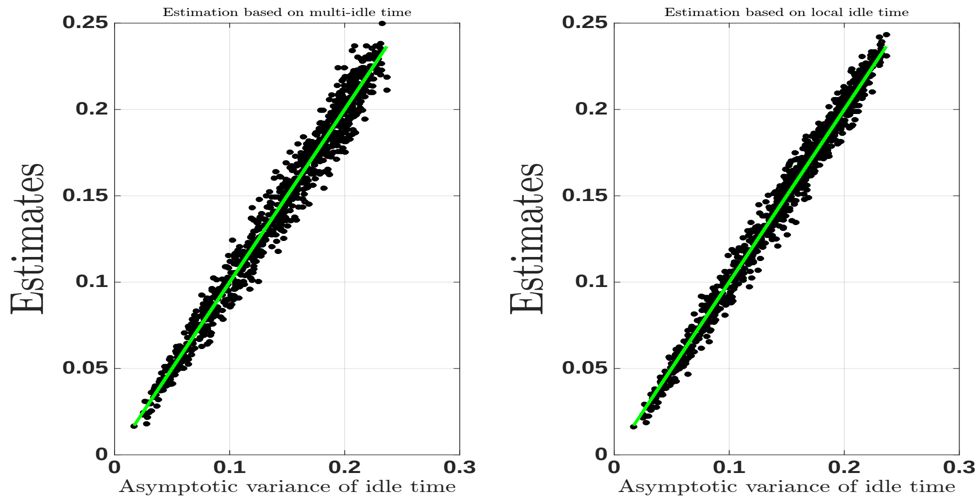

Figure 3: Scatter plot of the asymptotic variance of idle time, $\int_0^1 p_s(1-p_s)\,ds$ and its estimated values based on multi-idle time (left panel) and local idle time based on blocks of size $k_n = 13$ (right panel). The green line represents the true value.

The superiority of $U''(\Delta_n, f)^n$ over the difference estimator $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$ is robust across reasonable choices of k_n. Figure 4 shows the bias, standard deviation and the root mean squared error (RMSE) of $U''(\Delta_n, f)^n$ as a function of k_n, and compares them with the corresponding characteristics of the difference estimator. It turns out that the bias of $U''(\Delta_n, f)^n$ (left panel of Figure 4) increases with k_n and it is larger than the bias of $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$. The variance of $U''(\Delta_n, f)^n$ (central panel of Figure 4) is U-shaped with the minimum at around $k_n = 15$, which roughly corresponds to $k_n = n^{2/5}$. Even if k_n takes a large value, e.g. $k_n = 40$, which roughly corresponds to $k_n = 3/2\sqrt{n}$, the variance of $U''(\Delta_n, f)^n$ is smaller than the variance of $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$. For both estimators, the bias is an order of magnitude smaller than the variance, hence, RMSE of $U''(\Delta_n, f)^n$ (right panel of Figure 4) is dominated by the variance and it is smaller than RMSE of the difference estimator for all reasonable choices of k_n. Of course, the optimal choice of k_n, in general, depends on the properties of p_t, e.g. on its quadratic variation. However, the Monte Carlo illustration suggest that $U''(\Delta_n, f)^n$ remains reasonably precise even for suboptimal values of $k_n < \sqrt{n}$.

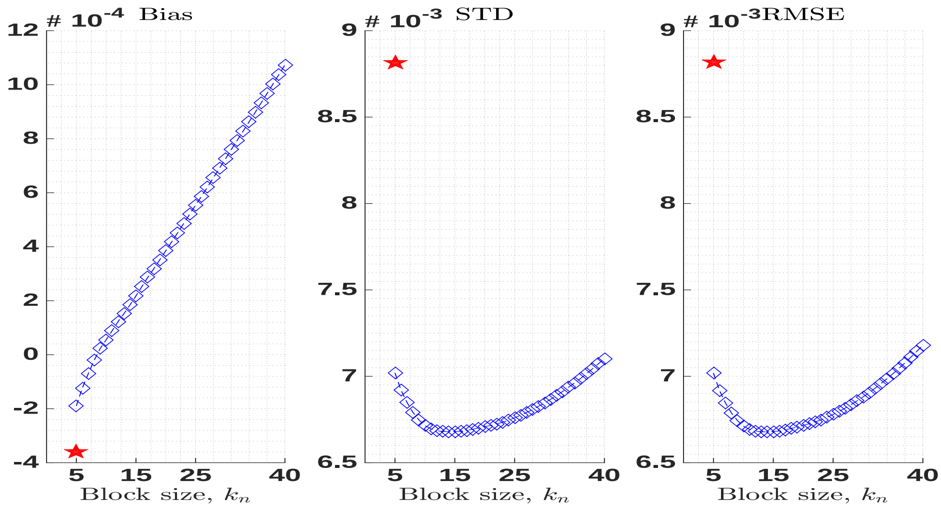

Figure 4: The bias, standard deviation (STD) and the root mean squared error (RMSE) of the estimators of $\int_0^1 p_s(1-p_s)\,ds$. The blue rombus correspond to $U''(\Delta_n, f)^n$ for different choices of k_n. The red stars represents the difference $\mathrm{IT}_n - \mathrm{MIT}_n^{(2)}$, which does not depend on k_n.

5.3 Sizes and powers of $\Psi_{n,m}$ and $\Phi_{n,k}$ tests

The test statistics $\Psi_{n,m}$ and $\Phi_{n,k}$ are both characterized by a choice variable, more precisely $\Psi_{n,m}$ depends on the number m of factors in the multi-idle time $\mathrm{MIT}_n^{(m)}$ defined in (5) while $\Phi_{n,k}$ depends on the number of lags k in the staggered multi-idle time $\mathrm{SIT}_n^{(k)}$ defined in (5). Asymptotically, the distribution of both $\Psi_{n,m}$ and $\Phi_{n,k}$ are unaffected by the value of m and k, as well as their divergence toward $+\infty$ under the respective alternative hypotheses. Nevertheless, in finite sample, both m and k can be chosen to trade-off size and power of the two tests. Following the procedures described in Section 5 we generate 10^4 replications of (rounded) price paths under Ω_0, $\mathcal{H}_0$ (which, clearly, is included in Ω_1) and $\mathcal{H}_1$. Since Ω_0 and $\mathcal{H}_0$ are, respectively, the null and the alternative for $\Psi_{n,m}$ while $\mathcal{H}_0$ and $\mathcal{H}_1$ are, respectively, the null and the alternative for $\Phi_{n,k}$, we can evaluate, for different choices of m and k, the size and power of both tests by computing their rejection rates under the proper set of artificial data. Figure 5 summarizes the results of this numerical experiment, in particular we report 5% rejections rates of both tests under their respective null and alternative. In the case of $\Psi_{n,m}$, a reasonable trade-off between size and power is attained taking m around 5, a choice that maximizes power and gives a conservative (less than the theoretical 5%) size. The case of $\Phi_{n,k}$ is quite different: the larger the value of the lag k the more distorted its size, while the power is quite high even for $k = 2$. Hence, in finite sample, a small value of k is advisable. Of course, the specific power and size of the tests depend on how the alternative is formulated. For example, an higher value for the parameter σ_u in (12) would deliver a more powerful $\Psi_{n,m}$.

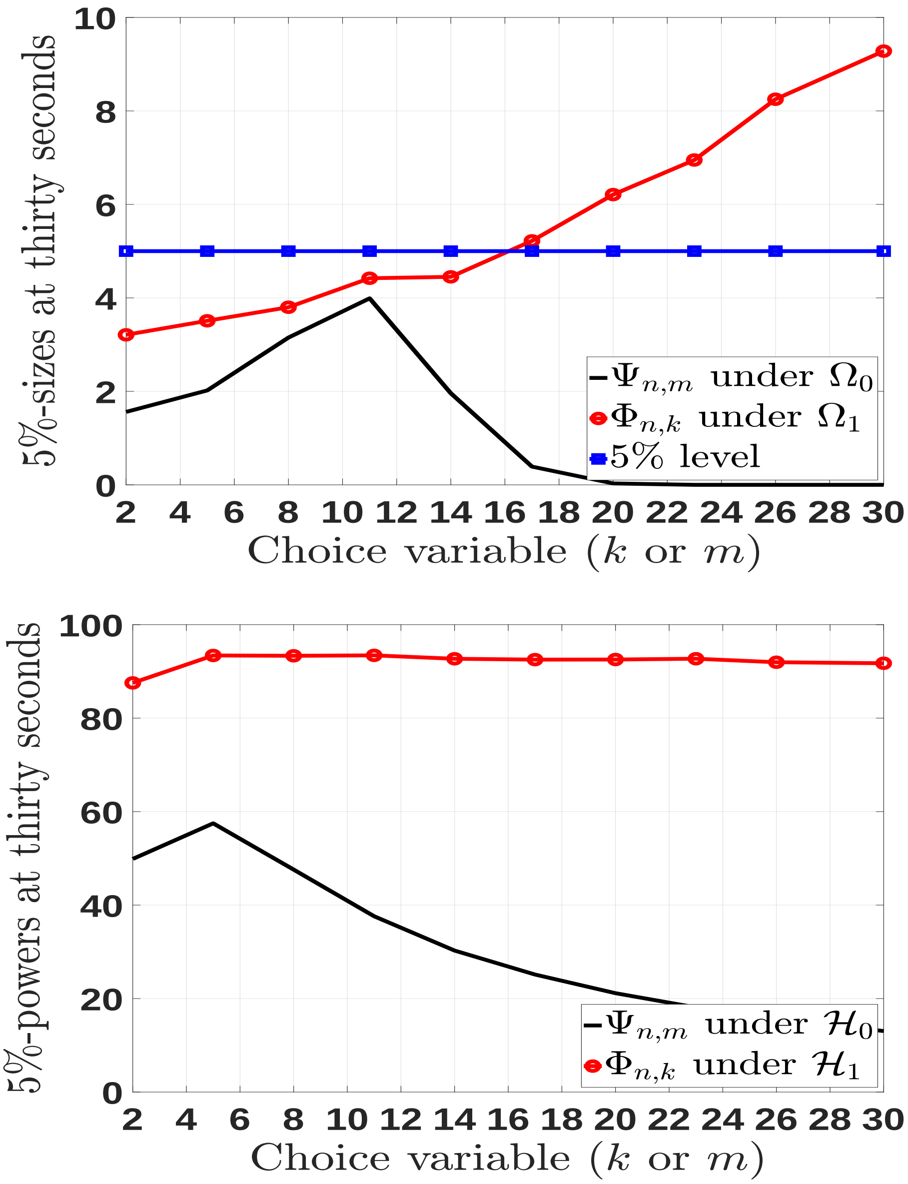

Figure 5: The top (resp. bottom) panel reports, as black thick line, the size and, as a red dotted line, the power of the test $\Psi_{n,m}$ (resp. $\Phi_{n,k}$) as a function of m (resp. k).

6 Empirical illustration

In this section we consider intraday price paths, sampled at 30-second intervals, for stocks traded on the New York Stock Exchange (NYSE). Our sample includes 248 most traded stocks. The observation period consists of 2246 trading days and ranges from 3 Jan 2006 to 31 Dec 2014. Each trading day includes 780 intraday observations recorded from 09 : 30 to 15 : 30. The days in which the trading session was interrupted prior to 15 : 30 are excluded from the consideration. For each day and stock we record zero returns defined as the absence of price adjustment during 30-second sampling intervals.

We start with specification tests. For each day and stock in our sample we compute Ψ and Φ tests for constancy and smoothness of the paths of p_t. Figure 6 shows kernel smooth density estimates of the test statistics of the two tests for the pooled data. The distribution of Ψ is clearly different from a standard normal, in particular, it is shifted to the right. This indicates that for the majority of days and stocks in our sample the constancy of p_t is rejected. The distribution of Φ test statistics is close to standard normal, but does not perfectly coincide with it. Hence, smoothness of the paths of p_t can not be rejected for the majority of days and stocks in our sample with rare exceptions, one of which is considered below. Overall, the specifications tests indicate that the probability of occurrence of zero returns is time-varying and most often its dynamics can be sufficiently well approximated by a smooth semimartingale model.

The most prominent example of a stock (in our sample) for which the smoothness of the probability of observing zeros is violated is Citigroup Inc. (C). Figure 7 shows the time series of daily Φ test statistics for this stock. It can be seen that the smoothness of p_t is systematically rejected during a particular sub-sample: from the beginning of 2009 until the middle of 2011. During this period Citigroup was reorganized into different operating units. This reorganisation might affect the liquidity of Citigroup stocks, which materializes in the change of the dynamics of zeros.

The intraday variation of zeros in our semimartingale model has two sources. The first is the deterministic dynamics captured by the drift component of p_t, while the second is due to the volatility of p_t. Figure 8 illustrates the deterministic component. For a selection of stocks, it shows averaged over the whole sample intraday local idle time estimates. For each stock local idle time exhibits emphatic intraday pattern. On average, the occurrence of zeros is almost twice less probable in the morning with respect to the noonday. For example, for Exxon Mobil Corporation (XOM) average local idle time is equal to 0.12 at 09 : 30, while it increases up to 0.24 at 12 : 30.

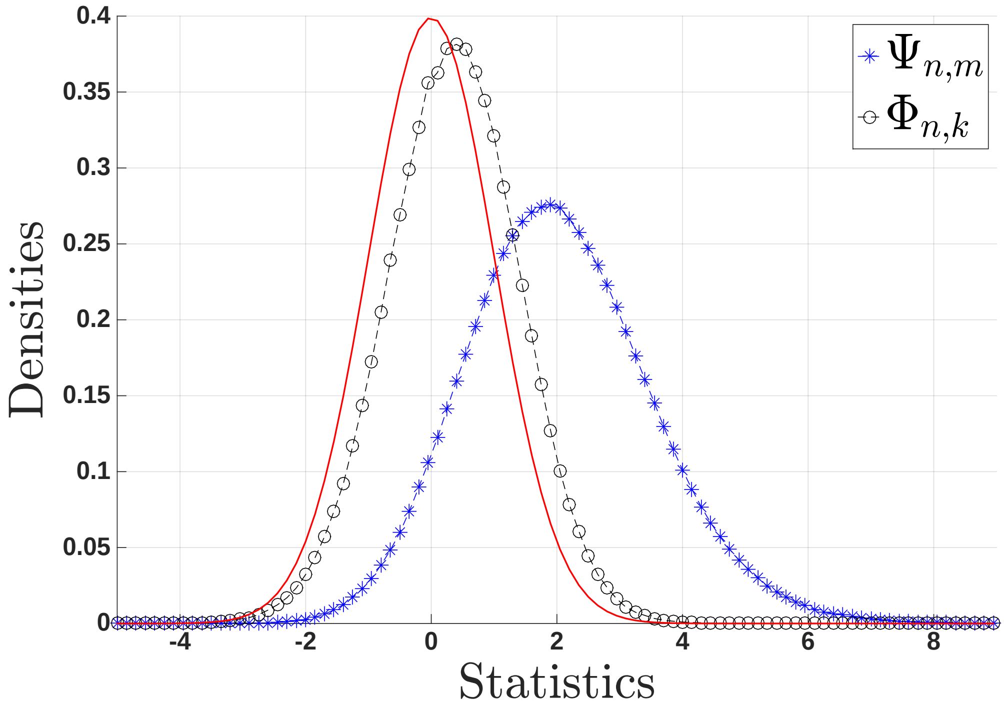

Figure 6: Kernel smooth density estimates of the daily test statistics of Ψ and Φ tests for constancy and smoothness of the paths of p_t respectively, computed for the pooled data.

Figure 9 illustrates the variation of zeros due to the stochastic volatility of p_t. Each panel of the figure compares estimated (using local idle time) paths of p_t for days with large and small volatility of p_t. For example, middle left panel shows the local idle times for PepsiCo Incorporation (PEP). In the morning the level of stalness is around 20% for both days with low and high volatility of p_t. For the first day, the level of staleness mildly fluctuates around the intraday pattern. For the day with high volatility of staleness, local idle time rises up to 90% by the noon. By 13 : 00 it declines back to the original level and continues fluctuating intensively by the end of the day. Together, Figures 8 and 9 indicate that both deterministic and stochastic components significantly contribute to the intraday variation of staleness.

Figure 10 presents a scatter plot of daily idle time and volatility of staleness for all considered stocks combined together. It shows the hump-shaped form of the dependence of the volatility of staleness on the level of staleness. The volatility of p_t is typically small if a stock is very actively traded (hence, idle time is close to zero) or if a stock price is very stale (idle time is close to 90%). The largest values of the volatility p_t are achieved for the days with medium level of staleness.

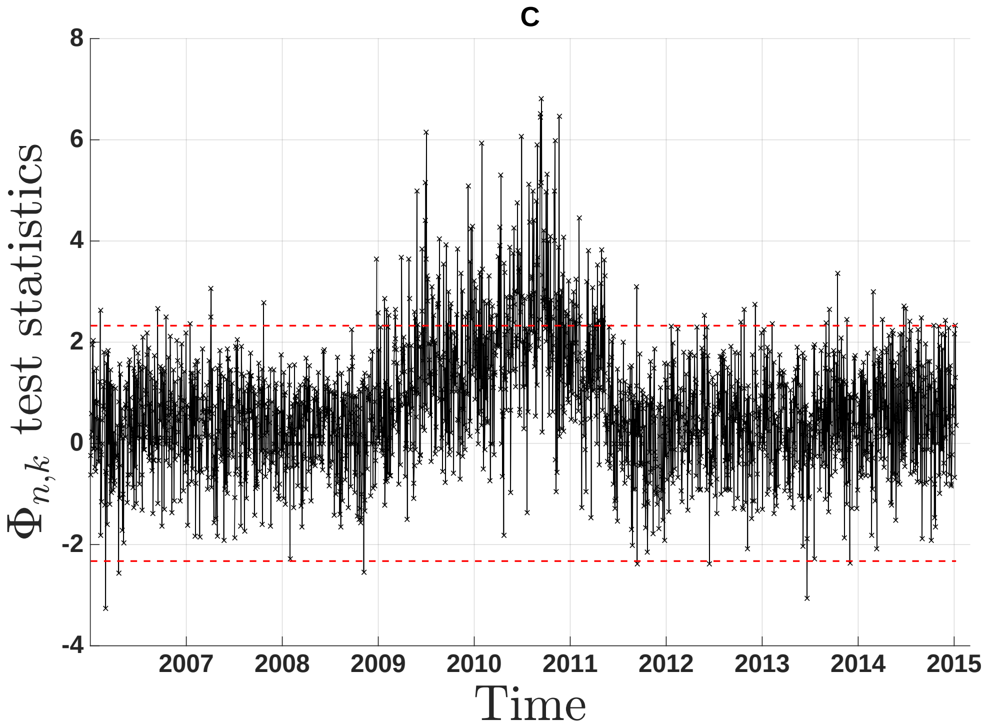

Figure 7: Daily test statistics of Φ test for smoothness of the paths of p_t for Citigroup Inc.

7 Conclusions

We introduce a general econometric framework, which incorporates the possibility of observing zero returns in the data generating process of stock prices. It extends widespread stochastic volatility models by allowing for staleness in price adjustments producing zero returns. The statistical properties of the staleness are controlled by the instantaneous probability of arrivals of stale prices, which is assumed to follow a continuous-time dynamics. Since price staleness is naturally linked to the absence of liquidity, our framework allows to conduct statistical analysis of liquidity in a way analogous to the analysis of integrated volatility. In particular, we develop asymptotic theory for several statistics, named *(m-)multi-idle time*, *staggered multi-idle time* and *local idle time*, instructive about the dynamic properties of the instantaneous probability of staleness. This allows to set up feasible confidence intervals for idle time, a liquidity measure introduced in Bandi et al. (2017), and to conduct nonparametric specification tests. We test whether the probability of observing zero returns is constant or time varying during the day and whether its dynamics can be described by a Brownian semimartingale. Application on NYSE stock prices shows that the probability of the occurrence of stale prices is time-varying and can be described with adequate realism by an $(0, 1)$-valued Itô

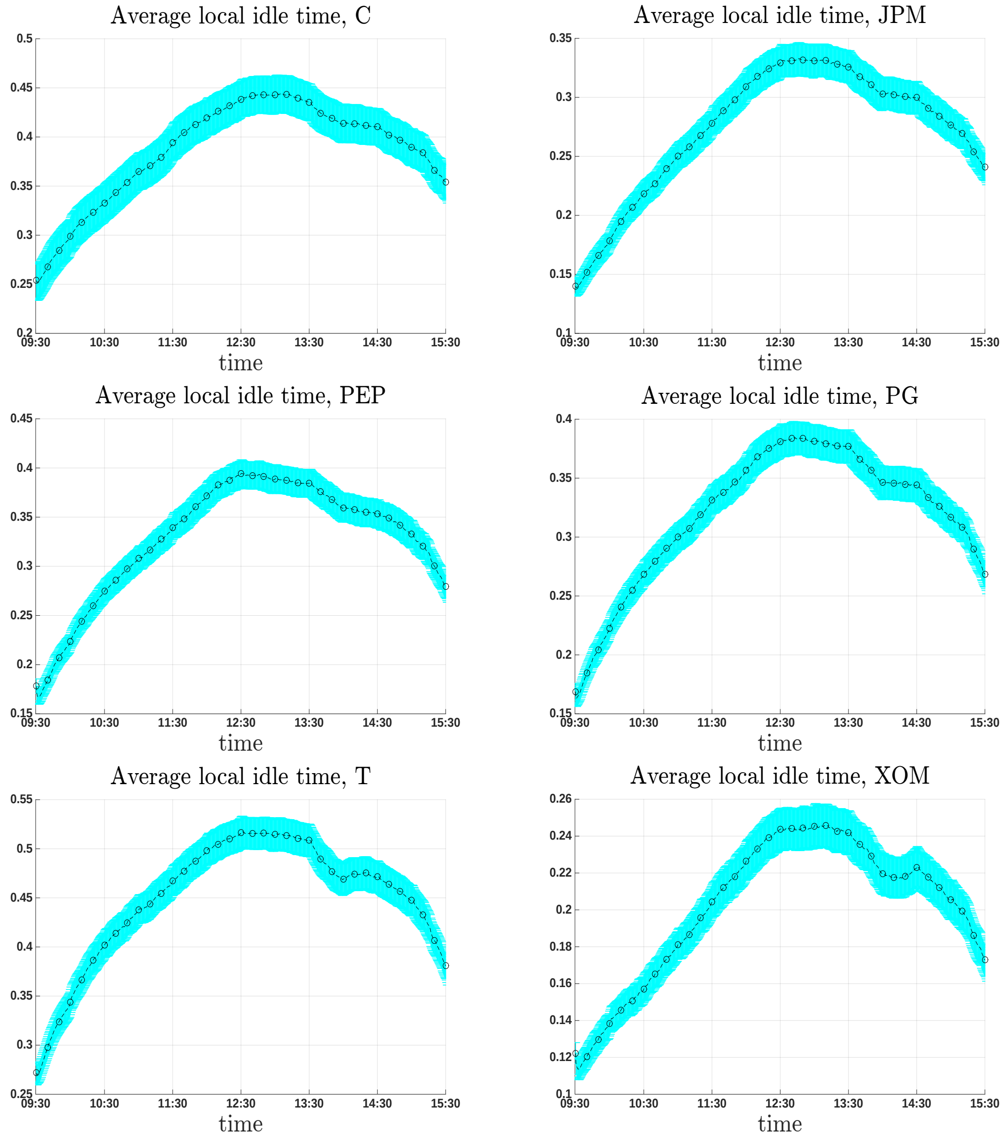

Figure 8: Averaged over the whole sample intraday local idle time estimates for a selection of stocks.

semimartingale.

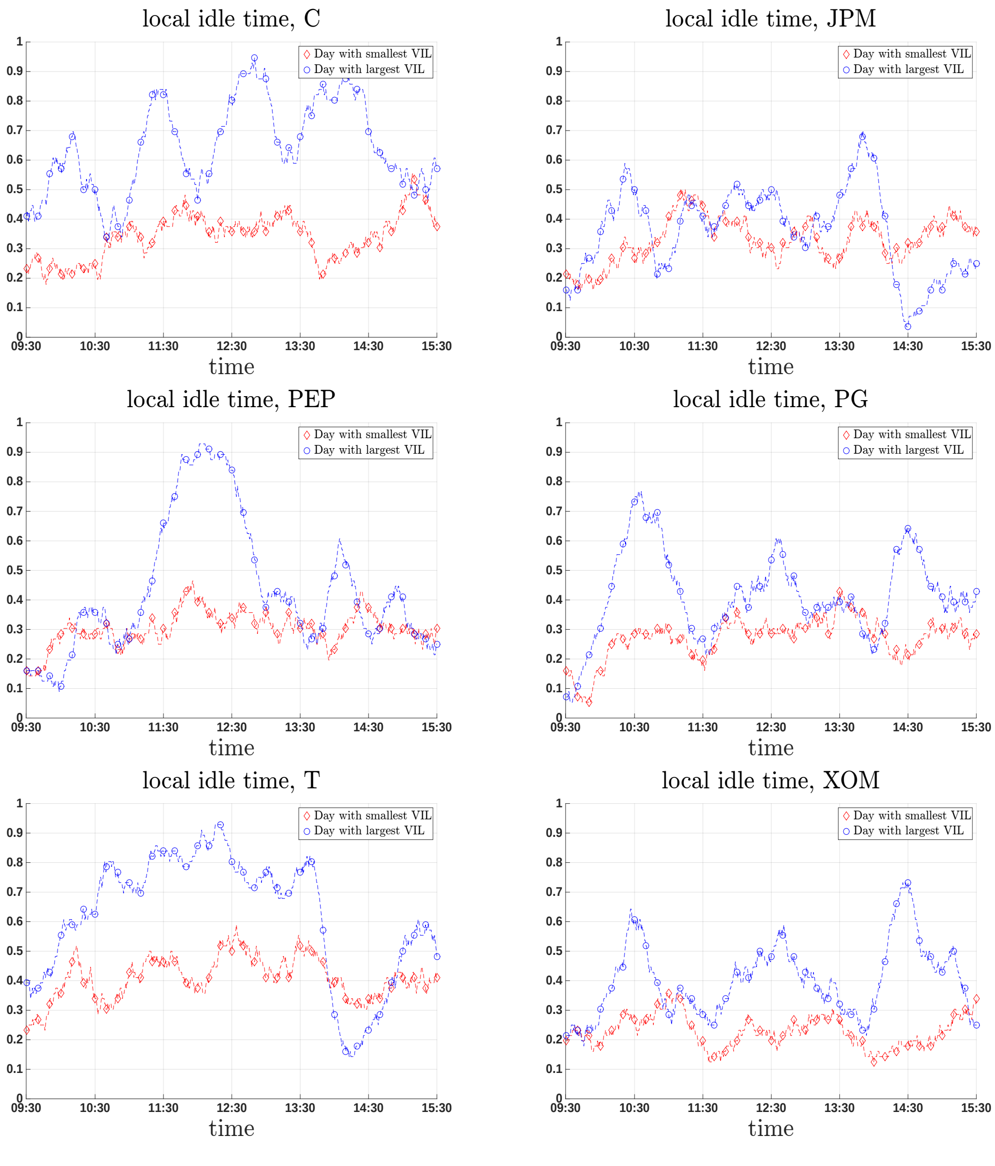

Figure 9: Local idle time estimates for days corresponding to the lowest (denoted by red rhombus) and the highest (denoted by blue circles) volatility of p_t for a selection of stocks.

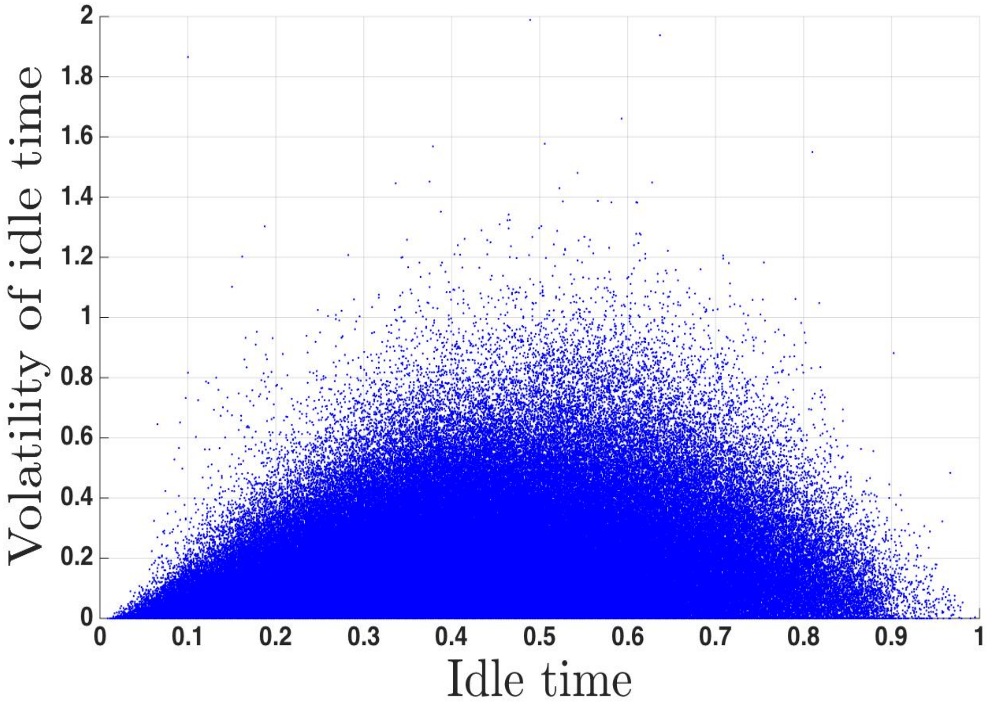

Figure 10: Scatter plot of daily idle time (horizontal axis) and volatility of p_t (vertical axis) for stocks combined together

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A Appendix: Proofs

The appendix is divided into four parts. Section A.1 introduces the notation and collects auxiliary results on the convergence of triangular arrays. Section A.2 is dedicated to the proofs of limiting results from Sections 3.1, 4.1 and 4.2. Section A.3 presents the proofs of Theorems 3.4 and 3.5. Finally, the proof of Theorem 3.6 is presented in Section A.4.

A.1 Notations and Auxiliary results

In what follows, we indicate with $t_{j,n} = j/n$, $j \in \{0, \ldots, n\}$ the deterministic equispaced partition of the interval $[0,1]$ and with $N_n(s) = \max\{j \mid t_{j,n} \le s\}$. Trivially $N_n(1) = n$. We use the symbol $\xrightarrow{p}$ for the convergence in probability, $\xrightarrow{u.c.p}$ for the uniform convergence in probability and $\overset{stably}{\Longrightarrow}$ for the stable convergence.

Now, we specify the structure of the σ-field $\mathcal{F}$. We have the following flows of information on $\mathcal{F}$: i) $(\mathcal{F}_t^{(p)})_{t\in[0,1]}$ is the natural filtration associated to the process p_t, ii) $\mathcal{U}_{t_{j,n}}$ is the σ-algebra generated by random variables $U_{0,n}, \ldots, U_{j,n}$, iii) $\mathcal{F}_{t_{j,n}} = \mathcal{F}_{t_{j,n}}^{(p)} \vee \mathcal{U}_{j,n}$ is a discrete time filtration associated to partitioning the interval $[0,1]$ with a descretization step $\Delta_n = 1/n$. Let $\mathcal{F}_\infty^{(p)} = \vee_{t\in[0,1]}\mathcal{F}_t^{(p)}$ be the smallest σ-algebra, which contains $\cup_{t\in[0,1]}\mathcal{F}_t^{(p)}$, $\mathcal{U}_\infty = \vee_{n=2}^{\infty}\mathcal{U}_{n,n}$, and $\mathcal{F}_{t_{n,n}} = \mathcal{F}_\infty^{(p)} \vee \mathcal{U}_{n,n}$. Then, we have: $\mathcal{F} = \mathcal{F}_\infty^{(p)} \vee \mathcal{U}_\infty$.

For sake of readability, we denote, for a generic index $j \in \{1, \ldots, n\}$, by $\mathbb{P}_j[\,\cdot\,]$, $\mathbb{E}_j[\,\cdot\,]$, $\mathbb{V}_j[\,\cdot\,]$ the conditional probability, the conditional expectation, and the conditional variance with respect to the filtration $\mathcal{F}_{t_{j,n}}$.

In what follows, our proofs and formalism will be inspired by those of Jacod (2012), Jacod and Protter (2012) and Aït-Sahalia and Jacod (2014). We say that a triangular array of random variables ξ_j^n, $j \in \{0, \ldots, n\}$, is asymptotically negligible (sometimes shortened, henceforth, in AN) if

$$\sum_{j=1}^{n} \xi_j^n \xrightarrow{u.c.p} 0,$$

that is,

$$\sup_{s\in[0,1]} \left| \sum_{j=1}^{N_n(s)} \xi_j^n \right| \xrightarrow{p} 0. \tag{13}$$

The following two remarks state simple properties that will be invoked repeatedly during the proofs

Remark 1. *Suppose that $\sum_{j=1}^{n} \left|\xi_j^n\right|$ converges to zero in $\mathbb{L}^1$, i.e.*

$$\mathbb{E}\left[\sum_{j=1}^{n} \left|\xi_j^n\right|\right] \to 0. \tag{14}$$

By standard argument, this implies that $\sum_{j=1}^{n} \left|\xi_j^n\right| \xrightarrow{p} 0$ and so it is sufficient to note that

$$\sup_{s\in[0,1]} \left| \sum_{j=1}^{N_n(s)} \xi_j^n \right| \le \sup_{s\in[0,1]} \sum_{j=1}^{N_n(s)} \left|\xi_j^n\right| = \sum_{j=1}^{n} \left|\xi_j^n\right| \xrightarrow{p} 0$$

to conclude that condition (14) is enough to guarantee that ξ_j^n is AN.

Remark 2. *Throughout the paper, we will use implicitly this simple fact. If $g(s)$ is a Riemann-integrable function on $[0,1]$ therefore*

$$\sup_{t\in[0,1]} \int_0^t |g(s)|\, ds = \int_0^1 |g(s)|\, ds,$$

whence for any sequence of function $g_n(s)$, uniform convergence on $[0,1]$ of the integral of $|g_n(s)|$ is equivalent to the convergence of $\int_0^1 |g_n(s)|\, ds$.

Finally, we remind the following two lemmas which give us a simple criterion to conclude that a triangular array is AN and are used repeatidly in the rest of the appendix. The first one is Lemma 4.1 of Jacod (2012) whereas the second is Lemma B.8 in of Aït-Sahalia and Jacod (2014).

Lemma 1. *Let ξ_j^n be a triangular array of $\mathcal{F}_{t_{j,n}}$-measurable random variables. If the following condition is satisfied*

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}\left[\left|\xi_j^n\right|\right] \xrightarrow{p} 0,$$

then $\sum_{j=1}^{n} \xi_j^n \xrightarrow{u.c.p} 0$, i.e. ξ_j^n is AN. Moreover, the same conclusion holds under the following two conditions

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}\left[\xi_j^n\right] \xrightarrow{u.c.p} 0, \tag{15}$$

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] \xrightarrow{p} 0. \tag{16}$$

As a consequence if $\mathbb{E}_{j-1}\left[\xi_j^n\right]=0$ then condition (16) is sufficient to guarantee that $\sum_{j=1}^{n} \xi_j^n \xrightarrow{u.c.p} 0$.

Lemma 2. *If $m_n, \ell_n \geq 1$ are arbitrary integers, and if for all $n \geq 1$ and $1 \leq i \leq m_n$ the variable ξ_j^n is $\mathcal{F}_{t_{j+\ell,n}}$-measurable, and if*

$$\sum_{j=1}^{m_n} \left|\mathbb{E}_{j-1}\left[\xi_j^n\right]\right| \xrightarrow{p} 0, \quad \ell_n \sum_{j=1}^{m_n} \mathbb{E}\left[\left|\xi_j^n\right|^2\right] \to 0,$$

then

$$\sup_{i \leq m_n} \left|\sum_{j=1}^{i} \xi_j^n\right| \xrightarrow{p} 0,$$

i.e. $\sum_{j=1}^{n} \xi_j^n \xrightarrow{u.c.p} 0$.

We now turn to characterising the *stable convergence* of triangular arrays (cfr. Podolskij and Vetter, 2010, Definition 1). For a sequence of random variables Y_n (representing the sequence of partial sums of a triangular array), the stable convergence is defined as follows:

Definition 1. *A sequence of random variables Y_n defined on $(\Omega, \mathcal{F}, \mathcal{P})$ is said to converge stably with limit Y defined on an extension of the original probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ if and only if for any bounded continuous function g and any bounded $\mathcal{F}$-measurable random variable Z it holds that*

$$\mathbb{E}\left[g(Y_n)Z\right] \to \mathbb{E}\left[g(Y)Z\right].$$

The classical stable Central Limit Theorem of Hall and Heyde (1980) is not valid for the triangular arrays considered in our paper. Indeed, by construction, we have that $\mathcal{F}_{t_{j,m}} \nsubseteq \mathcal{F}_{t_{j,n}}$ whenever $n > m$. As a consequence, the nesting assumption on the filtrations as in Theorem 3.2 of Hall and Heyde (1980) fails. However, a similar stable Central Limit Theorem hold.

Theorem A.1. *For any given integer ℓ consider the triangular array random variables*

$$\gamma_{j,n}^{(\ell)} = \varphi\left(\mathbb{B}_{j-\ell,n}, \ldots, \mathbb{B}_{j,n}, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right], \ldots, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+\ell,n}\right]\right)$$

where $\varphi : \mathbb{R}^{2\,\ell+1} \to \mathbb{R}$ is a locally bounded function of a finite number of variables. Define the centred triangular array $X_{j,n}^{(\ell)}$ as

$$X_{j,n}^{(\ell)} = \frac{1}{\sqrt{n}}\left(\gamma_{j,n}^{(\ell)} - \mathbb{E}_{j-1}\left[\gamma_{j,n}^{(\ell)}\right]\right)$$

and assume that

$$\sum_{j=1}^{n}\left(X_{j,n}^{(\ell)}\right)^2 \xrightarrow{p} \sigma^2, \tag{17}$$

for an a.s. finite random variable σ. Then, as $n \to \infty$,

$$\sum_{j=1}^{n} X_{j,n}^{(\ell)} \overset{stably}{\Longrightarrow} Z, \tag{18}$$

where Z is a random variable with characteristic function $\mathbb{E}\left[e^{-\frac{1}{2}\sigma^2 t^2}\right]$, defined on an extension of the original probability space.

Proof. The technicalities of the proof largely follow results in Hall and Heyde (1980), Lemma 3.1 and Theorem 3.2. Because of the locally boundedness of φ and the distributional assumptions on random variables $\mathbb{B}_{j-\ell,n}, \ldots, \mathbb{B}_{j+\ell,n}$, it is easy to check that $\max_{1\le j\le n}\left|X_{j,n}^{(\ell)}\right| \xrightarrow{p} 0$. Moreover, by hypothesis $\sum_{j=1}^{n}\left(X_{j,n}^{(\ell)}\right)^2 \xrightarrow{p} \sigma^2$ for an a.s. finite random variable σ. As a consequence (cfr. Lemma 3.1 in Hall and Heyde, 1980), to prove the statement above it is sufficient to prove that for all real t the random variable $T_n(t)$ defined as ($\imath = \sqrt{-1}$)

$$T_n(t) \doteq \prod_{j=1}^{n}\left(1 + \imath\, t\, X_{j,n}^{(\ell)}\right)$$

converges to 1 as $n \to \infty$ *weakly in* $\mathbb{L}^1$. By definition, this is equivalent to prove that for all $E \in \mathcal{F}$, $\mathbb{E}\left[T_n(t)\,\mathbb{I}(E)\right] \to \mathbb{P}[E]$, where $\mathbb{I}(E)$ is the indicator function of the event E. For a fixed $2 \le m \le n$, let $E_m \in \mathcal{F}_{t_{m,m}}$. We compute

$$\begin{aligned}
\mathbb{E}\left[T_n(t)\,\mathbb{I}(E_m)\right] &= \mathbb{E}\left[\mathbb{E}\left[T_n(t)\,\mathbb{I}(E_m)\,|\mathcal{F}_{t_{m,m}}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{j=1}^{n}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\mathbb{I}(E_m)\,\middle|\mathcal{F}_{t_{m,m}}\right]\right]\\
&= \mathbb{E}\left[\prod_{j\in \mathrm{I}_1}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\mathbb{I}(E_m)\,\mathbb{E}\left[\prod_{j\in \mathrm{I}_2\cup \mathrm{I}_3}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{t_{m,m}}\right]\right]\\
&= \mathbb{E}\left[\prod_{j\in \mathrm{I}_1}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\mathbb{I}(E_m)\,\mathbb{E}\left[\mathbb{E}\left[\prod_{j\in \mathrm{I}_2\cup \mathrm{I}_3}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{\infty}^{(p)}\right]\middle|\mathcal{F}_{t_{m,m}}\right]\right]\\
&= \mathbb{E}\left[\prod_{j\in \mathrm{I}_1}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\mathbb{I}(E_m)\,\mathbb{E}\left[\prod_{j\in \mathrm{I}_2}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{t_{m,m}}\right]\mathbb{E}\left[\prod_{j\in \mathrm{I}_3}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{\infty}^{(p)}\right]\right],
\end{aligned}\tag{19}$$

where I_1, I_2, I_3 are three sets of indexes such that $X_{j,n}^{(\ell)} \in \mathcal{F}_{t_{m,m}}$ for $j \in \mathrm{I}_1$, $X_{j,n}^{(\ell)} \in \mathcal{F}_{t_{m+\ell,m+\ell}}$ for $j \in \mathrm{I}_2$, and $X_{j,n}^{(\ell)} \in \left(\mathcal{F}_{t_{n,n}} \setminus \mathcal{F}_{t_{m+l,m+l}}\right)$ for $j \in \mathrm{I}_3$. In particular, $\left(\mathcal{F}_{t_{n,n}} \setminus \mathcal{F}_{t_{m+l,m+l}}\right)$ denotes the smallest σ-algebra containing all the events of $\mathcal{F}_{t_n,t_n}$ that are not included in $\mathcal{F}_{t_{m+\ell},t_{m+\ell}}$. First, we note that I_1 and I_2 includes at most a finite number of terms and that

$$\mathbb{E}\left[\prod_{j\in \mathrm{I}_3}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{\infty}^{(p)}\right] = \prod_{j\in \mathrm{I}_3}\mathbb{E}\left[\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\middle|\mathcal{F}_{\infty}^{(p)}\right] = 1,$$

because of the independence of the factors conditionally on $\mathcal{F}_{\infty}^{(p)}$ and the fact that, for each $j \in \{1, \ldots, n\}$, $X_{j,n}^{(l)}$ has expected value equal to one. Eq.(19) then becomes

$$\mathbb{E}\left[T_n(t)\,\mathbb{I}(E_m)\right] = \mathbb{E}\left[\mathbb{I}(E_m)\prod_{j\in \mathrm{I}_1\cup \mathrm{I}_2}\left(1+\imath\, t\, X_{j,n}^{(\ell)}\right)\right] = \mathbb{P}[E_m] + R_n$$

where the remainder term R_n consists of at most $2^{2|\mathrm{I}_1 \cup \mathrm{I}_2|} - 1$ terms of the form $\mathbb{E}\left[\mathbb{I}\left(E_m\right)(it)^r X_{j_1,n}^{(\ell)} \dots X_{j_r,n}^{(\ell)}\right]$, with $1 \le r \le |\mathrm{I}_1 \cup \mathrm{I}_2|$ and $j_1, \dots j_r \in \mathrm{I}_1 \cup \mathrm{I}_2$. Note that R_n converges to zero as $n \to \infty$. Consequently,

$$\mathbb{E}\left[T_n(t)\, \mathbb{I}\left(E_m\right)\right] \xrightarrow{p} \mathbb{P}\left[E_m\right].$$

Finally, let $\triangle$ denotes the symmetric difference. For any $E \in \mathcal{F}$ and any $\varepsilon > 0$ there exists an an m and an $E_m \in \mathcal{F}_{t_{m,m}}$, such that $\mathbb{P}\left[E \triangle E_m\right] \le \varepsilon$. Since T_n is uniformly integrable by assumption,

$$\left|\mathbb{E}\left[T_n(t)\, \mathbb{I}\left(E_m\right)\right] - \mathbb{E}\left[T_n(t)\, \mathbb{I}(E)\right]\right| \le \mathbb{E}\left[\left|T_n(t)\right| \mathbb{I}\left(E \triangle E_m\right)\right],$$

and $\sup_n \left|\mathbb{E}\left[T_n(t)\, \mathbb{I}\left(E_m\right)\right] - \mathbb{E}\left[T_n(t)\, \mathbb{I}(E)\right]\right|$ can be made arbitrarily small by choosing sufficiently small ε. Whence the thesis. □

We conclude this section with the following corollary, which will be used in the subsequent sections.

Corollary 3. *Let $\mathbf{X}_{j,n}^{(\ell)}$ a q-dimensional random vector with each component defined as $X_{j,n}^{(\ell)}$ in Theorem A.1, such that*

$$\sum_{j=1}^{n} \mathbf{X}_{j,n}^{(\ell)} \left(\mathbf{X}_{j,n}^{(\ell)}\right)' \xrightarrow{p} \Sigma, \tag{20}$$

for an a.s. finite positive definite random matrix $\Sigma = \{\sigma_{i,j}\}$. Then,

$$\sum_{j=1}^{n} \mathbf{X}_{j,n}^{(\ell)} \overset{stably}{\Longrightarrow} \mathcal{MN}(0, \Sigma),$$

where $\mathcal{MN}(0, \Sigma)$ is a q-dimensional mixed-normal random variable.

Proof. The condition (20) implies that

$$\sum_{j=1}^{n} \left(\mathbf{c}' \mathbf{X}_{j,n}^{(\ell)}\right)^2 \xrightarrow{p} \mathbf{c}' \Sigma \mathbf{c}.$$

for an arbitrary real valued vector $\mathbf{c} = (c_1, ..., c_q)'$. Consequently, by Theorem A.1, we have:

$$\sum_{j=1}^{n} \mathbf{c}' \mathbf{X}_{j,n}^{(\ell)} \overset{stably}{\Longrightarrow} \mathcal{MN}(0, \mathbf{c}' \Sigma \mathbf{c}),$$

where $\mathcal{MN}(0, \mathbf{c}' \Sigma \mathbf{c})$ denotes a mixed-normal random variable. Since $\mathbf{c}$ is arbitrary, the later convergence implies the statement of the Corollary. □

Remark 3. *The statement of Theorem A.1 remains true if the condition (17) is replaced by the analogous condition for conditional variances*

$$\sum_{j=1}^{n} \mathbb{E}\left[\left(X_{j,n}^{(\ell)}\right)^2 \,\middle|\, \mathcal{F}_{t_{j,n}}\right] \xrightarrow{p} \sigma^2.$$

A.2 Proofs of limit theorems from Sections 3.1, 4.1 and 4.2

The proofs of the limiting results from Sections 3.1, 4.1 and 4.2 follows directly from severals auxillary Lemmas on the limiting behaviour of triangular arrays of Bernoulli random variables presented below. In particular, Theorem 3.1 is a combination of Lemmas 8 and 6; Theorem 3.2 is a combination of Lemmas 4 and 6; Theorem 3.3 is a combination of Lemmas 3 and 7; Corollary 1 follows directly from Lemma 6; Corollary 2 follows directly from Lemmas 7 and 9.

We start with a remark about Assumption 2, which is repeatedly used in the subsequent proofs.

Remark 4. *Under Assumption 2,*

$$\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right] = p_{j-1,n} + O_p\left(\Delta_n^{1/2}\right). \tag{21}$$

Indeed,

$$\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{B}_{j,n} \,\middle|\, \mathcal{F}_{t_{j-1,n}} \vee \mathcal{F}^{(p)}_{t_{j,n}}\right]\right] = \mathbb{E}_{j-1}\left[p_{j,n}\right] = p_{j-1,n} + \mathbb{E}_{j-1}\left[p_{j,n} - p_{j-1,n}\right], \tag{22}$$

where

$$\left|\mathbb{E}_{j-1}\left[p_{j,n} - p_{j-1,n}\right]\right| \leq \mathbb{E}_{j-1}\left[\left|p_{j,n} - p_{j-1,n}\right|\right] \leq C\left(\Delta_n\right)^{1/2},$$

where the last inequality follows from standard estimates for semimartingales (Jacod, 2008). Moreover, by Proposition 1 of Barndorff-Nielsen et al. (2006),

$$\left|p_{j,n} - p_{j-1,n}\right| = O_p\left(\left(\Delta_n\left|\log \Delta_n\right|\right)^{1/2}\right),$$

which implies that, for every finite integer k,

$$p_{j+k} = p_{j-1} + O_p\left(k\left(\Delta_n\left|\log \Delta_n\right|\right)^{1/2}\right). \tag{23}$$

Lemma 3. *Under Assumption 2, as $n \to \infty$,*

$$\frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+k,n} \xrightarrow{u.c.p} \int_0^1 \left(p_s\right)^2 ds.$$

Proof. To prove the result above, we apply Lemma 1. The key assumption of this lemma is that the random variables defining the triangular array ξ_j^n must be $\mathcal{F}_{t_{j,n}}$-measurable. Thus, we make the following steps.

$$\begin{aligned}
\frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+k,n} &= \frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\left(\mathbb{B}_{j+k,n} - \mathbb{E}_j\left[\mathbb{B}_{j+k,n}\right]\right) + \sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{E}_j\left[\mathbb{B}_{j+k,n}\right] \\
&= \frac{1}{n-k}\sum_{j=1+k}^{n} \mathbb{B}_{j-k,n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-k}\left[\mathbb{B}_{j,n}\right]\right) + \frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{E}_j\left[\mathbb{B}_{j+k,n}\right] \\
&= \frac{1}{n-k}\sum_{j=1}^{n-1} \mathbb{B}_{j-k,n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-k}\left[\mathbb{B}_{j,n}\right]\right) + \frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{E}_j\left[\mathbb{B}_{j+k,n}\right] \\
&+ \frac{1}{n-k}\mathbb{B}_{n-k,n}\left(\mathbb{B}_{n,n} - \mathbb{E}_{n-k}\left[\mathbb{B}_{n,n}\right]\right) - \frac{1}{n-k}\sum_{j=1}^{k} \mathbb{B}_{j-k,n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-k}\left[\mathbb{B}_{j,n}\right]\right).
\end{aligned}$$

Because of the boundedness of the Bernoulli variates, for any k fixed, the last two terms are both $o_p(1)$. Thus, by setting $\xi_j^n \doteq \mathbb{B}_{j-k,n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-k}\left[\mathbb{B}_{j,n}\right]\right) + \mathbb{B}_{j,n}\mathbb{E}_j\left[\mathbb{B}_{j+k,n}\right]$, we write

$$\frac{1}{n-k}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\,\mathbb{B}_{j+k,n} = \frac{1}{n-k}\sum_{j=1}^{n-k} \xi_j^n + o_p(1).$$

Note that each ξ_j^n is now $\mathcal{F}_{t_{j,n}}$-measurable. Set now $\zeta_j^n \doteq (n-k)^{-1}\left(\xi_j^n - \mathbb{E}_{j-1}\left[\xi_j^n\right]\right)$. We show that

$$\sum_{j=1}^{n-k} \zeta_j^n \xrightarrow{u.c.p} 0.$$

Since ζ_j^n is a martingale difference, it is enough to prove that (cfr. condition (16))

$$\mathbb{E}\left[\sum_{j=1}^{n-k}|\zeta_j^n|^2\right]\to 0.$$

The previous result follows immediately from the boundedness of the Bernoulli variates. Indeed,

$$\mathbb{E}\left[\sum_{j=1}^{n-k}|\zeta_{j,n}|^2\right]=\mathbb{E}\left[\sum_{j=1}^{n-k}\left|\frac{1}{n-k}\left(\xi_j^n-\mathbb{E}_{j-1}\left[\xi_j^n\right]\right)\right|^2\right]=\frac{1}{(n-k)^2}\sum_{j=1}^{n-k}\mathbb{E}\left[\left|\xi_j^n-\mathbb{E}_{j-1}\left[\xi_j^n\right]\right|^2\right]\leq\frac{\mathrm{C}}{(n-k)}\to 0.$$

We have proved that

$$\frac{1}{n-k}\sum_{j=1}^{n-k}\xi_j^n\xrightarrow{u.c.p}\frac{1}{n-k}\sum_{j=1}^{n-k}\mathbb{E}_{j-1}\left[\xi_j^n\right],$$

thus, to conclude it is sufficient to prove that

$$\sum_{j=1}^{n-k}\frac{1}{n-k}\left(\mathbb{E}_{j-1}\left[\xi_j^n\right]-(p_{j-1,n})^2\right)\xrightarrow{u.c.p}0.$$

With abuse of notation, let $\zeta_{j-1}^n\doteq(n-k)^{-1}\left(\mathbb{E}_{j-1}\left[\xi_j^n\right]-(p_{j-1,n})^2\right)$ and we show that condition (14) hold for ζ_{j-1}^n. Write [6]

$$\begin{aligned}
\mathbb{E}_{j-1}\left[\xi_j^n\right] &= \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\mathbb{B}_{j+k,n}\right]=\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\,\mathbb{E}_{j+k-1}\left[\mathbb{B}_{j+k,n}\right]\right]\\
\text{(Using equation (21))} &= \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\,p_{j+k-1,n}\right]+O_p\left(\Delta_n^{1/2}\right)\\
\text{(Using equation (23))} &= \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\,p_{j-1,n}\right]+O_p\left(\Delta_n^{1/2}\right)\\
&= p_{j-1,n}\,\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]+O_p\left(\Delta_n^{1/2}\right)\\
&= p_{j-1,n}^2+O_p\left(\Delta_n^{1/2}\right)
\end{aligned}\tag{24}$$

Thus[7]

$$\sum_{j=1}^{n}\mathbb{E}\left[\left|\zeta_{j-1}^n\right|\right]=\sum_{j=1}^{n}\frac{1}{n-k}\,\mathbb{E}\left[\left|\mathbb{E}_{j-1}\left[\xi_j^n\right]-(p_{j-1,n})^2\right|\right]\longrightarrow 0.$$

In particular

$$\frac{1}{n-k}\sum_{j=1}^{n-k}\mathbb{E}_{j-1}\left[\xi_j^n\right]-\frac{1}{n-k}\sum_{j=1}^{n-k}(p_{j-1,n})^2\xrightarrow{u.c.p}0$$

---

[6]Note that the infinitesimal (in probability) term $O_p\left(\Delta_n^{1/2}\right)$ that appears in equation (21) of Remark **??** is exactly $\mathbb{E}_{j-1}\left[p_j-p_{j-1}\right]$, so that the $O_p\left(\Delta_n^{1/2}\right)$ that appears in the second of the equations (24) is $\mathbb{E}_{j+k-1}\left[p_{j+k}-p_{j+k-1}\right]$. Nevertheless, again using the tower rule and the Markov inequality we can prove that

$$\mathbb{E}_{j-1}\left[p_{j+k}-p_{j+k-1}\right]=O_p\left(\Delta_n^{1/2}\right),$$

whence the second and the third of the equations in (24).

[7]We note here that the absolute value of the difference $\left|\mathbb{E}_{j-1}\left[\xi_j^n\right]-\left(p_{j-1,n}^F\right)^2\right|$ is a term of the type $\left|\mathbb{E}_{j-1}\left[p_{j+k}-p_{j+k-1}\right]\right|$ for some k. Hence the summation in equation (24) is bounded by a constant times $\Delta_n^{1/2}$, which converges to zero.

Finally, by Riemann integrability we have, path-wise on Ω,

$$\frac{1}{n-k}\sum_{j=1}^{n-k}\left(p_{j-1,n}\right)^2 \to \int_0^1 \left(p_s\right)^2\, ds,$$

whence the thesis. $\square$

Lemma 4. *Under Assumption 2, as $n\to\infty$,*

$$\frac{1}{n}\sum_{j=1}^{n}\prod_{i=0}^{m-1}\mathbb{B}_{i+j,n} \overset{u.c.p}{\longrightarrow} \int_0^1 \left(p_s\right)^m ds$$

Proof. Consider the following quantity

$$A_n \;=\; \frac{1}{n}\sum_{j=1}^{n}\prod_{i=0}^{m-1}\mathbb{B}_{i+j,n}-\frac{1}{n}\sum_{j=1}^{n}\left(p_{j-1,n}\right)^m=\frac{1}{n}\sum_{j=1}^{n}\left[\mathbb{B}_{j,n}\mathbb{B}_{j+1,n}\cdots\mathbb{B}_{j+(m-1),n}-\left(p_{j-1,n}\right)^m\right]$$

We show that $A_n \overset{u.c.p}{\longrightarrow} 0$. To do so, we rewrite the quantity A_n as a sum of a $\mathcal{F}_{t_{j,n}}$-measurable quantity and a negligible term. We introduce the following quantity

$$\varsigma_{j,\ell}^{(m)}=\mathbb{B}_{j,n}\mathbb{B}_{j+1,n}\cdots\mathbb{B}_{j+\ell-1,n}\left(\mathbb{B}_{j+\ell,n}-p_{j-1,n}\right)\left(p_{j-1,n}\right)^{m-\ell-1}$$

and we show that A_n can be rewritten in the following way

$$A_n=\frac{1}{n}\sum_{j=m}^{n}\sum_{\ell=0}^{m-1}\varsigma_{j-\ell,\ell}^{(m)}+\frac{\mathcal{R}_n}{n} \tag{25}$$

where $\mathcal{R}_n/n$ is asymptotically negligible. Let us consider the following expressions

$$\begin{aligned}
\psi_{j,1} &= \mathbb{B}_{j,n}-p_{j-1,n}\doteq\varsigma_{j,0}^{(1)}\\
\psi_{j,2} &= \mathbb{B}_{j,n}\mathbb{B}_{j+1,n}-p_{j-1,n}^2=\mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n}-p_{j-1,n}\right)+\left(\mathbb{B}_{j,n}-p_{j-1,n}\right)p_{j-1,n}\doteq\varsigma_{j,1}^{(2)}+\varsigma_{j,0}^{(2)}\\
\psi_{j,3} &= \mathbb{B}_{j,n}\mathbb{B}_{j+1,n}\left(\mathbb{B}_{j+2,n}-p_{j-1,n}\right)+\mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n}-p_{j-1,n}\right)p_{j-1,n}+\left(\mathbb{B}_{j,n}-p_{j-1,n}\right)p_{j-1,n}^2\doteq\varsigma_{j,2}^{(3)}+\varsigma_{j,1}^{(3)}+\varsigma_{j,0}^{(3)},
\end{aligned}$$

and similarly for each fixed m. Then $A_n=n^{-1}\sum_{j=1}^{n}\psi_{j,m}$ becomes

$$A_n=\frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=0}^{m-1}\varsigma_{j,\ell}^{(m)}=\frac{1}{n}\sum_{j=m}^{n}\sum_{\ell=0}^{m-1}\varsigma_{j,\ell}^{(m)}+\frac{1}{n}\sum_{j=1}^{m-1}\sum_{\ell=0}^{m-1}\varsigma_{j,\ell}^{(m)}=\frac{1}{n}\sum_{j=m}^{n}\sum_{\ell=0}^{m-1}\varsigma_{j-\ell,\ell}^{(m)}+\frac{1}{n}\underbrace{\sum_{j=m}^{n}\sum_{\ell=0}^{m-1}\left(\varsigma_{j,\ell}^{(m)}-\varsigma_{j-\ell,\ell}^{(m)}\right)}_{\mathcal{R}_1}+\frac{1}{n}\underbrace{\sum_{j=1}^{m-1}\sum_{\ell=0}^{m-1}\varsigma_{j,\ell}^{(m)}}_{\mathcal{R}_2}.$$

We show now that both $\mathcal{R}_1/n$ and $\mathcal{R}_2/n$ are $o_p(1)$. Since m is fixed, by the boundedness of the Bernoulli variables we have $\mathcal{R}_2/n=o_p(1)$. Now, considering that all the terms with $\ell=0$ in $\mathcal{R}_{1,n}$ are identically zero, we get

$$\begin{aligned}
\mathcal{R}_1 &= \sum_{\ell=1}^{m-1}\sum_{j=m}^{n}\left(\varsigma_{j,\ell}^{(m)}-\varsigma_{j-\ell,\ell}^{(m)}\right)=\sum_{\ell=1}^{m-1}\left(\sum_{j=m}^{n}\varsigma_{j,\ell}^{(m)}-\sum_{j=m}^{n}\varsigma_{j-\ell,\ell}^{(m)}\right)=\sum_{\ell=1}^{m-1}\left(\sum_{j=m}^{n}\varsigma_{j,\ell}^{(m)}-\sum_{j=m-\ell}^{n-\ell}\varsigma_{j,\ell}^{(m)}\right)\\
&= \sum_{\ell=1}^{m-1}\left(\underbrace{\sum_{j=n-\ell+1}^{n}\varsigma_{j,\ell}^{(m)}}_{\ell\text{ addends}}-\underbrace{\sum_{j=m-\ell}^{m-1}\varsigma_{j,\ell}^{(m)}}_{\ell\text{addends}}\right),
\end{aligned}$$

hence, as for $\mathcal{R}_2$, for given m the number of addends in $\mathcal{R}_1$ is independent of n (and bounded) so that $\mathcal{R}_1/n = o_p(1)$. Thus, by setting $\mathcal{R}_n \doteq \mathcal{R}_1 + \mathcal{R}_2$ the decomposition in (25) hold, i.e.

$$A_n = \frac{1}{n}\sum_{j=m}^{n}\sum_{\ell=0}^{m-1}\varsigma_{j-\ell,\ell}^{(m)} + o_p(1)$$

To conclude, we have to show that A_n is AN. Before proceeding, for sake of clarity, we briefly describe how we achieve this result. Let us set $\zeta_j^n = \frac{1}{n}\varsigma_{j-\ell,\ell}^{(m)}$, for fixed ℓ and m. We note that to prove the asymptotically negligibility of A_n it is sufficient to prove that ζ_j^n is AN. By Lemma 1 this amounts to show that the following two conditions are satisfied

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}\left[\zeta_j^n\right] = \sum_{j=1}^{n}\frac{1}{n}\mathbb{E}_{j-1}\left[\varsigma_{j-\ell,\ell}^{(m)}\right] \xrightarrow{u.c.p} 0 \tag{26}$$

and

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}\left[\left(\zeta_j^n\right)^2\right] \xrightarrow{p} 0. \tag{27}$$

In particular, to prove Eq.(26) we set $\xi_j^n = n^{-1}\mathbb{E}_{j-1}\left[\varsigma_{j-\ell,\ell}^{(m)}\right]$ and, by using again Lemma 1, we show that

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}\left[\left|\xi_j^n\right|\right] \xrightarrow{p} 0. \tag{28}$$

Thus, we start from assertion in (28) and we prove

$$\begin{aligned}
\sum_{j=1}^{n}\mathbb{E}_{j-1}\left[\left|\xi_j^n\right|\right] &= \sum_{j=1}^{n}\mathbb{E}_{j-1}\left[\left|\frac{1}{n}\mathbb{E}_{j-1}\left[\varsigma_{j-\ell,\ell}^{(m)}\right]\right|\right] = \sum_{j=1}^{n}\frac{1}{n}\left|\mathbb{E}_{j-1}\left[\varsigma_{j-\ell,\ell}^{(m)}\right]\right| \\
&= \sum_{j=1}^{n}\frac{1}{n}\left|\mathbb{E}_{j-1}\left[\mathbb{B}_{j-\ell,n}\cdot\ldots\cdot\mathbb{B}_{j-1,n}\left(p_{j-\ell-1,n}\right)^{m-\ell-1}\left(\mathbb{B}_{j,n}-p_{j-\ell-1,n}\right)\right]\right| \\
&= \sum_{j=1}^{n}\frac{1}{n}\left|\mathbb{B}_{j-\ell,n}\cdot\ldots\cdot\mathbb{B}_{j-1,n}\left(p_{j-\ell-1,n}\right)^{m-\ell-1}\mathbb{E}_{j-1}\left[\left(\mathbb{B}_{j,n}-p_{j-\ell-1,n}\right)\right]\right| \\
&= \sum_{j=1}^{n}\frac{1}{n}\left|\mathbb{B}_{j-\ell,n}\cdot\ldots\cdot\mathbb{B}_{j-1,n}\left(p_{j-1,n}\right)^{m-\ell-1}\mathbb{E}_{j-1}\left[\left(p_{j,n}-p_{j-l-1,n}\right)\right]\right| \\
&\le \sum_{j=1}^{n}\frac{1}{n}\mathbb{E}_{j-1}\left[\left|p_{j,n}-p_{j-\ell-1,n}\right|\right] \le \sum_{j=1}^{n}\frac{1}{n}C\Delta_n^{1/2} \le C\Delta_n^{1/2},
\end{aligned}$$

At this point, it is enough to prove the convergence in Eq.(27). This is an easy check because of the boundedness of the Bernoulli variates, i.e.

$$\sum_{i=1}^{n}\mathbb{E}_{j-1}\left[\left(\zeta_j^n\right)^2\right] = \frac{1}{n^2}\mathbb{E}_{j-1}\left[\left(\varsigma_{j-\ell,\ell}^{(m)}\right)^2\right] \le K\Delta_n \longrightarrow 0,$$

which implies the asymptotic negligibility of A_n. Finally, by Riemann integrability,

$$\frac{1}{n}\sum_{j=1}^{n}\left(p_{j-1,n}\right)^m \longrightarrow \int_0^1 \left(p_s\right)^m ds,$$

which completes the proof. $\square$

Before proceeding, we state and prove another useful lemma.

Lemma 5. *Under Assumption 2, for any finite numbers $\ell, d \geq 0$ and powers $q_1, \ldots, q_d \geq 0$, as $n \to \infty$,*

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n} \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\right)^{q_1} \cdots \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+d,n}\right]\right)^{q_d} \xrightarrow{p} \int_0^1 p_s^{\ell+v} ds,$$

where $v = q_1 + \ldots + q_d$.

Proof. First, by Remark (**??**),

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n} \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\right)^{q_1} \cdots \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+d,n}\right]\right)^{q_d} = \frac{1}{n}\sum_{j=1}^{n} \mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n}\, p_{j-1,n}^{v} + O_p\left(\Delta^{1/2}\right),$$

Next, by conditioning on $\mathcal{F}_{\infty}^{(p)}$ and using the law of iterated expectations,

$$\mathbb{E}\left[\mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n}\, p_{j-1,n}^{v} - p_{j-\ell,n} \cdots p_{j,n}\, p_{j-1,n}^{v}\right] = 0.$$

Hence, by Theorem 2.13 in Hall and Heyde (1980)[8] applied to the martingale difference $X_{j,n}^{(\ell)} = \mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n} p_{j-1,n}^{v} - p_{j-\ell,n} \cdots p_{j,n} p_{j-1,n}^{v}$

$$\frac{1}{n}\sum_{j=1}^{n} \left(\mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j,n}\, p_{j-1,n}^{v} - p_{j-\ell,n} \cdots p_{j,n}\, p_{j-1,n}^{v}\right) \xrightarrow{p} 0.$$

Using Remark (**??**) again,

$$\frac{1}{n}\sum_{j=1}^{n} p_{j-\ell,n} \cdots p_{j,n}\, p_{j-1,n}^{v} = \frac{1}{n}\sum_{j=1}^{n} p_{j-1,n}^{\ell+v} + O_p\left(\Delta^{1/2}\right).$$

Finally, by Riemann integrability we have, path-wise on Ω,

$$\frac{1}{n}\sum_{j=1}^{n} p_{j-1,n}^{\ell+v} \longrightarrow \int_0^1 p_s^{\ell+v} ds,$$

which completes the proof. □

Lemma 6. *Let $m \geq 2$ be a given integer number. Under Assumption 2, as $n \to \infty$,*

$$\sqrt{n}\begin{bmatrix} \mathrm{IT}_n - \int_0^1 p_s\, ds \\ \mathrm{MIT}_n^{(m)} - \int_0^1 (p_s)^m\, ds \end{bmatrix} \overset{stably}{\Longrightarrow} \mathcal{MN}\left(0, \Sigma\right) \tag{29}$$

where

$$\mathrm{IT}_n = \frac{1}{n}\sum_{j=1}^{n} \mathbb{B}_{j,n} \qquad \mathrm{MIT}_n^{(m)} = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=0}^{m-1} \mathbb{B}_{j+i,n}$$

and $\mathcal{MN}(0,\Sigma)$ denotes the mixed-normal distribution with covariance matrix Σ

$$\Sigma = \begin{bmatrix} \int_0^1 p_s\,(1-p_s)\, ds & \int_0^1 m\, p_s^m\,(1-p_s)\, ds \\ \int_0^1 m\, p_s^m\,(1-p_s)\, ds & \int_0^1 p_s^m\, \frac{p_s^m\,(2\,m+1) - p_s^{m+1}\,(2\,m-1) - (1+p_s)}{1-p_s}\, ds \end{bmatrix}.$$

[8] The hypothesis of the Theorem are readily satisfied because of the boundedness of the Bernoulli random variables with $\mathbb{B}_n = n$.

Proof. We consider the following decomposition

$$\sqrt{n}\begin{bmatrix} \mathrm{IT}_n - \int_0^1 p_s\, ds \\ \mathrm{MIT}_n^{(m)} - \int_0^1 (p_s)^m\ ds \end{bmatrix} = A_1 + A_2,$$

where

$$A_1 = \frac{1}{\sqrt{n}}\sum_{j=1}^n \begin{bmatrix} \mathbb{B}_{j,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}] \\ \prod_{i=0}^{m-1}\mathbb{B}_{j+i,n} - \prod_{i=0}^{m-1}\mathbb{E}_{j+i-1}[\mathbb{B}_{j+i,n}] \end{bmatrix}, \quad A_2 = \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{bmatrix} \mathbb{E}_{j-1}[\mathbb{B}_{j,n}] - \int_0^1 p_s\, ds \\ \prod_{i=0}^{m-1}\mathbb{E}_{j+i-1}[\mathbb{B}_{j+i,n}] - \int_0^1 (p_s)^m\, ds \end{bmatrix}.$$

A_2 is asymptotically negligible (which can be proven as in Lemma **??** above). Thus, it is enough to prove that $A_1 \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma)$. To do so, we rewrite the quantity A_1 as a sum of a $\mathcal{F}_{t_{j,n}}$-measurable quantity and a negligible term. We introduce the following quantity

$$\zeta_{j,\ell}^{(m)} = \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\cdots\mathbb{B}_{j+\ell-1,n}\left(\mathbb{B}_{j+\ell,n} - \mathbb{E}_{j+\ell-1}[\mathbb{B}_{j+\ell,n}]\right)\mathbb{E}_{j+\ell}[\mathbb{B}_{j+\ell+1,n}]\cdots\mathbb{E}_{j+m-2}[\mathbb{B}_{j+m-1,n}]$$

and we consider the following expression

$$\varphi_{j,m} = \prod_{i=0}^{m-1}\mathbb{B}_{j+i,n} - \prod_{i=0}^{m-1}\mathbb{E}_{j+i-1}[\mathbb{B}_{j+i,n}]$$

for a generic m. Note that $\varphi_{j,m} = \sum_{\ell=0}^{m-1}\zeta_{j,\ell}^{(m)}$. Indeed

$$\begin{aligned}
\varphi_{j,1} &= \mathbb{B}_{j,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}] \equiv \zeta_{j,0}^{(1)} \\
\varphi_{j,2} &= \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_j[\mathbb{B}_{j+1,n}] \\
&= \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n} - \mathbb{B}_{j,n}\,\mathbb{E}_j[\mathbb{B}_{j+1,n}] + \mathbb{B}_{j,n}\,\mathbb{E}_j[\mathbb{B}_{j+1,n}] - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_j[\mathbb{B}_{j+1,n}] \\
&= \mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n} - \mathbb{E}_j[\mathbb{B}_{j+1,n}]\right) + \mathbb{B}_{j,n}\,\mathbb{E}_j[\mathbb{B}_{j+1,n}] - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_i[\mathbb{B}_{j+1,n}] \\
&= \underbrace{\mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n} - \mathbb{E}_j[\mathbb{B}_{j+1,n}]\right)}_{\zeta_{j,1}^{(2)}} + \underbrace{\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\right)\mathbb{E}_j[\mathbb{B}_{j+1,n}]}_{\zeta_{j,0}^{(2)}} \\
\varphi_{j,3} &= \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\,\mathbb{B}_{j+2,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_j[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] \\
&= \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\,\mathbb{B}_{j+2,n} - \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\,\mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] + \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\,\mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] \\
&- \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_j[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] \\
&= \mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\left(\mathbb{B}_{j+2,n} - \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}]\right) + \\
&\quad +\mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\,\mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] - \mathbb{B}_{j,n}\,\mathbb{E}_i[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] + \\
&\quad +\mathbb{B}_{j,n}\,\mathbb{E}_j[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\ \mathbb{E}_i[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}] \\
&= \underbrace{\mathbb{B}_{j,n}\,\mathbb{B}_{j+1,n}\left(\mathbb{B}_{j+2,n} - \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}]\right)}_{\zeta_{i,2}^{(3)}} + \underbrace{\mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n} - \mathbb{E}_j[\mathbb{B}_{j+1,n}]\right)\mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}]}_{\zeta_{j,1}^{(3)}} + \\
&\quad + \underbrace{\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}[\mathbb{B}_{j,n}]\right)\mathbb{E}_j[\mathbb{B}_{j+1,n}]\ \mathbb{E}_{j+1}[\mathbb{B}_{j+2,n}]}_{\zeta_{j,0}^{(3)}},
\end{aligned}$$

and so on and so forth for every m. So the second component of A_1, $A_1(2) = n^{-1/2}\sum_{j=1}^n \varphi_{j,m}$, can be rewritten as

$$
\begin{aligned}
A_1(2) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{\ell=0}^{m-1} \zeta_{j,\ell}^{(m)} &= \frac{1}{\sqrt{n}} \sum_{j=m}^{n} \sum_{\ell=0}^{m-1} \zeta_{j,\ell}^{(m)} + \sum_{j=1}^{m-1} \sum_{\ell=0}^{m-1} \zeta_{j,\ell}^{(m)} \\
&= \frac{1}{\sqrt{n}} \sum_{j=m}^{n} \sum_{\ell=0}^{m-1} \zeta_{j-\ell,\ell}^{(m)} + \frac{1}{\sqrt{n}} \underbrace{\sum_{j=m}^{n} \sum_{\ell=0}^{m-1} \left(\zeta_{j,\ell}^{(m)} - \zeta_{j-\ell,\ell}^{(m)} \right)}_{\mathcal{R}_1} + \frac{1}{\sqrt{n}} \underbrace{\sum_{j=1}^{m-1} \sum_{\ell=0}^{m-1} \zeta_{j,\ell}^{(m)}}_{\mathcal{R}_2}
\end{aligned}
$$

Reasoning as in Lemma **??** one can prove that both $\mathcal{R}_1/\sqrt{n}$ and $\mathcal{R}_2/\sqrt{n}$ are $o_p(1)$. To render $A_1(2)$ $\mathcal{F}_{t_{j,n}}$-measurable a further step is necessary. We define

$$
\widetilde{\zeta}_{j-\ell,\ell}^{(m)} = \mathbb{B}_{j-\ell,n} \, \mathbb{B}_{j-\ell+1,n} \cdots \mathbb{B}_{j-1,n} \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right] \right) \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right],
$$

and consider

$$
\begin{aligned}
\mathcal{R}_3 &= \sum_{j=m}^{n} \sum_{\ell=0}^{m-1} \left(\zeta_{j-\ell,\ell}^{(m)} - \widetilde{\zeta}_{j-\ell,\ell}^{(m)} \right) \\
&= \sum_{j=m}^{n} \sum_{\ell=0}^{m-1} \mathbb{B}_{j-\ell,n} \, \mathbb{B}_{j-\ell+1,n} \cdots \mathbb{B}_{j-1,n} \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right] \right) \times \\
&\times \; \left(\mathbb{E}_{j}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-\ell+m-2}\left[\mathbb{B}_{j-\ell+m-1,n}\right] - \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right] \right)
\end{aligned}
$$

Notice, that, using Remark **??**, for all $i > 0$,

$$
\begin{aligned}
\left| \mathbb{E}_{j+i-1}\left[\mathbb{B}_{j+i,n}\right] - \mathbb{E}_{j-1}\left[\mathbb{B}_{j+i,n}\right] \right| &= \left| p_{j+i-1,n} - p_{j-1,n} + O_p\left(\Delta_n^{1/2}\right) \right| \\
&= \left| p_{j+i-1,n} - p_{j+i-2,n} + p_{j+i-2,n} - p_{j+i-3,n} + \cdots + p_{j,n} - p_{j-1,n} + O_p\left(\Delta_n^{1/2}\right) \right| \\
&\leq O_p\left((i+1)\, \Delta_n^{1/2} \right) \qquad (30)
\end{aligned}
$$

Now note that, using the triangular inequality and a recursive decomposition, for any set of bounded random variables $x_1, \ldots, x_{m-\ell-1}$, $y_1, \ldots, y_{m-\ell-1}$ we obtain (to reduce notation we put $M = m - \ell - 1$)

$$
\begin{aligned}
\left| x_1 \cdots x_M - y_1 \cdots y_M \right| &= \left| x_1 \cdots x_{M-1} \left(x_M - y_M \right) + \left(x_1 \cdots x_{M-1} - y_1 \ldots y_{M-1} \right) y_M \right| \\
&\leq \left| x_1 \cdots x_{M-1} \left(x_M - y_M \right) \right| + \left| \left(x_1 \cdots x_{M-1} - y_1 \cdots y_{M-1} \right) y_M \right| \\
&\leq K \left| \left(x_M - y_M \right) \right| + K \left| \left(x_1 \cdots x_{M-1} - y_1 \cdots y_{M-1} \right) \right| \\
&\leq \cdots \\
&\leq K \sum_{k=1}^{M} \left| x_k - y_k \right|,
\end{aligned}
$$

where the constant K changes from line to line. Applying this inequality to the difference $\left| \zeta_{j-\ell,\ell}^{(m)} - \widetilde{\zeta}_{j-\ell,\ell}^{(m)} \right|$, we obtain:

$$\left|\zeta_{j-\ell,\ell}^{(m)}-\widetilde{\zeta}_{j-\ell,\ell}^{(m)}\right| \leq K \sum_{i=1}^{m-\ell-1}\left|\mathbb{E}_{j-1}\left[\mathbb{B}_{j+i}, n\right]-\mathbb{E}_{j+i-1}\left[\mathbb{B}_{j+i}, n\right]\right|.$$

Consequently, since m is finite, the inequality (30) implies that $\mathcal{R}_3/\sqrt{n} = o_p(1)$. In particular, A_1 can be represented as

$$A_1 = \frac{1}{\sqrt{n}}\sum_{j=m}^{n}\eta_j + \frac{1}{\sqrt{n}}R_n = \frac{1}{\sqrt{n}}\sum_{j=m}^{n}\begin{bmatrix}\eta_j(1)\\ \eta_j(2)\end{bmatrix} + \frac{1}{\sqrt{n}}\begin{bmatrix}R_n(1)\\ R_n(2)\end{bmatrix},$$

with

$$\eta_j(1) \doteq \mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right], \quad \eta_j(2) \doteq \sum_{\ell=0}^{m-1}\widetilde{\zeta}_{j-\ell,\ell}^{(m)},$$

and where the reminders are given by

$$R_n(1) = \sum_{j=1}^{m-1}\left(\mathbb{B}_{j\Delta,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j\Delta,n}\right]\right), \quad R_n(2) = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3.$$

Note that since also the first component of R_n consists of a finite number of bounded terms, $R_n/\sqrt{n}$ is asymptotically negligible: It is enough to establish the following convergence convergence

$$\frac{1}{\sqrt{n}}\sum_{j=m}^{n}\eta_j \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma).$$

To establish the previous convergence, we use Corollary 3. We have to find two functions $\varphi^{(1)}$ and $\varphi^{(2)}$ such that

$$\eta_j(1) = \varphi^{(1)}\left(\mathbb{B}_{j-m+1,n},\ldots,\mathbb{B}_{j,n},\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right],\ldots,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\right) - \mathbb{E}_{j-1}\left[\varphi^{(1)}\left(\mathbb{B}_{j-m+1,n},\ldots,\mathbb{B}_{j,n},\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right],\ldots,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\right)\right]$$

and similarly for $\eta_j(2)$. The case of $\eta_j(1)$ is trivial since it is enough to define $\varphi^{(1)}(x_1) \doteq x_1$ to have the identity $\eta_j(1) = \varphi^{(1)}(\mathbb{B}_j) - \mathbb{E}_{j-1}\left[\varphi^{(1)}(\mathbb{B}_j)\right]$. For what concerns $\eta_j(2)$ note that

$$\begin{aligned}
\eta_j(2) &= \sum_{\ell=0}^{m-1}\widetilde{\zeta}_{j-\ell,\ell}^{(m)} = \sum_{\ell=0}^{m-1}\mathbb{B}_{j-\ell,n}\,\mathbb{B}_{j-\ell+1,n}\cdots\mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right]\\
&= \left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]+\\
&\quad +\mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-2,n}\right]+\ldots\\
&\quad +\mathbb{B}_{j-m+1,n}\,\mathbb{B}_{j-m+2,n}\cdots\mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\\
&= \mathbb{B}_{j,n}\,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]+\mathbb{B}_{j-1,n}\,\mathbb{B}_{j,n}\,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-2,n}\right]+\ldots\\
&\quad +\mathbb{B}_{j-m+1,n}\,\mathbb{B}_{j-m+2,n}\cdots\mathbb{B}_{j-1,n}\,\mathbb{B}_{j,n}-\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\cdots\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\right)+\ldots\\
&\quad +\mathbb{B}_{j-m+1,n}\,\mathbb{B}_{j-m+2,n}\cdots\mathbb{B}_{j-1,n}\,\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\\
&= \varphi^{(2)}\left(\mathbb{B}_{j-m+1,n},\ldots,\mathbb{B}_{j,n},\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right],\ldots,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\right)\\
&- \mathbb{E}_{j-1}\left[\varphi^{(2)}\left(\mathbb{B}_{j-m+1,n},\ldots,\mathbb{B}_{j,n},\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right],\ldots,\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\right)\right]
\end{aligned}$$

where $\varphi^{(2)} : \mathbb{R}^{2\,(m-1)+1} \to \mathbb{R}$ takes the following form

$$\varphi^{(2)}\left(x_1, \cdots, x_m, \cdots, x_{2(m-1)+1}\right) \doteq x_m\, x_{m+1} \cdots x_{2(m-1)+1} + x_{m-1}\, x_m \cdots x_{2(m-1)} + \ldots + x_1\, x_2 \cdots x_m.$$

We now proceed by noticing that, for all j, the vector η_j is $\mathcal{F}_{t_{j,n}}$-measurable and bounded, whence

$$\sum_{j=m}^{n} \mathbb{E}_{j-1}\left[\left\|\frac{1}{\sqrt{n}}\eta_j\right\|^4\right] \xrightarrow{p} 0,$$

and $\mathbb{E}_{j-1}\left[\eta_j\,(1)\right] = 0$. To see that also $\mathbb{E}_{j-1}\left[\eta_j\,(2)\right] = 0$ it is better to write down $\mathbb{E}_{j-1}\left[\eta_j\,(2)\right]$ explicitly

$$\mathbb{E}_{j-1}\left[\eta_j\,(2)\right] = \sum_{\ell=0}^{m-1} \mathbb{E}_{j-1}\left[\widetilde{\zeta}^{(m)}_{j-\ell,\ell}\right]$$

$$= \sum_{\ell=0}^{m-1} \mathbb{B}_{j-\ell,n}\, \mathbb{B}_{j-\ell+1,n} \cdots \mathbb{B}_{j-1,n}\, \underbrace{\mathbb{E}_{j-1}\left[\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\right]}_{=0}\, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+2,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j-\ell+m-1,n}\right].$$

Consequently, it is enough to show that $n^{-1} \sum_{i=m}^{n} \mathbb{E}_{j-1}\left[\eta_j \eta_j'\right] \xrightarrow{p} \Sigma$. Consider each component of the matrix $\eta_j \eta_j'$ separately.

$$\eta_j(1)\eta_j(1) = \mathbb{B}_{j,n} - 2\mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right] + \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2.$$

By Lemma 5,

$$\frac{1}{n}\sum_{i=m}^{n} \mathbb{E}_{i-1}\left[\eta_i(1)\eta_i(1)\right] \xrightarrow{p} \int_0^1 \left(p_s - p_s^2\right)\, ds.$$

Now consider the product

$$\eta_j(2)\,\eta_j(2) = \sum_{\ell=0}^{m-1}\left(\widetilde{\zeta}^{(m)}_{j-\ell,\ell}\right)^2 + 2\sum_{\ell=0}^{m-1}\sum_{\ell'=\ell+1}^{m-1} \widetilde{\zeta}^{(m)}_{j-\ell,\ell}\, \widetilde{\zeta}^{(m)}_{j-\ell',\ell'} = \sum_{\ell=0}^{m-1}\left(\widetilde{\zeta}^{(m)}_{j-\ell,\ell}\right)^2 + 2\sum_{\ell=0}^{m-1}\sum_{k=1}^{m-\ell-1} \widetilde{\zeta}^{(m)}_{j-\ell,\ell}\, \widetilde{\zeta}^{(m)}_{j-\ell-k,\ell+k}$$

We note that

$$\left(\widetilde{\zeta}^{(m)}_{j-\ell,\ell}\right)^2 = \underbrace{\mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j-1,n}}_{\ell \text{ factors}} \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2 \underbrace{\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right]\right)^2}_{m-\ell-1 \text{ factors}}.$$

and

$$\begin{aligned}
&\widetilde{\zeta}^{(m)}_{j-\ell,\ell}\, \widetilde{\zeta}^{(m)}_{j-\ell-k,\ell+k} \\
= \;& \mathbb{B}_{j-\ell,n}\, \mathbb{B}_{j-\ell+1,n} \cdots \mathbb{B}_{j-1,n}\, \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right] \times \\
& \times \mathbb{B}_{j-\ell-k,n}\, \mathbb{B}_{j-\ell-k+1,n} \cdots \mathbb{B}_{j-\ell,n} \cdots \mathbb{B}_{j-1,n}\, \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-k-1,n}\right] \\
= \;& \underbrace{\mathbb{B}_{j-\ell-k,n} ... \mathbb{B}_{j-1,n}}_{\ell+k \text{ factors}} \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2 \times \\
& \underbrace{\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-k-1,n}\right]\right)^2}_{m-(\ell+k)-1 \text{ factors}} \underbrace{\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-k,n}\right] \cdots \mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-\ell-1,n}\right]}_{k \text{ factors}},
\end{aligned}$$

Consequently, using Lemma 5,

$$\frac{1}{n}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j(2)\eta_j(2)\right]\xrightarrow{p}\Sigma_{22}\doteq\int_0^1\left(\sum_{\ell=0}^{m-1}p_s^{2m-\ell-1}(1-p_s)+2\sum_{\ell=0}^{m-1}(m-\ell-1)\,p_s^{2m-\ell-1}(1-p_s)\right)ds,$$

which, after some standard algebra becomes,

$$\begin{aligned}\Sigma_{22} &= \int_0^1 p_s^{2m-1}\,(1-p_s)\left(\sum_{\ell=0}^{m-1}p_s^{-\ell}+2\sum_{\ell=0}^{m-1}(m-\ell-1)\,p_s^{-\ell}\right)ds\\ &= \int_0^1\frac{p_s^m\,(1+p_s-(2m(1-p_s)+1+p_s)p_s^m)}{1-p_s}\,ds\\ &= \int_0^1 p_s^m\,\frac{p_s^m\,(2\,m+1)-p_s^{m+1}\,(2\,m-1)-(1+p_s)}{1-p_s}\,ds \qquad (31)\end{aligned}$$

Finally,

$$\begin{aligned}\eta_j(1)\eta_j(2) &= \left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]...\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right]\\ &+\ \mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]...\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-2,n}\right]\\ &+\ \ldots\\ &+\ \mathbb{B}_{j-m-1,n}...\mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2.\end{aligned}$$

Applying Lemma 5 again,

$$\frac{1}{n}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j(1)\eta_j(2)\right]\xrightarrow{p}\int_0^1 m\,p_s^m\,(1-p_s)\,ds,$$

which completes the proof. $\square$

Lemma 7. *Under Assumption 2, as $n\to\infty$ and $k>1$*

$$\sqrt{n}\begin{bmatrix}\mathrm{SIT}_n^{(1)}-\int_0^1(p_s)^2\,ds\\ \mathrm{SIT}_n^{(k)}-\int_0^1(p_s)^2\,ds\end{bmatrix}\overset{stably}{\Longrightarrow}\mathcal{MN}\left(0,\Sigma\right)$$

where

$$\mathrm{SIT}_n^{(1)}=\frac{1}{n}\sum_{j=1}^{n}\mathbb{B}_{j,n}\mathbb{B}_{j+1,n}\qquad \mathrm{SIT}_n^{(k)}=\frac{1}{n}\sum_{j=1}^{n}\mathbb{B}_{j,n}\mathbb{B}_{j+k,n}$$

and $\mathcal{MN}\left(0,\Sigma\right)$ denotes the mixed-normal with covariance matrix Σ whose elements are

$$\Sigma=\begin{bmatrix}\int_0^1 p_s^2+2p_s^3-3p_s^4 & \int_0^1 4p_s^3\,(1-p_s)\,ds\\ \int_0^1 4p_s^3\,(1-p_s)\,ds & \int_0^1 p_s^2+2p_s^3-3p_s^4\end{bmatrix}$$

Proof. We consider the following decomposition

$$\sqrt{n}\begin{bmatrix}\mathrm{SIT}_n^{(1)}-\int_0^1(p_s)^2\,ds\\ \mathrm{SIT}_n^{(k)}-\int_0^1(p_s)^2\,ds\end{bmatrix}=A_1+A_2,$$

where

$$A_1 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \begin{bmatrix} \mathbb{B}_{j,n}\mathbb{B}_{j+1,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\mathbb{E}_{j}\left[\mathbb{B}_{j+1,n}\right] \\ \mathbb{B}_{j,n}\mathbb{B}_{j+k,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\mathbb{E}_{j+k-1}\left[\mathbb{B}_{j+k,n}\right] \end{bmatrix} \quad A_2 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \begin{bmatrix} \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\mathbb{E}_{j}\left[\mathbb{B}_{j+1,n}\right] - \int_0^1 p_s^2\, ds \\ \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\mathbb{E}_{j+k-1}\left[\mathbb{B}_{j+k,n}\right] - \int_0^1 p_s^2\, ds \end{bmatrix}$$

A_2 is asymptotically negligible (which can be proven as in Lemma **??** above): It is so sufficient to show that $A_1 \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma)$. Before proceeding, we note the following things. The first component of A_1, $A_1(1)$, coincides with the quantity $\varphi_{j,m}$ as in the previous Lemma by setting $m=2$. The second component, instead, coincides with $\varphi_{j,m}$ if we set $k = m-1$ and $\mathbb{B}_{j+i} = 1$ for all $i \in \{1,\ldots,m-1\}$. Mimicking the steps done in the previous lemma, it is not difficult to see that A_1 can be explicitly written as

$$A_1 = \frac{1}{\sqrt{n}} \sum_{j=m}^{n} \eta_j + \frac{1}{\sqrt{n}} R_n = \frac{1}{\sqrt{n}} \sum_{j=m}^{n} \begin{bmatrix} \eta_j(1) \\ \eta_j(2) \end{bmatrix} + \frac{1}{\sqrt{n}} \begin{bmatrix} R_n(1) \\ R_n(2) \end{bmatrix}$$

with

$$\begin{aligned} \eta_j(1) &= \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] + \mathbb{B}_{j-1,n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right) \\ \eta_j(2) &= \left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)\mathbb{E}_{j-1}\left[\mathbb{B}_{j+m-1,n}\right] + \mathbb{B}_{j-(m-1),n}\left(\mathbb{B}_{j,n} - \mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right) \end{aligned}$$

and the reminder $R_n/\sqrt{n}$ is asymptotically negligible. Thus, we need to determine the following convergence

$$\frac{1}{\sqrt{n}} \sum_{j=m}^{n} \eta_j \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma).$$

To do so, we want to use the Corollary 3. In particular we have to determine the functions $\varphi^{(1)}$ and $\varphi^{(2)}$. It is convenient to rewrite $\eta_j(1)$ and $\eta_j(2)$ as follows.

$$\begin{aligned} \eta_j(1) &= \mathbb{B}_{j-1,n}\mathbb{B}_{j,n} + \mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right] - \mathbb{E}_{j-1}\left[\mathbb{B}_{j-1,n}\mathbb{B}_{j,n} + \mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\right] \\ \eta_j(2) &= \mathbb{B}_{j-(m-1),n}\mathbb{B}_{j,n} + \mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right] - \mathbb{E}_{j-1}\left[\mathbb{B}_{j-(m-1),n}\mathbb{B}_{j,n} + \mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right]\right] \end{aligned}$$

It follows that $\eta_j(1)$ can be rewritten in terms of a function $\varphi^{(1)}: \mathbb{R}^3 \to \mathbb{R}$ in the following way

$$\eta_j(1) = \varphi^{(1)}\left(\mathbb{B}_{j-1,n}, \mathbb{B}_{j,n}, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\right) - \mathbb{E}_{j-1}\left[\varphi^{(1)}\left(\mathbb{B}_{j-1,n}, \mathbb{B}_{j,n}, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\right)\right]$$

with $\varphi^{(1)}(x_1,x_2,x_3) = x_1x_2 + x_2x_3$. On the other hand $\eta_j(2)$ can be rewritten in terms of a function $\varphi^{(2)}: \mathbb{R}^{2(m-1)+1} \to \mathbb{R}$ in the following way

$$\begin{aligned} \eta_j(2) &= \varphi^{(2)}\left(\mathbb{B}_{j-(m+1),n}, \cdots, \mathbb{B}_{j,n}, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right], \cdots, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)}\right]\right) \\ &- \mathbb{E}_{j-1}\left[\varphi^{(2)}\left(\mathbb{B}_{j-(m+1),n}, \cdots, \mathbb{B}_{j,n}, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right], \cdots, \mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)}\right]\right)\right] \end{aligned}$$

with $\varphi^{(1)}\left(x_1,\cdots,x_{2(m-1)+1}\right) = x_1x_{m+2} + x_{m+2}x_{2(m-1)+1}$. Now, we proceed by noticing that for all j the vector η_j is $\mathcal{F}_{t_{j,n}}$ measurable and bounded. In particular

$$\sum_{j=m}^{n} \mathbb{E}_{j-1}\left[\left\|\frac{1}{\sqrt{n}}\eta_j\right\|^4\right] \xrightarrow{p} 0.$$

Besides, we have that $\mathbb{E}_{j-1}\left[\eta_j(1)\right] = \mathbb{E}_{j-1}\left[\eta_j(2)\right] = 0$. We proceed now to show that $n^{-1}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j\eta_j^{'}\right] \xrightarrow{p} \Sigma$. For the first component we inherit the result of the previous Lemma and we conclude that

$$\frac{1}{n}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j\left(1\right)\eta_j\left(1\right)\right]\xrightarrow{p}\Sigma_{11}\doteq\int_0^1 p_s^2+2\,p_s^3-3\,p_s^4\,ds$$

We consider now $\eta_j\left(2\right)\eta_j\left(2\right)$. Using standard algebra, we obtain

$$\begin{aligned}\eta_j\left(2\right)\eta_j\left(2\right) &= \left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)},n\right]\right)^2\mathbb{B}_{j,n}-2\mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right]\right)^2+\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right]\right)^2\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2\\ &+ \mathbb{B}_{j+(m-1),n}\mathbb{B}_{j,n}-2\mathbb{B}_{j+(m-1),n}\mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]+\mathbb{B}_{j+(m-1),n}\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2\\ &+ 2\mathbb{B}_{j+(m-1),n}\mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)},n\right]\\ &- 4\mathbb{B}_{j+(m-1),n}\mathbb{B}_{j,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)},n\right]\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]+2\mathbb{B}_{j+(m-1),n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1)},n\right]\left(\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2.\end{aligned}$$

Applying Lemma 3 we obtain

$$\frac{1}{n}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j\left(2\right)\eta_j\left(2\right)\right]\xrightarrow{p}\Sigma_{22}\doteq\int_0^1 p_s^2+2\,p_s^3-3\,p_s^4\,ds$$

Finally, we have the following

$$\begin{aligned}&\eta_j\left(1\right)\eta_j\left(2\right)\\ =\;&\left(\mathbb{B}_{j,n}-\mathbb{E}_{j-1}\left[\mathbb{B}_{j,n}\right]\right)^2\cdot\\ \cdot\;&\left(\mathbb{B}_{j-1,n}\mathbb{B}_{j-(m-1),n}+\mathbb{B}_{j-1,n}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right]+\mathbb{B}_{j-(m-1)}\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]+\mathbb{E}_{j-1}\left[\mathbb{B}_{j+1,n}\right]\mathbb{E}_{j-1}\left[\mathbb{B}_{j+(m-1),n}\right]\right)\end{aligned}$$

Using again Lemma 3 we have

$$\frac{1}{n}\sum_{j=m}^{n}\mathbb{E}_{j-1}\left[\eta_j\left(1\right)\eta_j\left(2\right)\right]\xrightarrow{p}\Sigma_{12}\doteq\int_0^1 4\,p_s^3\left(1-p_s\right)\,ds,$$

which completes the proof. □

Lemma 8. *Assume that p_t has Riemann integrable paths. Then, as $n\to\infty$,*

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{B}_{j,n}\xrightarrow{u.c.p}\int_0^1 p_s\,ds.$$

Proof. Consider the decomposition:

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{B}_{j,n}=\frac{1}{n}\sum_{j=1}^{n}\left(\mathbb{B}_{j,n}-p_{j,n}\right)+\frac{1}{n}\sum_{j=1}^{n}p_{j,n}.$$

By Riemann integrability,

$$\frac{1}{n}\sum_{j=1}^{n}p_{j,n}\longrightarrow\int_0^1 p_s\,ds,$$

hence, to conclude, it is enough to show that the array $n^{-1}\sum_{j=1}^{n}\left(\mathbb{B}_{j,n}-p_{j,n}\right)$ is AN. We use Lemma 1 and we define $\xi_j^n\doteq n^{-1}\left(\mathbb{B}_{j,n}-p_{j,n}\right)$. Since

$$\mathbb{E}_{j-1}\left[\xi_j^n\right]=\mathbb{E}_{j-1}\left[\frac{1}{n}\left(\mathbb{B}_{j,n}-p_{j,n}\right)\right]=\frac{1}{n}\mathbb{E}\left[\mathbb{E}\left[\mathbb{B}_{j,n}-p_{j,n}\,\middle|\,\mathcal{F}_{t_{j-1,n}}\vee\mathcal{F}^{(p)}_{t_{j,n}}\right]\right]=0,$$

it is enough to prove that $\sum_{j=1}^{n} \mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] \xrightarrow{p} 0$. By boundedness of Bernoulli random variables and the probability process, for some constant $C > 0$, we have:

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] = \sum_{j=1}^{n} \mathbb{E}\left[\left|\frac{1}{n}\left(\mathbb{B}_{j,n} - p_{j,n}\right)\right|^2\right] \leq \frac{C}{n} \longrightarrow 0,$$

which completes the proof. $\square$

Lemma 9. *Assume that p_t has Riemann integrable paths and $|\mathbb{E}_t\left[|p_{t+\Delta_n} - p_t|\right]| = K\Delta_n^q + o_p(\Delta_n^{q+\epsilon})$ pointwise on Ω, for some $0 < q < \frac{1}{2}$, $K, \epsilon > 0$. Then, as $n \to \infty$,*

$$\sqrt{n}\left|\frac{1}{n}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+1,n} - \frac{1}{n}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+k,n}\right| \xrightarrow{p} \infty.$$

Proof. First, consider $D^{(n)} = \Delta_n^{1-q} \sum\limits_{j=1}^{n-k} \mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n} - \mathbb{B}_{j+k,n}\right)$, which can be decomposed as the sum of the two terms:

$$D^{(n)} = D_1^{(n)} + D_2^{(n)},$$

where

$$D_1^{(n)} = \Delta_n^{1-q} \sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{E}_{j+1}\left[\mathbb{B}_{j+1,n} - \mathbb{B}_{j+k,n}\right],$$

$$D_2^{(n)} = \Delta_n^{1-q} \sum_{j=1}^{n-k} \mathbb{B}_{j,n}\left(\mathbb{B}_{j+1,n} - \mathbb{B}_{j+k,n} - \mathbb{E}_{j+1}\left[\mathbb{B}_{j+1,n} - \mathbb{B}_{j+k,n}\right]\right).$$

By assumption, for the first term we have:

$$\left|D_1^{(n)}\right| \leq K \sum_{j=1}^{n-k} \mathbb{B}_{j,n}\Delta_n + o_p(\Delta_n^{\epsilon}) \xrightarrow{p} \int_0^1 K\, p_s\, ds,$$

where the convergence follows from Lemma 8. The second term can be expressed as $D_2^{(n)} = \sum\limits_{j=1}^{n-k} d_{j,n}$, where

$$\mathbb{E}_{j-1}\left[d_{j,n}\right] = 0,$$

and

$$\mathbb{E}\left[\left(d_{j,n}\right)^2\right] \leq C\Delta_n^{1-2q} \to 0.$$

Hence, by Lemma 2, $D_2^{(n)}$ is asymptotically negligible. Consequently, $D^{(n)} \xrightarrow{p} \int\limits_0^1 K\, p_s\, ds = \mathit{const}$, which implies that:

$$\sqrt{n}\left|\frac{1}{n}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+1,n} - \frac{1}{n}\sum_{j=1}^{n-k} \mathbb{B}_{j,n}\mathbb{B}_{j+k,n}\right| = \Delta_n^{q-1/2}\left|D^{(n)}\right| \xrightarrow{p} \infty.$$

$\square$

A.3 Proofs of Theorems 3.4 and 3.5 from Section 3.2

For an arbitrary sequence of integers k_n such that $k_n \to \infty$ and $k_n \Delta_n = \frac{k_n}{n} \to 0$, let

$$\alpha_j^n \doteq \frac{1}{k_n} \sum_{i=0}^{k_n-1} \left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right), \qquad \beta_j^n \doteq \frac{1}{k_n} \sum_{i=0}^{k_n-1} \left(p_{j+i,n} - p_{j-1,n}\right),$$

and set $h_n = n - k_n$. Note that

$$\widehat{p}_j\left(k_n\right) - p_{j-1,n} = \alpha_j^n + \beta_j^n, \quad j \in \{1, \dots, h_n + 1\}.$$

The auxiliary results for the proofs of Theorems 3.4 and 3.5 are summarised by the following Lemma. delete at the end the numbers

Lemma 10. *Under Assumptions 1, 2 and 3, for $C > 0$ and for all $q \geq 2$, we have*

$$\text{(A1)} \quad \mathbb{E}_{j-1}\left[\sup_{s \in [0,\Delta_n]} \left|p_{j-1,n+s} - p_{j-1,n}\right|^q\right] \leq C \cdot \Delta_n^{1 \wedge (q/2)} \tag{32}$$

$$\text{(A2)} \quad \left|\mathbb{E}_{j-1}\left[p_{j,n} - p_{j-1,n}\right]\right| \leq C \cdot \Delta_n \tag{33}$$

$$\text{(A3)} \quad \left|\mathbb{E}_{j-1}\left[\beta_j^n\right]\right| \leq C \cdot k_n \Delta_n \tag{34}$$

$$\text{(A4)} \quad \mathbb{E}_{j-1}\left[\left|\beta_j^n\right|^q\right] \leq C \cdot \left(k_n \Delta_n\right)^{q/2} \tag{35}$$

$$\text{(A5)} \quad \left|\mathbb{E}_{j-1}\left[\alpha_j^n\right]\right| = 0 \tag{36}$$

$$\text{(A6)} \quad \mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|^q\right] \leq C k_n^{-q/2} \tag{37}$$

$$\text{(A7)} \quad \left|\mathbb{E}_{j-1}\left[\left(\alpha_i^n\right)^2 - \frac{1}{k_n} p_{i-1,n}\left(1 - p_{j-1,n}\right)\right]\right| \leq C \cdot \Delta_n \tag{38}$$

$$\text{(A8)} \quad \left|\mathbb{E}_{j-1}\left[\alpha_j^n \beta_j^n\right]\right| = 0 \tag{39}$$

Proof. The proof of (A1)-(A4) follows the same arguments as in the proof of results of Appendix A and Lemma B-4 of Aït-Sahalia and Jacod (2012). In order to complete the proof of the Lemma, we need to prove (A5)-(A8). Equality (A5) easily follows by conditioning on the path of the process p_t.

$$\left|\mathbb{E}_{j-1}\left[\alpha_j^n\right]\right| = \left|\frac{1}{k_n} \sum_{j=0}^{k_n-1} \mathbb{E}_{j-1}\left[\mathbb{B}_{j+i,n} - p_{j+i,n}\right]\right| = 0.$$

To prove the other relations, we first observe that conditioning on the path $(p_t)_{t\in[0,1]}$ we have

$$\begin{aligned} \mathbb{E}_{i-1}\left[\left(\alpha_j^n\right)^2\right] &= \frac{1}{k_n^2} \mathbb{E}_{j-1}\left[\sum_{i=0}^{k_n-1} \left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right)^2\right] + \frac{2}{k_n} \mathbb{E}_{j-1}\left[\sum_{i=0}^{k_n-2} \sum_{m=1}^{k_n-1-i} \left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right)\left(\mathbb{B}_{j+i+m,n} - p_{j+i+m,n}\right)\right] \\ &= \frac{1}{k_n^2} \sum_{i=0}^{k_n-1} \mathbb{E}_{j-1}\left[\left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right)^2\right] = \frac{1}{k_n^2} \sum_{i=0}^{k_n-1} \mathbb{E}_{j-1}\left[p_{j+i,n}\left(1 - p_{j+i,n}\right)\right] \leq \frac{C}{k_n}, \end{aligned} \tag{40}$$

where the last inequality is due to the fact that $p_t \in (0,1)$. Moreover, we have

$$\mathbb{E}_{j-1}\left[\left(\alpha_j^n\right)^2 - \frac{1}{k_n} p_{j-1,n}(1 - p_{j-1,n})\right] = \frac{1}{k_n^2} \sum_{i=0}^{k_n-1} \mathbb{E}_{j-1}\left[p_{j+i,n} - p_{j-1,n}\right] - \frac{1}{k_n^2} \sum_{i=0}^{k_n-1} \mathbb{E}_{j-1}\left[p_{j+i,n}^2 - p_{j-1,n}^2\right].$$

By applying triangular inequality, we obtain

$$\left|\mathbb{E}_{j-1}\left[\left(\alpha_j^n\right)^2-\frac{1}{k_n}p_{j-1,n}(1-p_{j-1,n})\right]\right|\leq\frac{1}{k_n^2}\sum_{i=0}^{k_n-1}\left|\mathbb{E}_{j-1}\left[p_{j+i,n}-p_{j-1,n}\right]\right|+\frac{1}{k_n^2}\sum_{i=0}^{k_n-1}\left|\mathbb{E}_{j-1}\left[p_{j+i,n}^2-p_{j-1,n}^2\right]\right|.$$

Hence, (38) follows from (33), whereas (37) from Hölder's inequality and (40). Finally, (39) is obtained by conditioning on the path $(p_t)_{t\in[0,1]}$ and by using Eq.(36). $\square$

Proof of Theorem 3.4. For any $t>0$, define a function of t, $\widehat{p}(k_n,t)$, as

$$\widehat{p}(k_n,t)\doteq\widehat{p}_j(k_n),\quad t\in((j-2)\Delta_n,(j-1)\Delta_n].$$

First, we prove that $\widehat{p}(k_n,t)$ converges in probability to p_t for every $t\in[0,1]$. For any $t\in[0,1]$ and j_t such that $t\in((j_t-2)\Delta_n,(j_t-1)\Delta_n]$, we have:

$$(j+1)\Delta_n\leq(j_t+j)\Delta_n-t\leq(j+2)\Delta_n.$$

Second, we have

$$\begin{aligned}\mathbb{E}\left[\left(\widehat{p}(k_n,t)-p_t\right)^2\right]&=\mathbb{E}\left[\left(\frac{1}{k_n}\sum_{i=0}^{k_n-1}\left(\mathbb{B}_{j_t+i,n}-p_t\right)\right)^2\right]=\mathbb{E}\left[\frac{1}{k_n^2}\sum_{i=0}^{k_n-1}\left(\mathbb{B}_{j_t+i,n}-p_t\right)^2+\frac{1}{k_n^2}\sum_{i\neq i'}\left(\mathbb{B}_{j_t+i,n}-p_t\right)\left(\mathbb{B}_{j_t+i',n}-p_t\right)\right]\\&=\mathbb{E}\left[\frac{1}{k_n^2}\sum_{i=0}^{k_n-1}\left(\mathbb{B}_{j_t+i,n}-p_t\right)^2\right]+\mathbb{E}\left[\frac{1}{k_n^2}\sum_{i\neq i'}\left(\mathbb{B}_{j_t+i,n}-p_t\right)\left(\mathbb{B}_{j_t+i',n}-p_t\right)\right].\end{aligned}$$

The first of the two terms converges to zero by boundedness of $\mathbb{B}_{j_t+i,n}$ and p_t. Concerning the second, we have that, by conditioning on $(p_t)_{t\in[0,1]}$ and (33)

$$\left|\mathbb{E}\left[\left(\mathbb{B}_{j_t+i,n}-p_t\right)\left(\mathbb{B}_{j_t+i',n}-p_t\right)\right]\right|=\left|\mathbb{E}\left[p_{(j_t+i)\Delta_n}-p_t\right]\mathbb{E}\left[p_{(j_t+i')\Delta_n}-p_t\right]\right|\leq C(k_n\Delta_n)^2,$$

hence,

$$\left|\mathbb{E}\left[\frac{1}{k_n^2}\sum_{j\neq j'}\left(\mathbb{B}_{i_t+j,n}-p_t\right)\left(\mathbb{B}_{i_t+j',n}-p_t\right)\right]\right|\leq C(k_n\Delta_n)^2\longrightarrow 0.$$

Thus, $\widehat{p}(k_n,t)\xrightarrow{p}p_t$ for each $t\in[0,1]$. Now, we write $U\left(\Delta_n,f\right)^n$ as

$$U\left(\Delta_n,f\right)^n=\Delta_n f(\widehat{p}_1(k_n))+\int_0^{h_n\Delta_n}f(\widehat{p}(k_n,t))\,ds.$$

and we compute

$$
\begin{aligned}
\mathbb{E}\left[\left|U\left(\Delta_n, f\right)^n-\int_0^1 f(p_s)\, ds\right|\right] &= \Delta_n\, \mathbb{E}\left[\left|f(\widehat{p}_1(k_n))-\int_0^1 f(p_s)\, ds\right|\right]+\int\limits_0^{h_n\Delta_n} a_s\, ds \\
&= \Delta_n\, \mathbb{E}\left[\left|\int_0^1 \left(f(\widehat{p}_1(k_n))-f(p_s)\right) ds\right|\right]+\int\limits_0^{h_n\Delta_n} a_s\, ds \\
&\leq \Delta_n\, \mathbb{E}\left[\int_0^1 \left|\left(f\left(\widehat{p}_1\left(k_n\right)\right)-f(p_s)\right)\right| ds\right]+\int\limits_0^{h_n\Delta_n} a_s\, ds \\
&\leq C\, \Delta_n+\int\limits_0^{h_n\Delta_n} a_n(s)\, ds.
\end{aligned}
$$

where $a_n(s) \doteq \mathbb{E}\left[|f(\widehat{p}(k_n,s))-f(p_s)|\right]$, C is a suitable constant and we used the locally boundedness of $f(\cdot)$ and the boundedness of p_s and $\widehat{p}(k_n,s)$. By continuous mapping theorem, condition $\widehat{p}(k_n,t) \xrightarrow{p} p_t$ implies that, for a given $s \in [0,1]$

$$
f(\widehat{p}(k_n,s)) \xrightarrow{p} f(p_s), \tag{41}
$$

Nonetheless, since the sequence of random variables $f(\widehat{p}(k_n,s))$ is uniformly integrable (using, again, the locally boundedness of $f(\cdot)$ and the boundedness of $\widehat{p}(k_n,s)$) then the convergence in Eq. (41) is also in $\mathbb{L}^1$ norm and so $a_n(s) \longrightarrow 0$ for each s. Besides, since $a_n(s)$ is uniformly bounded in (n,s), $U\left(\Delta_n, f\right)^n \overset{u.c.p.}{\longrightarrow} \int_0^1 f\left(p_s\right) ds$ by dominated convergence theorem (cfr. Jacod and Protter, 2012, Theorem 9.4.1). □

Proof of Theorem 3.5. First, consider the following decomposition:

$$
\begin{aligned}
\frac{1}{\sqrt{\Delta_n}}\left(U'\left(\Delta_n,f\right)_1^n-U\left(f\right)_1\right) &= \sqrt{\Delta_n}\sum_{j=1}^{h_n+1}\left(f\left(\widehat{p}_j\left(k_n\right)\right)-\frac{1}{2\,k_n}f''\left(\widehat{p}_j\left(k_n\right)\right)\widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right)\right)-\frac{1}{\sqrt{\Delta_n}}\int_0^1 f\left(p_s\right) ds \\
&= \sum_{r=1}^{4} U\left(r\right)_1^n,
\end{aligned}
$$

with

$$
\begin{aligned}
U\left(1\right)_1^n &= \frac{1}{\sqrt{\Delta_n}}\sum_{j=1}^{h_n+1}\int_{(j-1)\Delta_n}^{j\Delta_n}\left(f\left(p_{j-1,n}\right)-f\left(p_s\right)\right) ds-\frac{1}{\sqrt{\Delta_n}}\int_{(h_n+1)\,\Delta_n}^{1} f\left(p_s\right) ds \\
U\left(2\right)_1^n &= \sqrt{\Delta_n}\sum_{j=1}^{h_n+1} f'\left(p_{j-1,n}\right)\beta_j^n \\
U\left(3\right)_1^n &= \sqrt{\Delta_n}\sum_{j=1}^{h_n+1}\left(f\left(\widehat{p}_j\left(k_n\right)\right)-f\left(p_{j-1,n}\right)-f'\left(p_{j-1,n}\right)\left(\alpha_j^n+\beta_j^n\right)-\frac{1}{2\,k_n}f''\left(\widehat{p}_j\left(k_n\right)\right)\widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right)\right) \\
U\left(4\right)_1^n &= \sqrt{\Delta_n}\sum_{j=1}^{h_n+1} f'\left(p_{j-1,n}\right)\alpha_j^n.
\end{aligned}
$$

At this point, the rest of the proof is divided into four parts. In the firsts three we prove that $U\left(k\right)^n$, $k=1,2,3$, is AN, whereas in the last part we show that $U(4)^n \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma)$.

Part 1: Proof of the AN of $U\left(1\right)_1^n$

Remember that $h_n = n-k_n$ and that $n = 1/\Delta_n$, whence $1-\left(h_n+1\right)\,\Delta_n = 1-\left(n-k_n+1\right)\,\Delta_n = k_n\,\Delta_n-\Delta_n$. Since

$f(p_s)$ is bounded, for the second term of $U(1)_1^n$ we have

$$\left|\frac{1}{\sqrt{\Delta_n}}\int\limits_{(h_n+1)\,\Delta_n}^{1} f\left(p_s\right)\, ds\right| \leq C k_n \sqrt{\Delta_n} \longrightarrow 0.$$

The first term of $U(1)_1^n$ can be expressed as $\sum\limits_{j=1}^{h_n+1} \xi_j^n$, where

$$\xi_j^n = \frac{1}{\sqrt{\Delta_n}} \int_{(j-1)\Delta_n}^{j\Delta_n} \left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\, ds.$$

Since the process $f(p_t)$ is bounded semimartingale by using inequality (33) we get

$$\begin{aligned}
\left|\mathbb{E}\left[\xi_j^n\right]\right| &= \frac{1}{\sqrt{\Delta_n}}\left|\mathbb{E}\left[\int_{(j-1)\Delta_n}^{j\Delta_n} \left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\, ds\right]\right| = \frac{1}{\sqrt{\Delta_n}}\left|\int_{(j-1)\Delta_n}^{j\Delta_n} \mathbb{E}\left[\mathbb{E}_{j-1}\left[\left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\right]\, ds\right]\right| \\
&\leq \frac{1}{\sqrt{\Delta_n}} \int_{(j-1)\Delta_n}^{j\Delta_n} \left|\mathbb{E}\left[\mathbb{E}_{j-1}\left[\left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\right]\right]\right| ds \leq \frac{1}{\sqrt{\Delta_n}} \int_{(j-1)\Delta_n}^{j\Delta_n} \mathbb{E}\left[\left|\mathbb{E}_{j-1}\left[\left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\right]\right|\right] ds \\
&\leq \frac{C}{\sqrt{\Delta_n}}\,\Delta_n^2 = C\left(\Delta_n\right)^{3/2} \longrightarrow 0,
\end{aligned}$$

while, using inequality (32) and Holder's inequality, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\left|\xi_j^n\right|^2\right] &= \frac{1}{\Delta_n}\mathbb{E}\left[\left(\int_{(j-1)\Delta_n}^{j\Delta_n} \left(f\left(p_{j-1,n}\right) - f\left(p_s\right)\right)\, ds\right)^2\right] \\
&= \frac{1}{\Delta_n}\mathbb{E}\left[\int\limits_{(j-1)\Delta_n}^{j\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n} \left(f(p_{j-1,n}) - f(p_q)\right)\left(f(p_{j-1,n}) - f(p_s)\right)\, ds\, dq\right] \\
&= \frac{1}{\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n} \mathbb{E}\left[\left(f(p_{j-1,n}) - f(p_q)\right)\left(f(p_{j-1,n}) - f(p_s)\right)\right]\, ds\, dq \\
&\leq \frac{1}{\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n} \sqrt{\mathbb{E}\left[\left|f(p_{j-1,n}) - f(p_q)\right|^2\right]\mathbb{E}\left[\left|f(p_{j-1,n}) - f(p_s)\right|^2\right]}\, ds\, dq \\
&\leq \frac{1}{\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n}\int\limits_{(j-1)\Delta_n}^{j\Delta_n} C\,\Delta_n\, ds\, dq \leq C\,\Delta_n^2 \longrightarrow 0.
\end{aligned}$$

Consequently, by Lemma 2, $U(1)_1^n$ is AN.

Part 2: Proof of the AN of $U\left(2\right)_1^n$

Using Lemma 10 and boundedness of $f'\left(p_{j-1,n}\right)$, we obtain

$$\sum_{j=1}^{h_n+1}\left|\mathbb{E}_{j-1}\left[\sqrt{\Delta_n} f'\left(p_{j-1,n}\right)\beta_j^n\right]\right| \leq C\sum_{j=1}^{h_n+1}\sqrt{\Delta_n}\left|\mathbb{E}_{j-1}\left[\beta_j^n\right]\right| \leq C\sum_{j=1}^{h_n+1} k_n\left(\Delta_n\right)^{3/2} \longrightarrow 0,$$

and

$$\sum_{j=1}^{h_n+1}\mathbb{E}_{j-1}\left[\left|\sqrt{\Delta_n} f'\left(p_{j-1,n}\right)\beta_j^n\right|^2\right] \leq C\sum_{j=1}^{h_n+1}\mathbb{E}_{j-1}\left[\Delta_n\left|\beta_j^n\right|^2\right] \leq C\sum_{j=1}^{h_n+1} k_n\left(\Delta_n\right)^2 = C\left(n-k_n\right)k_n\,\Delta_n^2 \leq C\,k_n\,\Delta_n \longrightarrow 0,$$

and so

$$k_n \sum_{j=1}^{h_n+1} \mathbb{E}\left[\left|\sqrt{\Delta_n} f'\left(p_{j-1,n}\right) \beta_j^n\right|^2\right] \leq C\, k_n^2\, \Delta_n \longrightarrow 0.$$

Consequently, by applying Lemma 2 we get that $U(2)_1^n$ is AN.

Part 3: Proof of the AN of $U\left(3\right)_1^n$

As a first step, we rewrite $U\left(3\right)_1^n$ as $U(3)_1^n = \sum_{j=1}^{h_n+1} \sum_{k=1}^{4} v_j^n(k)$ with $v_n^j\left(k\right)$, $k=1,\ldots,4$, suitably defined triangular arrays. To do so, we remind that

$$\alpha_j^n + \beta_j^n = \frac{1}{k_n} \sum_{i=0}^{k_n-1} \left(\mathbb{B}_{j+i,n} - p_{j-1,n}\right) = \widehat{p}_j\left(k_n\right) - p_{j-1,n}.$$

Using Taylor expansion of $f\left(p\right)$ around $p_0 = p_{j-1,n}$ and computing the expansion in $p = \widehat{p}_j\left(k_n\right)$, we obtain

$$f\left(\widehat{p}_j\left(k_n\right)\right) - f\left(p_{j-1,n}\right) - f'\left(p_{j-1,n}\right)\left(\alpha_j^n + \beta_j^n\right) = \frac{1}{2} f''\left(p_{j-1,n}\right)\left(\alpha_j^n + \beta_j^n\right)^2 + \frac{1}{6} f'''\left(p_j^\star\right)\left(\alpha_j^n + \beta_j^n\right)^3,$$

where $p_j^\star$ is a point between $p_{j-1,n}$ and $p_{j-1,n} + \alpha_j^n + \beta_j^n$. Then, we have

$$\begin{aligned}
\frac{1}{2} f''\left(p_{j-1,n}\right)\left(\alpha_j^n + \beta_j^n\right)^2 &= \frac{1}{2} f''\left(p_{j-1,n}\right)\left((\alpha_j^n)^2 + 2\alpha_j^n \beta_j^n - \frac{1}{k_n} p_{j-1,n}(1-p_{j-1,n})\right) \\
&+ \frac{1}{2\, k_n} f''\left(p_{j-1,n}\right) p_{j-1,n}(1-p_{j-1,n}) + \frac{1}{2} f''\left(p_{j-1,n}\right)\left(\beta_j^n\right)^2.
\end{aligned}$$

Consequently, $U(3)_1^n$ can be represented as $U(3)_1^n = \sum_{j=1}^{h_n+1} \sum_{k=1}^{4} v_j^n(k)$, where

$$\begin{aligned}
v_j^n(1) &= \frac{\sqrt{\Delta_n}}{2} f''\left(p_{j-1,n}\right)\left((\alpha_j^n)^2 + 2\alpha_j^n \beta_j^n - \frac{1}{k_n} p_{j-1,n}(1-p_{j-1,n})\right), \\
v_j^n(2) &= \frac{\sqrt{\Delta_n}}{2\, k_n} f''\left(p_{j-1,n}\right) p_{j-1,n}(1-p_{j-1,n}) - \frac{\sqrt{\Delta_n}}{2\, k_n} f''\left(\widehat{p}_j\left(k_n\right)\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right), \\
v_j^n(3) &= \frac{\sqrt{\Delta_n}}{2} f''\left(p_{j-1,n}\right)\left(\beta_j^n\right)^2, \\
v_j^n(4) &= \frac{\sqrt{\Delta_n}}{6} f'''\left(p_j^\star\right)\left(\alpha_j^n + \beta_j^n\right)^3.
\end{aligned}$$

We have to prove that all the triangular arrays $v_j^n(k)$ are AN for $k = 1,2,3,4$. First, consider $v_i^n(1)$. Inequalities (38) and (39) imply that $\left|\mathbb{E}_{j-1}\left[v_j^n(1)\right]\right| \leq C\, \Delta_n^{3/2}$, and so

$$\sum_{j=1}^{h_n+1} \left|\mathbb{E}_{j-1}\left[v_j^n(1)\right]\right| \leq C\, \Delta_n^{1/2} \xrightarrow{p} 0. \tag{42}$$

Besides

$$\begin{aligned}
v_j^n(1)^2 &= \frac{\Delta_n}{4} f''\left(p_{j-1,n}\right)^2 \Big((\alpha_j^n)^4 + 4\left(\alpha_j^n \beta_j^n\right)^2 + \frac{1}{k_n^2} p_{j-1,n}^2(1-p_{j-1,n})^2 + \\
&\quad +4\left(\alpha_j^n\right)^3 \beta_j^n - 2\,\frac{(\alpha_j^n)^2}{k_n} p_{j-1,n}(1-p_{j-1,n}) - \frac{4\,\alpha_j^n\, \beta_j^n}{k_n} p_{j-1,n}(1-p_{j-1,n})\Big) \\
&\leq \frac{\Delta_n}{4} f''\left(p_{j-1,n}\right)^2 \Big((\alpha_j^n)^4 + 4\left(\alpha_j^n \beta_j^n\right)^2 + \frac{1}{k_n^2} p_{j-1,n}^2(1-p_{j-1,n})^2 + \\
&\quad +4\left|\left(\alpha_j^n\right)^3 \beta_j^n\right| + 2\,\frac{(\alpha_j^n)^2}{k_n} p_{j-1,n}(1-p_{j-1,n}) - \frac{4\,\alpha_j^n\, \beta_j^n}{k_n} p_{j-1,n}(1-p_{j-1,n})\Big).
\end{aligned}$$

Now, in computing $\mathbb{E}\left[v_j^n(1)^2\right]$ we consider that

- Inequality (37) implies that
$$\mathbb{E}_{j-1}\left[(\alpha_j^n)^4\right] \leq C\, k_n^{-2},$$
and that
$$\mathbb{E}_{j-1}\left[\frac{(\alpha_j^n)^2}{k_n} p_{j-1,n}(1-p_{j-1,n})\right] \leq C\, k_n^{-2}.$$
- Cauchy-Schwartz inequality plus (37) and (35) imply that
$$\mathbb{E}_{j-1}\left[\left(\alpha_j^n \beta_j^n\right)^2\right] \leq \left(\mathbb{E}_{j-1}\left[\left(\alpha_j^n\right)^4\right]\right)^{1/2} \left(\mathbb{E}_{j-1}\left[\left(\beta_j^n\right)^4\right]\right)^{1/2} \leq C\, \Delta_n$$
and that
$$\left|\mathbb{E}_{j-1}\left[\left(\alpha_j^n\right)^3 \beta_j^n\right]\right| \leq \left(\mathbb{E}_{j-1}\left[\left(\alpha_j^n\right)^6\right]\right)^{1/2} \left(\mathbb{E}_{j-1}\left[\left(\beta_j^n\right)^2\right]\right)^{1/2} \leq C\, k_n^{-1}\, \Delta_n^{1/2}.$$
- Equation (39) implies $\mathbb{E}_{j-1}\left[\alpha_j^n\, \beta_j^n k_n p_{j-1,n}(1-p_{j-1,n})\right] = 0$

Summing up
$$\mathbb{E}_{j-1}\left[v_j^n(1)^2\right] \leq C\, \Delta_n \left(\frac{1}{k_n^2} + \Delta_n + \frac{\sqrt{\Delta_n}}{k_n}\right)$$
whence
$$k_n \sum_{j=1}^{h_n} \mathbb{E}\left[v_j^n(1)^2\right] \longrightarrow 0. \tag{43}$$

Summing up, the limits in (42) and (43) imply, through Lemma 2, that $v_j^n(1)$ is AN. Now, consider $v_j^n(4)$. Since both p_t and $\widehat{p}_i(k_n)$ are in $[0,1]$, $|f'''(p_i^\star)| \leq C$, for some constant $C > 0$, hence, we have:

$$\sum_{j=1}^{h_n+1} \left|\frac{\sqrt{\Delta_n}}{6} f'''\left(p_j^\star\right)\left(\alpha_j^n + \beta_j^n\right)^3\right| \leq C \sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\left|\left(\alpha_j^n + \beta_j^n\right)^3\right| = C \sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\left(\left|\alpha_j^n\right|^3 + 3\left|\alpha_j^n\right|\left|\beta_j^n\right|^2 + 3\left|\alpha_j^n\right|^2\left|\beta_j^n\right| + \left|\beta_j^n\right|^3\right).$$

Using estimates from the preliminary results and Cauchy-Schwartz inequality, we have the following implications.

- inequality (37) implies
$$\sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|^3\right] \leq C \cdot k_n^{-3/2}(\Delta_n)^{-1/2} \xrightarrow{p} 0,$$
- inequalities (37) and (35), plus Cauchy-Schwartz, imply
$$\sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|^2\left|\beta_j^n\right|\right] \leq C \sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\sqrt{\mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|^4\right]\mathbb{E}_{j-1}\left[\left|\beta_j^n\right|^2\right]} \leq C\, k_n^{-1/2} \xrightarrow{p} 0,$$
and
$$\sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|\left|\beta_j^n\right|^2\right] \leq C \sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\sqrt{\mathbb{E}_{j-1}\left[\left|\alpha_j^n\right|^2\right]\mathbb{E}_{j-1}\left[\left|\beta_j^n\right|^4\right]} \leq C \cdot (k_n\, \Delta_n)^{1/2} \xrightarrow{p} 0,$$
- inequality (35) implies
$$\sum_{j=1}^{h_n+1} \sqrt{\Delta_n}\mathbb{E}_{j-1}\left[\left|\beta_j^n\right|^3\right] \leq C \cdot k_n^{3/2}\Delta_n \xrightarrow{p} 0.$$

Whence
$$\sum_{j=1}^{h_n+1} \left|\mathbb{E}_{j-1}\left[v_j^n\,(4)\right]\right| \xrightarrow{p} 0. \tag{44}$$

Now consider

$$\begin{aligned}
\sum_{j=1}^{h_n+1} v_j^n(4)^2 &\leq C \sum_{j=1}^{h_n+1} \Delta_n \Big(\left|\alpha_j^n\right|^6 + 9\left|\alpha_j^n\right|^2 \left|\beta_j^n\right|^4 + 9\left|\alpha_j^n\right|^4 \left|\beta_j^n\right|^2 + \left|\beta_j^n\right|^6 + 6\left|\alpha_j^n\right|^4 \left|\beta_j^n\right|^2 + \\
&+ 6\left|\alpha_j^n\right|^5 \left|\beta_j^n\right| + 2\left|\alpha_j^n\right|^3 \left|\beta_j^n\right|^3 + 18\left|\alpha_j^n\right|^3 \left|\beta_j^n\right|^3 + 6\left|\alpha_j^n\right| \left|\beta_j^n\right|^5 + 6\left|\alpha_j^n\right|^2 \left|\beta_j^n\right|^4 \Big). \\
&= C \sum_{j=1}^{h_n+1} \Delta_n \Big(\left|\alpha_j^n\right|^6 + 15\left|\alpha_j^n\right|^2 \left|\beta_j^n\right|^4 + 15\left|\alpha_j^n\right|^4 \left|\beta_j^n\right|^2 + \left|\beta_j^n\right|^6 + 6\left|\alpha_j^n\right|^5 \left|\beta_j^n\right| + 20\left|\alpha_j^n\right|^3 \left|\beta_j^n\right|^3 + 6\left|\alpha_j^n\right| \left|\beta_j^n\right|^5 \Big).
\end{aligned}$$

inequalities (37) and (35), respectively, imply

$$k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right|^6\right] \leq C\, k_n^{-2} \longrightarrow 0,$$

$$k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\beta_j^n\right|^6\right] \leq C\, \left(k_n^{4/3}\,\Delta_n\right)^3 \longrightarrow 0,$$

and, using also Cauchy-Schwartz, they imply

$$\begin{aligned}
k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right|^2 \left|\beta_j^n\right|^4\right] &\leq C\, (k_n\,\Delta_n)^2 \longrightarrow 0 \\
k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right|^4 \left|\beta_j^n\right|^2\right] &\leq C\, k_n^{-2}\,\Delta_n \longrightarrow 0 \\
k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right|^5 \left|\beta_j^n\right|\right] &\leq C\, k_n^{-1}\,\Delta_n^{1/2} \longrightarrow 0 \\
k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right| \left|\beta_j^n\right|^5\right] &\leq C\, \left(k_n^{6/5}\,\Delta_n\right)^{5/2} \longrightarrow 0 \\
k_n \sum_{j=1}^{h_n+1} \Delta_n \mathbb{E}\left[\left|\alpha_j^n\right|^3 \left|\beta_j^n\right|^3\right] &\leq C\, \left(k_n^{2/3}\,\Delta_n\right)^{3/2} \longrightarrow 0
\end{aligned}$$

Consequently,

$$k_n \sum_{j=1}^{h_n+1} \mathbb{E}\left[v_j^n(4)^2\right] \longrightarrow 0. \tag{45}$$

As before, the limits in (44) and (45) imply, through Lemma 2, that $v_j^n(4)$ is AN. Similarly, for $v_j^n(3)$ we have

$$\sum_{j=1}^{h_n+1} \mathbb{E}_{j-1}\left[\left|\frac{\sqrt{\Delta_n}}{2} f''(p_\prime)\left(\beta_j^n\right)^2\right|\right] \leq C \cdot k_n \sqrt{\Delta_n} \xrightarrow{p} 0, \tag{46}$$

besides

$$k_n \sum_{j=1}^{h_n+1} \mathbb{E}\left[\left|\frac{\Delta_n}{4}\left(f''(p_{j-1,n})\right)^2 \left(\beta_j^n\right)^4\right|\right] \leq C \cdot \left(k_n^{3/2}\,\Delta_n\right)^2 \longrightarrow 0, \tag{47}$$

hence, the limits in (46) and (47) imply, through Lemma 2, that $v_j^n(3)$ is AN. Finally, consider $v_j^n(2)$. Using Taylor's expansion, we have (remember that $\widehat{p}_j(k_n) - p_{j-1,n} = \alpha_j^n + \beta_j^n$)

$$f''\left(\widehat{p}_j(k_n)\right) = f''(p_{j-1,n}) + f'''\left(p_j^\star\right)\left(\alpha_j^n + \beta_j^n\right).$$

Consequently $v_j^n(2)$ takes the form

$$\begin{aligned}
v_j^n(2) &= \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) p_{(j-1)\Delta_n}(1-p_{(j-1)\Delta_n}) - \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(\widehat{p}_j\left(k_n\right)\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right) \\
&= \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) p_{(j-1)\Delta_n}(1-p_{(j-1)\Delta_n}) - \frac{\sqrt{\Delta_n}}{2\,k_n}\left(f''\left(p_{j-1,n}\right) + f'''\left(p_j^\star\right)\left(\alpha_j^n+\beta_j^n\right)\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right) \\
&= \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) p_{(j-1)\Delta_n} - \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) p^2_{(j-1)\Delta_n} \\
&- \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) \widehat{p}_j\left(k_n\right) + \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right) \widehat{p}_j\left(k_n\right)^2 - \frac{\sqrt{\Delta_n}}{2\,k_n} f'''\left(p_j^\star\right)\left(\alpha_j^n+\beta_j^n\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right) \\
&= -\frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right)\left(\widehat{p}_j(k_n) - p_{j-1,n}\right) + \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right)\left(\widehat{p}_j(k_n)^2 - p^2_{j-1,n}\right) \\
&- \frac{\sqrt{\Delta_n}}{2\,k_n} f'''\left(p_j^\star\right)\left(\alpha_j^n+\beta_j^n\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right) \\
&= \underbrace{-\frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right)\left(\alpha_j^n+\beta_j^n\right)}_{\mathcal{A}_{j,n}} + \underbrace{\frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right)\left(\widehat{p}_j(k_n)^2 - p^2_{j-1,n}\right)}_{\mathcal{B}_{j,n}} \\
&- \underbrace{\frac{\sqrt{\Delta_n}}{2\,k_n} f'''\left(p_j^\star\right)\left(\alpha_j^n+\beta_j^n\right) \widehat{p}_j\left(k_n\right)\left(1-\widehat{p}_j\left(k_n\right)\right)}_{\mathcal{C}_{j,n}}.
\end{aligned}$$

Using Lemma 10, we have

$$\begin{aligned}
&\sum_{j=1}^{h_n}\left|\mathbb{E}_{j-1}\left[\frac{\sqrt{\Delta_n}}{k_n} f''\left(p_{(j-1)\Delta_n}\right)\alpha_j^n\right]\right| = 0, \\
&k_n \sum_{j=1}^{h_n}\mathbb{E}\left[\frac{\Delta_n}{k_n^2}\left(f''\left(p_{(j-1)\Delta_n}\right)\right)^2\left|\alpha_j^n\right|^2\right] \le C\,k_n^{-2}, \\
&\sum_{j=1}^{h_n}\left|\mathbb{E}_{j-1}\left[\frac{\sqrt{\Delta_n}}{k_n} f''\left(p_{(j-1)\Delta_n}\right)\beta_j^n\right]\right| \le C\,\Delta_n^{1/2}, \\
&k_n \sum_{j=1}^{h_n}\mathbb{E}\left[\frac{\Delta_n}{k_n^2}\left(f''\left(p_{(j-1)\Delta_n}\right)\right)^2\left|\beta_j^n\right|^2\right] \le C\,\Delta_n,
\end{aligned}$$

which imply, through Lemma 2, that $\mathcal{A}_{j,n}$ is AN. Now since

$$\mathcal{B}_{j,n} = \frac{\sqrt{\Delta_n}}{2\,k_n} f''\left(p_{j-1,n}\right)\left(\alpha_j^n+\beta_j^n\right)\left(\widehat{p}_j(k_n)+p_{j-1,n}\right) = \mathcal{A}_{j,n}\ \left(\widehat{p}_j(k_n)+p_{j-1,n}\right)$$

and being $(\widehat{p}_j(k_n)+p_{j-1,n})$ bounded, we can apply to $\mathcal{B}_{j,n}$ the same reasoning used for $\mathcal{A}_{j,n}$, whence $\mathcal{B}_{j,n}$ is AN. An identical reasoning applies to $\mathcal{C}_{j,n}$, which is then AN as well.

Part 4: Proof of the convergence $U_1^n(4) \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma)$

Recall that $U(4)_1^n$ is defined as

$$U(4)_1^n = \frac{\sqrt{\Delta_n}}{k_n}\sum_{j=1}^{h_n+1} f'\left(p_{j-1,n}\right)\sum_{i=0}^{k_n-1}\mathbb{B}_{j+i,n}.$$

For the sake of readability define, temporarily, the variables

$$a_{j-1} = f'\left(p_{j-1,n}\right), \mathbb{B}_{j+i} = \mathbb{B}_{j+i,n} - p_{j+i,n}$$

so that

$$U\left(4\right)_1^n = \frac{\sqrt{\Delta_n}}{k_n}\sum_{j=1}^{n-k_n+1} a_{j-1}\sum_{i=0}^{k_n-1}\mathbb{B}_{j+i}.$$

The convolution of summation in $U\left(4\right)_1^n$ can be re-written as

$$\begin{aligned}
\sum_{j=1}^{n-k_n+1} a_{j-1}\sum_{i=0}^{k_n-1}\mathbb{B}_{j+i} &= a_0\left(\mathbb{B}_1+\mathbb{B}_2+\cdots+\mathbb{B}_{k_n}\right)+a_1\left(\mathbb{B}_2+\mathbb{B}_3+\cdots+\mathbb{B}_{k_n+1}\right)+\cdots \\
&\cdots +a_{k_n-1}\left(\mathbb{B}_{k_n}+\mathbb{B}_{k_n+1}+\cdots+\mathbb{B}_{2\,k_n-1}\right)+a_{k_n}\left(\mathbb{B}_{k_n+1}+\mathbb{B}_{k_n+2}+\cdots+\mathbb{B}_{2\,k_n}\right)+\cdots \\
&\cdots +a_{n-k_n-1}\left(\mathbb{B}_{n-k_n}+\mathbb{B}_{n-k_n+1}+\cdots+\mathbb{B}_{n-1}\right)+a_{n-k_n}\left(\mathbb{B}_{n-k_n+1}+\mathbb{B}_{n-k_n+1}+\cdots+\mathbb{B}_{n}\right) \\
&= \mathbb{B}_1\,a_0+\mathbb{B}_2\left(a_0+a_1\right)+\mathbb{B}_3\left(a_0+a_1+a_2\right)+\cdots+\mathbb{B}_{k_n}\left(a_0+a_1+a_2+\ldots+a_{k_n-1}\right) \\
&+ \mathbb{B}_{k_n+1}\left(a_1+a_2+a_3+\ldots+a_{k_n}\right)+\mathbb{B}_{k_n+2}\left(a_2+a_3+a_4+\ldots+a_{k_n+1}\right)+\cdots \\
&+ \mathbb{B}_{n-k_n+1}\left(a_{n-2\,k_n+1}+a_{n-2\,k_n+1}+\ldots+a_{n-k_n}\right) \\
&+ \mathbb{B}_{n-k_n+2}\left(a_{n-2\,k_n+2}+a_{n-2\,k_n+3}+\ldots+a_{n-k_n}\right)+\cdots+\mathbb{B}_{n-1}\left(a_{n-k_n-1}+a_{n-k_n}\right)+\mathbb{B}_n\,a_{n-k_n} \\
&= \sum_{j=1}^{k_n}\mathbb{B}_j\sum_{i=0}^{j-1}a_i+\sum_{j=k_n+1}^{n-k_n+1}\mathbb{B}_j\sum_{i=j-k}^{j-1}a_i+\sum_{j=n-k_n+1}^{n}\mathbb{B}_j\sum_{i=j-k_n}^{n-k_n}a_i \\
(i\to j-i-1)\quad &= \sum_{j=1}^{k_n}\mathbb{B}_j\sum_{j=0}^{i-1}a_{j-i-1}+\sum_{j=k_n+1}^{n-k_n+1}\mathbb{B}_j\sum_{i=0}^{k_n-1}a_{j-i-1}+\sum_{j=n-k_n+1}^{n}\mathbb{B}_j\sum_{i=j-n+k_n-1}^{k_n-1}a_{j-i-1} \\
&= \sum_{j=1}^{n}\sum_{i=j-n+k_n-1\vee 0}^{(j-1)\wedge(k_n-1)}a_{j-i-1}\mathbb{B}_j.
\end{aligned}$$

Hence,

$$\begin{aligned}
U\left(4\right)^n &= \sqrt{\Delta_n}\sum_{j=1}^{n}\frac{1}{k_n}\sum_{i=j-n+k_n-1\vee 0}^{(j-1)\wedge(k_n-1)}f'\left(p_{j-i-1,n}\right)\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right) \\
&= \sqrt{\Delta_n}\sum_{j=1}^{n}\left(\left(\frac{1}{k_n}\sum_{i=j-n+k_n-1\vee 0}^{(j-1)\wedge(k_n-1)}f'\left(p_{j-i-1,n}\right)\right)-f'\left(p_{j-1,n}\right)+f'\left(p_{j-1,n}\right)\right)\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right) \\
&= \sqrt{\Delta_n}\sum_{j=1}^{n}f'\left(p_{j-1,n}\right)\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right)+\sqrt{\Delta_n}\sum_{j=1}^{n}w_j^n\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right),
\end{aligned}$$

where

$$w_j^n = \frac{1}{k_n}\sum_{i=j-n+k_n-1\vee 0}^{(j-1)\wedge(k_n-1)}f'\left(p_{j-i-1,n}\right)-f'\left(p_{j-1,n}\right).$$

By conditioning on $(p_t)_{t\in[0,1]}$, $\mathbb{E}\left[w_j^n\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right)\right]=0$. Next, by (**??**),

$$\left|w_j^n\right|\leq C\sup_{s\in[(j-1)\Delta_n,(j+k_n-1)\Delta_n]}\left|p_s-p_{j-1,n}\right|.$$

Hence, inequality (32) implies that $\mathbb{E}\left[\left|w_j^n\right|^2\right]\leq C\sqrt{\Delta_n}$ when $k_n\leq j\leq\lfloor 1/\Delta_n\rfloor-k_n$ and $\left|w_j^n\right|\leq C$ always. Therefore, since both $\mathbb{B}_{j,n}$ and p_t are bounded,

$$\mathbb{E}_{j-1}\left[\left|\sqrt{\Delta_n}w_j^n\left(\mathbb{B}_{j,n}-p_{j\Delta_n}\right)\right|^2\right]\leq\begin{cases}C\Delta_n^{3/2} & k_n\leq j\leq h_n,\\ C\Delta_n & \text{otherwise.}\end{cases}$$

Consequently, $\sum_{j=1}^{\lfloor 1/\Delta_n \rfloor} \mathbb{E}_{j-1}\left[\left|\sqrt{\Delta_n} w_j^n \left(\mathbb{B}_{j,n} - p_{j\Delta_n}\right)\right|^2\right] \longrightarrow 0$, which, by Lemma 2, implies that $\sqrt{\Delta_n}\sum_{j=1}^{\lfloor 1/\Delta_n \rfloor} w_j^n \left(\mathbb{B}_{j,n} - p_{j\Delta_n}\right)$ is AN. Now, set $\xi_j^n = \sqrt{\Delta_n} f'\left(p_{j-1,n}\right)\left(\mathbb{B}_{j,n} - p_{j\Delta_n}\right)$. Clearly, $\mathbb{E}\left[\xi_j^n\right] = 0$ and we have:

$$\mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] = \Delta_n \left(f'\left(p_{j-1,n}\right)\right)^2 \mathbb{E}_{j-1}\left[p_{j\Delta_n} - \left(p_{j\Delta_n}\right)^2\right].$$

Since, $\left(f'\left(p_{j-1,n}\right)\right)^2$ is bounded, using (33) we have:

$$\left|\mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] - \Delta_n \left(f'\left(p_{j-1,n}\right)\right)^2 \left(p_{j-1,n} - \left(p_{j-1,n)^2}\right)\right| \leq C(\Delta_n)^2.$$

Hence,

$$\sum_{j=1}^{\lfloor 1/\Delta_n \rfloor} \mathbb{E}_{j-1}\left[\left(\xi_j^n\right)^2\right] \xrightarrow{P} \int_0^1 f'\left(p_s\right)^2 p_s(1-p_s)\, ds.$$

Consequently,

$$\sum_{j=1}^{\lfloor 1/\Delta_n \rfloor} \xi_j^n \overset{stably}{\Longrightarrow} \mathcal{MN}(0,\Sigma),$$

which completes the proof. □

A.4 Proof of Theorem 3.6 from Section 3.3

For any process X, denote the increments by $\Delta_j^n X = X_{(j+1)\,\Delta_n} - X_{j\,\Delta_n}$. Set $k_n = \theta\left\lfloor\sqrt{n}\right\rfloor$ and define

$$\mathrm{IV}_n = \sum_{i=1}^{n-2k_n+1} \left(\widehat{p}_{i+k_n}\left(k_n\right) - \widehat{p}_i\left(k_n\right)\right)^2.$$

Then, we have to prove that, as $n \to \infty$,

$$k_n^{-1}\,\mathrm{IV}_n \xrightarrow{p} \frac{2}{3}\int_0^1 \nu_s^2\, ds + \frac{2}{\theta^2}\int_0^1 p_s\left(1-p_s\right)\, ds.$$

We have

$$\widehat{p}_j\left(k_n\right) = \frac{1}{k_n}\sum_{i=0}^{k_n-1}\left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right) + \frac{1}{k_n}\sum_{i=0}^{k_n-1} p_{j+i,n}.$$

Consequently, the difference between $\widehat{p}_{j+k_n}\left(k_n\right)$ and $\widehat{p}_j\left(k_n\right)$ can be expressed as

$$\widehat{p}_{j+k_n}\left(k_n\right) - \widehat{p}_j\left(k_n\right) = \frac{1}{k_n}\sum_{i=0}^{2k_n-1}\epsilon(1)_i\left(\mathbb{B}_{j+i,n} - p_{j+i,n}\right) + \frac{1}{k_n}\sum_{i=0}^{k_n-1}\left(p_{j+i+k_n,n} - p_{j+i,n}\right), \tag{48}$$

where, for $m \in \{0,\ldots,2k_n-1\}$,

$$\epsilon(1)_m = \begin{cases} -1, & 0 \leq m < k_n, \\ +1, & k_n \leq m < 2k_n. \end{cases}$$

Then, using telescopic sums, notice that

$$\left(p_{j+i+k_n,n} - p_{j+i,n}\right) = \sum_{\ell=0}^{k_n-1}\Delta_{j+i+\ell,n}p.$$

Now note that the sum $S_{j,n}=\sum_{i=0}^{k_n-1}\left(p_{j+i+k_n,n}-p_{j+i,n}\right)$, collecting identical terms, becomes

$$
\begin{aligned}
S_{j,n} \;&=\; \Delta_j^n p+\Delta_{j+1}^n p+\Delta_{j+2}^n p+\ldots+\Delta_{j+k_n-1}^n p \\
&\qquad +\Delta_{j+1}^n p+\Delta_{j+2}^n p+\ldots+\Delta_{j+k_n-1}^n p+\Delta_{j+k_n}^n p \\
&\qquad\qquad +\Delta_{j+2}^n p+\ldots+\Delta_{j+k_n-1}^n p+\Delta_{j+k_n}^n p+\Delta_{j+k_n+1}^n p \\
&\qquad\qquad\qquad \vdots \\
&\qquad\qquad\qquad +\Delta_{j+k_n-1}^n p+\Delta_{j+k_n}^n p+\Delta_{j+k_n+1}^n p+\ldots+\Delta_{j+2\,k_n-2}^n p \\
&= \underbrace{\Delta_j^n p+2\,\Delta_{j+1}^n p+3\,\Delta_{j+2}^n p+\ldots+k_n\,\Delta_{j+k_n-1}^n p}_{k_n \text{ terms}}+\underbrace{(k_n-1)\;\Delta_{j+k_n}^n p+\ldots+\Delta_{j+2\,k_n-2}^n p}_{k_n-1 \text{ terms}},
\end{aligned}
$$

which can be re-written as

$$
\frac{1}{k_n}\sum_{i=0}^{k_n-1}\left(p_{j+i+k_n,n}-p_{j+i,n}\right) \;=\; \frac{1}{k_n}\sum_{i=0}^{2k_n-1}\epsilon(2)_i\,\left(p_{j+i+1,n}-p_{j+i,n}\right),
$$

where, for $i\in\{0,\ldots,2k_n-1\}$,

$$
\epsilon(2)_i=(i+1)\wedge(2k_n-i-1),
$$

and, in particular, $\epsilon(2)_{2k_n-1}=0$. Now expression (48) becomes

$$
\widehat{p}_{j+k_n}(k_n)-\widehat{p}_j(k_n)=\frac{1}{k_n}\sum_{j=0}^{2k_n-1}\left(\epsilon(2)_i\left(\mathbb{B}_{i+j,n}-p_{j+i,n}\right)+\epsilon(2)_i\,\left(p_{j+i+1,n}-p_{j+i,n}\right)\right),
$$

whence

$$
\begin{aligned}
\left(\widehat{p}_{j+k_n}(k_n)-\widehat{p}_j(k_n)\right)^2 \;&=\; \frac{1}{k_n^2}\sum_{i=0}^{2k_n-1}\Bigg(\epsilon(2)_i^2\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)^2+\epsilon(2)_i^2\,\left(p_{j+i+1,n}-p_{j+i,n}\right)^2 \\
&+\; 2\,\epsilon(2)_i\,\epsilon(2)_i\,\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(p_{j+i+1,n}-p_{j+i,n}\right)\Bigg) \\
&+\; 2\sum_{j=0}^{2\,k_n-2}\sum_{\ell'=j+1}^{2\,k_n-1}\Bigg(\epsilon(2)_i\epsilon(1)_\ell\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(\mathbb{B}_{j+\ell,n}-p_{j+\ell,n}\right) \\
&+\; \epsilon(2)_i\,\epsilon(2)_\ell\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(p_{j+\ell+1,n}-p_{j+\ell,n}\right) \\
&+\; \epsilon(1)_\ell\,\epsilon(2)_i\left(\mathbb{B}_{j+\ell,n}-p_{j+\ell,n}\right)\left(p_{j+i+1,n}-p_{j+i,n}\right) \\
&+\; \epsilon(2)_i\,\epsilon(2)_\ell\,\left(p_{j+i+1,n}-p_{j+i,n}\right)\left(p_{j+\ell+1,n}-p_{j+\ell,n}\right)\Bigg)
\end{aligned}
\tag{49}
$$

So, setting

$$
\zeta(1)_j=\mathbb{B}_{j,n}-p_{j,n},\;\;\zeta(2)_j=p_{j+1,n}-p_{j,n}.
$$

we have the following more compact expression

$$
\left(\widehat{p}_{j+k_n}(k_n)-\widehat{p}_j(k_n)\right)^2=\frac{1}{k_n^2}\sum_{u,v=1}^{2}\left(\sum_{i=0}^{2k_n-1}\epsilon(u)_i\epsilon(v)_i\zeta(u)_{j+i}\zeta(v)_{j+i}+2\sum_{i=0}^{2k_n-2}\sum_{l=j+1}^{2k_n-1}\epsilon(u)_i\epsilon(v)_l\zeta(u)_{j+i}\zeta(v)_{i+l}\right).
$$

Consequently, IV_n can be expressed as

$$
\mathrm{IV}_n=\sum_{s=1}^{7}\sum_{i=1}^{n-2k_n+1}v_i^n(s),
$$

where

$$
\begin{aligned}
v_i^n(1) &= \frac{1}{k_n^2}\sum_{i=0}^{2k_n-1}\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)^2, \quad v_i^n(2) = \frac{1}{k_n^2}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^2\left(p_{j+i+1,n}-p_{j+i,n}\right)^2, \\
v_i^n(3) &= \frac{2}{k_n^2}\sum_{i=0}^{2k_n-1}\epsilon(1)_i\epsilon(2)_i\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(p_{j+i+1,n}-p_{j+i,n}\right), \\
v_i^n(4) &= \frac{2}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\epsilon(1)_i\epsilon(1)_l\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(\mathbb{B}_{j+l,n}-p_{j+l,n}\right), \\
v_i^n(5) &= \frac{2}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=j+1}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l\left(p_{j+i+1,n}-p_{j+i,n}\right)\left(p_{j+l+1,n}-p_{j+l,n}\right), \\
v_i^n(6) &= \frac{2}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=j+1}^{2k_n-1}\epsilon(1)_i\epsilon(2)_l\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)\left(p_{j+l+1,n}-p_{j+l,n}\right), \\
v_i^n(7) &= \frac{2}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=j+1}^{2k_n-1}\epsilon(2)_i\epsilon(1)_l\left(p_{j+i+1,n}-p_{j+i,n}\right)\left(\mathbb{B}_{j+l+1,n}-p_{j+l,n}\right).
\end{aligned}
$$

Consequently, in order to study the convergence of IV_n in probability, we need study the convergence of the sums $\sum_{j=1}^{n-2k_n+1} v_j^n(s)$ for $s=1,\dots 7$. In what follows we use the abbreviation $g_n = n-2k_n+1$. For sake or readability, we divide the rest of the proof in seven parts.

Part 1: Proof of the convergence in probability of $v_i^n(1)$

The quantity $\frac{1}{k_n}\sum_{j=1}^{g_n} v_j^n(1)$ can be decomposed as

$$
\frac{1}{k_n}\sum_{j=1}^{g_n} v_j^n(1) = \sum_{j=1}^{g_n} d_{j,1}^{(n)} + \sum_{j=1}^{g_n} d_{j,2}^{(n)},
$$

where

$$
d_{j,1}^{(n)} = \frac{1}{k_n^3}\sum_{j=0}^{2k_n-1}\Big(\left(\mathbb{B}_{j+i,n}-p_{j+i,n}\right)^2 - p_{i-1,n}\left(1-p_{i-1,n}\right)\Big), \quad d_{j,2}^{(n)} = \frac{1}{k_n^3}\sum_{j=0}^{2k_n-1} p_{i-1,n}\left(1-p_{i-1,n}\right).
$$

First, we show that $\sum_{j=1}^{g_n} d_{j,1}^{(n)}$ is AN. We have

$$
\begin{aligned}
\sum_{j=1}^{g_n}\left|\mathbb{E}_{j-1}\left[d_{j,1}^{(n)}\right]\right| &= \sum_{j=1}^{g_n}\frac{1}{k_n^3}\sum_{j=0}^{2k_n-1}\left|\mathbb{E}_{j-1}\left[p_{j+i,n}-p_{j-1,n}+p_{j-1,n}^2-p_{j+i,n}^2\right]\right| \\
&\le \sum_{j=1}^{g_n}\frac{1}{k_n^3}\sum_{i=0}^{2k_n-1}\left(\left|\mathbb{E}_{j-1}\left[p_{j+i,n}-p_{j-1,n}\right]\right| + \left|\mathbb{E}_{j-1}\left[p_{j+i,n}^2-p_{j-1,n}^2\right]\right|\right) \\
&= \sum_{j=1}^{g_n}\frac{1}{k_n^3}\sum_{i=0}^{2k_n-1}\left(\left|\mathbb{E}_{j-1}\left[p_{j+i,n}-p_{j-1,n}\right]\right| + \left|\mathbb{E}_{j-1}\left[\left(p_{j+i,n}+p_{j-1,n}\right)\left(p_{j+i,n}-p_{j-1,n}\right)\right]\right|\right) \\
&\le C\sum_{j=1}^{g_n}\frac{1}{k_n^3}\sum_{i=0}^{2k_n-1} k_n\Delta_n = C\,\frac{k_n\Delta_n(2k_n-1)g_n}{k_n^3} \sim \frac{1}{k_n} \longrightarrow 0,
\end{aligned}
$$

where we use conditioning on $(p_t)_{t\in[0,1]}$, triangular inequality and Lemma 10. Next, using the boundedness of p_t, we obtain

$$k_n \sum_{j=1}^{g_n} \mathbb{E}_{j-1}\left[\left|d_{j,1}^{(n)}\right|^2\right] \leq k_n \sum_{j=1}^{g_n} \frac{1}{k_n^6}\left(\sum_{i=0}^{2k_n-1} C\right)^2 = C\,\frac{(2k_n-1)^2\,g_n}{k_n^5} \sim \frac{1}{k_n^3\,\Delta_n} \longrightarrow 0.$$

Consequently, by Lemma 2, $\sum\limits_{j=1}^{g_n} d_{j,1}^{(n)}$ is AN. Now, consider $\sum\limits_{j=1}^{g_n} d_{j,2}^{(n)}$. We have

$$\sum_{j=1}^{g_n} d_{j,2}^{(n)} = \frac{2}{k_n^2}\sum_{j=1}^{g_n}\frac{1}{2k_n}\sum_{j=0}^{2k_n-1} p_{j-1,n}(1-p_{j-1,n}) = \frac{2}{\theta^2}\sum_{j=1}^{g_n} p_{j-1,n}(1-p_{j-1,n})\frac{1}{\lfloor\sqrt{n}\rfloor^2} \longrightarrow \frac{2}{\theta^2}\int_0^1 p_s(1-p_s)\,ds,$$

where the convergence is point-wise, by Riemann integrability. Combining the two results, we obtain:

$$\frac{1}{k_n}\sum_{j=1}^{g_n} v_j^n(1) \stackrel{u.c.p.}{\longrightarrow} \frac{2}{\theta^2}\int_0^1 p_s(1-p_s)\,ds. \tag{50}$$

Part 1: Proof of the convergence in probability of $v_i^n\,(2)$

First note that $v_j^n\,(2)$ can be written as

$$v_j^n\,(2) = \frac{1}{k_n^2}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^2\left(\Delta_{j+i}^n p\right)^2 = \frac{1}{k_n^2}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^2\left(\Delta_j^n p\right)^2 + \frac{1}{k_n^2}\sum_{i=1}^{2k_n-1}\epsilon(2)_i^2\left[\left(\Delta_{j+i}^n p\right)^2 - \left(\Delta_j^n p\right)^2\right],$$

so that the sum over the index i of all the $v_j^n\,(2)$ becomes

$$\frac{1}{k_n}\sum_{j=0}^{g_n} v_j^n\,(2) = \underbrace{\frac{1}{k_n^3}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^2\sum_{j=0}^{g_n}\left(\Delta_j^n p\right)^2}_{\mathcal{A}_n} + \underbrace{\frac{1}{k_n^3}\sum_{j=0}^{g_n}\sum_{i=1}^{2k_n-1}\epsilon(2)_i^2\left[\left(\Delta_{j+i}^n p\right)^2 - \left(\Delta_j^n p\right)^2\right]}_{\mathcal{B}_n}.$$

Now we want to prove that $\mathcal{A}_n$ converges in probability to a finite quantity, while $\mathcal{B}_n$ is AN. Using the definition of the integers coefficients $\epsilon(2)_i$ it is easy to show that

$$\frac{1}{k_n^3}\sum_{j=0}^{2\,k_n-1}\epsilon(2)_i^2 = \frac{1}{3\,k_n^3}\left(2k_n^3 + k_n\right) \longrightarrow \frac{2}{3}.$$

Hence, the standard theory of realized volatility for the semimartingale

$$p_t = p_0 + \int_0^t \mu_s\,ds + \int_0^t \nu_s\,dW_s$$

now implies that

$$\mathcal{A}_n \stackrel{p}{\longrightarrow} \frac{2}{3}\int_0^1 \nu_s^2\,ds.$$

Concerning $\mathcal{B}_n$, we write it as

$$\mathcal{B}_n = \sum_{j=0}^{g_n}\vartheta_{j+1,n} \text{ with } \vartheta_{j+1,n} = \frac{1}{k_n^3}\sum_{i=1}^{2k_n-1}\epsilon(2)_i^2\left[\left(\Delta_{j+i}^n p\right)^2 - \left(\Delta_j^n p\right)^2\right],$$

and, by Markov inequality, the Itô isometry and the boundedness of[9] ν

$$\int_0^{\Delta} \nu_s \, dW_s = \nu_0 \left(W_\Delta - W_0 \right) + O_p(\Delta^{1/2}), \tag{51}$$

whence, considering also that $\int_0^t \mu_s \, ds$ is $O_p\left(\Delta_n\right)$ for bounded μ , we have

$$\begin{aligned} p_{j+1,n} - p_{j,n} &= \int_{j\Delta_n}^{(j+1)\,\Delta_n} \mu_s \, ds + \int_{j\Delta_n}^{(j+1)\,\Delta_n} \nu_s \, dW_s = \left(\nu_{j,n} + O_p\left(\sqrt{\Delta_n} \right) \right) \left(W_{j+1,n} - W_{j,n} \right) + O_p(\Delta_n) \\ &= \nu_{j,n} \left(W_{j+1,n} - W_{j,n} \right) + O_p\left(\Delta_n^{1/2} \right) \left(W_{j+1,n} - W_{j,n} \right) + O_p\left(\Delta_n \right). \end{aligned}$$

The square of the increment $\Delta_j^n p = \left(p_{j+1,n} - p_{j,n} \right)$ then becomes

$$\begin{aligned} \left(\Delta_j^n p \right)^2 &= \nu_{j,n}^2 \left(\Delta_j^n W \right)^2 + \left(\Delta_j^n W \right)^2 O_p(\Delta_n) + O_p\left(\Delta_n^2 \right) + \left(\Delta_j^n W \right)^2 O_p\left(\Delta_n^{1/2} \right) + \left(\Delta_j^n W \right) O_p\left(\Delta_n \right) + \left(\Delta_j^n W \right) O_p\left(\Delta_n^{3/2} \right) \\ &= \nu_{j,n}^2 \left(\Delta_j^n W \right)^2 + O_p\left(\Delta_n^2 \right) + \left(\Delta_j^n W \right)^2 O_p\left(\Delta_n^{1/2} \right) + \left(\Delta_j^n W \right) O_p\left(\Delta_n \right), \end{aligned}$$

which, by preserving only the leading terms, can be further simplified into

$$\left(\Delta_j^n p \right)^2 = \nu_{j\,\Delta_n}^2 \left(\Delta_j^n W \right)^2 + O_p\left(\Delta_n^{1/2} \right) \left(\Delta_j^n W \right)^2 + \nu_{j,n} \left(\Delta_j^n W \right) O_p\left(\Delta_n \right), \tag{52}$$

so that

$$\mathbb{E}_j \left[\left(\Delta_j^n p \right)^2 \right] = \nu_{j,n}^2 \, \Delta_n + O_p(\Delta_n^{3/2}).$$

Now consider the same increment shifted by i units

$$\begin{aligned} \left(\Delta_{j+i}^n p \right)^2 &= \nu_{i+j,n}^2 \left(\Delta_{j+i}^n W \right)^2 + O_p\left(\Delta_n^2 \right) + \left(\Delta_{j+i}^n W \right)^2 O_p\left(\Delta_n^{1/2} \right) + \left(\Delta_{j+i}^n W \right) O_p\left(\Delta_n \right) \\ &= \left(\nu_{j\,\Delta_n}^2 + O_p\left(j\,\Delta_n \right) + O_p\left(\sqrt{j\,\Delta_n} \right) \right) \left(\Delta_{j+i}^n W \right)^2 + O_p\left(\Delta_n^2 \right) + \left(\Delta_{j+i}^n W \right)^2 O_p\left(\Delta_n^{1/2} \right) + \left(\Delta_{j+i}^n W \right) O_p\left(\Delta_n \right) \\ &= \nu_{j\,\Delta_n}^2 \left(\Delta_{j+i}^n W \right)^2 + \left(\Delta_{j+i}^n W \right)^2 O_p\left(\sqrt{j\,\Delta_n} \right) + O_p\left(\Delta_n^2 \right) + \left(\Delta_{j+i}^n W \right)^2 O_p\left(\Delta_n^{1/2} \right) + \left(\Delta_{j+i}^n W \right) O_p\left(\Delta_n \right), \end{aligned}$$

which, by preserving only the leading terms, can be further simplified into

$$\left(\Delta_{j+i}^n p \right)^2 = \nu_{j,n}^2 \left(\Delta_{j+i}^n W \right)^2 + \left(\Delta_{j+i}^n W \right)^2 O_p\left(\sqrt{i\,\Delta_n} \right) \tag{53}$$

and so

$$\mathbb{E}_j \left[\left(\Delta_{j+i}^n p \right)^2 \right] = \nu_{j,n}^2 \, \Delta_n + O_p\left(i^{1/2} \, \Delta_n^{3/2} \right).$$

Therefore the $\mathcal{F}_{t_{i,n}}$-conditional expected value of the difference between $\left(\Delta_{j+i}^n p \right)^2$ and $\left(\Delta_j^n p \right)^2$ has the following order in probability

$$\mathbb{E}_j \left[\left(\Delta_{j+i}^n p \right)^2 - \left(\Delta_j^n p \right)^2 \right] = O_p\left(i^{1/2} \, \Delta_n^{3/2} \right),$$

implying that

$$\sum_{j=0}^{g_n} \mathbb{E}_j \left[\vartheta_{j+1,n} \right] = \frac{1}{k_n^3} \sum_{j=0}^{g_n} \sum_{i=0}^{2k_n-1} \epsilon(2)_i^2 O_p\left(i^{1/2} \, \Delta_n^{3/2} \right) = O_p\left(\left(k_n \, \Delta_n \right)^{1/2} \right) \xrightarrow{p} 0,$$

[9]Here we follow the standard approach

$$\mathbb{P}\left(\left| \frac{1}{\sqrt{\Delta}} \left[\int_0^{\Delta} \nu_s \, dW_s - \nu_0 (W_\Delta - W_0) \right] \right| > M \right) \leq \frac{1}{M^2 \Delta} \mathbb{E}\left(\left| \int_0^{\Delta} (\nu_s - \nu_0) \, dW_s \right|^2 \right) = \frac{1}{M^2 \Delta} \mathbb{E}\left(\int_0^{\Delta} (\nu_s - \nu_0)^2 \, ds \right),$$

and then the identity (51) follows from the boundedness of ν.

which is the first of the two conditions in Lemma 2 that guarantee AN. To prove that also the second condition is satisfied consider

$$k_n\,\vartheta^2_{j+1,n} = \underbrace{\frac{1}{k_n^5}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^4\left[\left(\Delta^n_{j+i}p\right)^2-\left(\Delta^n_j p\right)^2\right]^2}_{\mathcal{C}_{i,n}} + \underbrace{\frac{2}{k_n^5}\sum_{i=0}^{2k_n-2}\sum_{\ell=j+1}^{2k_n-1}\epsilon(2)_i^2\,\epsilon(2)_\ell^2\left[\left(\Delta^n_{j+i}p\right)^2-\left(\Delta^n_j p\right)^2\right]\left[\left(\Delta^n_{j+\ell}p\right)^2-\left(\Delta^n_j p\right)^2\right]}_{\mathcal{D}_{i,n}}.$$

From (52) we get

$$\begin{aligned}\left(\Delta^n_j p\right)^4 &= \nu^4_{j,n}\left(\Delta^n_j W\right)^4 + O_p(\Delta_n)\left(\Delta^n_j W\right)^4 + \nu^2_{j,n}\left(\Delta^n_j W\right)^2 O_p\left(\Delta_n^2\right) + 2\,\nu^2_{j,n}\left(\Delta^n_j W\right)^4 O_p\left(\Delta_n^{1/2}\right)\\ &+ 2\,\nu^3_{j,n}\left(\Delta^n_j W\right)^3 O_p\left(\Delta_n\right) + 2\,\nu_{j,n}\left(\Delta^n_j W\right)^3 O_p\left(\Delta_n^{3/2}\right)\end{aligned}$$

and hence

$$\mathbb{E}_j\left[\left(\Delta^n_j p\right)^4\right] = 3\,\nu^4_{j,n}\,\Delta_n^2 + O_p(\Delta_n^3) + O_p(\Delta_n^3) + O_p(\Delta_n^{5/2}) = 3\,\nu^4_{j,n}\,\Delta_n^2 + O_p(\Delta_n^{5/2}).$$

Similarly from (53) we get

$$\left(\Delta^n_{j+i}p\right)^4 = \nu^4_{j,n}\left(\Delta^n_{j+i}W\right)^4 + \left(\Delta^n_{j+i}W\right)^4 O_p\left(j\,\Delta_n\right) + 2\,\nu^2_{j,n}\left(\Delta^n_{j+i}W\right)^4 O_p\left(\sqrt{i\,\Delta_n}\right)$$

and hence

$$\mathbb{E}_j\left[\left(\Delta^n_{j+i}p\right)^4\right] = 3\,\nu^4_{j,n}\,\Delta_n^2 + O_p\left(i\,\Delta_n^3\right) + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right) = 3\,\nu^4_{j,n}\,\Delta_n^2 + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right).$$

Summing up the two fourth powers so obtained

$$\mathbb{E}_j\left[\left(\Delta^n_{j+i}p\right)^4 + \left(\Delta^n_{j+i}p\right)^4\right] = 6\,\nu^4_{j,n}\,\Delta_n^2 + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right).$$

Finally consider that

$$\begin{aligned}\left(\Delta^n_{j+i}p\right)^2\left(\Delta^n_j p\right)^2 &= \left(\nu^2_{j,n}\left(\Delta^n_{j+i}W\right)^2 + \left(\Delta^n_{j+i}W\right)^2 O_p\left(\sqrt{i\,\Delta_n}\right)\right)\times\\ &\quad\left(\nu^2_{j,n}\left(\Delta^n_j W\right)^2 + \left(\Delta^n_j W\right)^2 O_p\left(\Delta_n^{1/2}\right) + \nu_{j,n}\left(\Delta^n_j W\right) O_p\left(\Delta_n\right)\right)\\ &= \nu^4_{j,n}\left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right)^2 + \nu^2_{j,n}\left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right)^2 O_p\left(\Delta_n^{1/2}\right) + \nu^3_{j,n}\left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right) O_p\left(\Delta_n\right)\\ &+ \nu^2_{j,n}\left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right)^2 O_p\left(\sqrt{i\,\Delta_n}\right) + \left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right)^2 O_p\left(i^{1/2}\,\Delta_n\right)\\ &+ \nu_{j,n}\left(\Delta^n_{j+i}W\right)^2\left(\Delta^n_j W\right) O_p\left(i^{1/2}\,\Delta_n^{3/2}\right),\end{aligned}$$

whence

$$\mathbb{E}_j\left[\left(\Delta^n_{j+i}p\right)^2\left(\Delta^n_j p\right)^2\right] = \nu^4_{j,n}\,\Delta_n^2 + O_p\left(\Delta_n^{5/2}\right) + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right) + O_p\left(i^{1/2}\,\Delta_n^{3}\right) = \nu^4_{j,n}\,\Delta_n^2 + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right).$$

and so

$$\mathbb{E}_j\left[\left(\Delta^n_{j+i}p\right)^4 + \left(\Delta^n_{j+i}p\right)^4 - 2\left(\Delta^n_{j+i}p\right)^2\left(\Delta^n_j p\right)^2\right] = 6\,\nu^4_{j,n}\,\Delta_n^2 + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right),$$

which implies

$$\sum_{i=0}^{g_n}\mathbb{E}\left[\mathcal{C}_{j,n}\right] = \frac{1}{k_n^5}\sum_{i=0}^{g_n}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^4\,\mathbb{E}\left[\left(\left(\Delta^n_{j+i}p\right)^2-\left(\Delta^n_j p\right)^2\right)^2\right] = \frac{1}{k_n^5}\sum_{i=0}^{g_n}\sum_{i=0}^{2k_n-1}\epsilon(2)_i^4\,\mathbb{E}\left[\nu^4_{j,n}\,\Delta_n^2 + O_p\left(i^{1/2}\,\Delta_n^{5/2}\right)\right] = O\left(\Delta_n\right)\longrightarrow 0.$$

Concerning $\mathcal{C}_{j,n}$, first call $\varepsilon_{i,\ell}^2 = \epsilon(2)_i^2\,\epsilon(2)_\ell^2$ and then note that

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{D}_{j,n}|\right] &= \frac{2}{k_n^5}\sum_{i=0}^{2k_n-2}\sum_{\ell=i+1}^{2k_n-1}\varepsilon_{i,\ell}^2\,\mathbb{E}\left[\left|\left(\Delta_{j+i}^n p\right)^2-\left(\Delta_j^n p\right)^2\right|\,\left|\left(\Delta_{j+\ell}^n p\right)^2-\left(\Delta_j^n p\right)^2\right|\right]\\
&\le \frac{2}{k_n^5}\sum_{i=0}^{2k_n-2}\sum_{\ell=i+1}^{2k_n-1}\varepsilon_{j,\ell}^2\left(\mathbb{E}\left[\left(\left(\Delta_{j+i}^n p\right)^2-\left(\Delta_j^n p\right)^2\right)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\left(\Delta_{j+\ell}^n p\right)^2-\left(\Delta_j^n p\right)^2\right)^2\right]\right)^{\frac{1}{2}}\\
&= \frac{2}{k_n^5}\sum_{i=0}^{2k_n-2}\sum_{\ell=i+1}^{2k_n-1}\varepsilon_{j,\ell}^2\left(\mathbb{E}\left[6\,\nu_{j,n}^4\,\Delta_n^2+O_p\left(j^{\frac{1}{2}}\,\Delta_n^{\frac{5}{2}}\right)\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[6\,\nu_{j,n}^4\,\Delta_n^2+O_p\left(\ell^{\frac{1}{2}}\,\Delta_n^{\frac{5}{2}}\right)\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\epsilon(2)_i^2\,\epsilon(2)_\ell^2 \le C\,k_n^4$ we get

$$\mathbb{E}\left[|\mathcal{D}_{j,n}|\right] \le C\,k_n\,\Delta_n^2$$

so that

$$\sum_{j=0}^{g_n}\mathbb{E}\left[|\mathcal{D}_{j,n}|\right] \le C\,k_n\,\Delta_n \to 0,$$

and hence, in conclusion, $\mathcal{B}_n$ is AN.

Part 1: Proof of the convergence in probability of $v_i^n\,(3)$

In what follows we call

$$\zeta\,(1)_j \doteq \mathbb{B}_{j,n}-p_{j,n} \text{ and } \zeta\,(2)_j \doteq p_{j+1,n}-p_{j,n}.$$

The quantity $v_j^n\,(3)$ can be rewritten as

$$v_j^n\,(3) = \frac{2}{k_n^2}\sum_{i=0}^{2k_n-1}\epsilon\,(1)_i\,\epsilon\,(2)_i\,\zeta\,(1)_{j+i}\,\zeta\,(2)_{j+i}\,.$$

So the quantity $\frac{1}{k_n}\sum_{i=0}^{g_n}v_i^n\,(3)$ becomes

$$\frac{1}{k_n}\sum_{i=0}^{g_n}v_j^n\,(3) = \sum_{i=0}^{g_n}\frac{2}{k_n^3}\sum_{i=0}^{2k_n-1}\epsilon\,(1)_i\,\epsilon\,(2)_i\,\zeta\,(1)_{j+i}\,\zeta\,(2)_{j+i}\,.$$

First, we observe that, conditionally on (p_t), we have that $\mathbb{E}\left[\zeta\,(1)_j\right]=0$ and so $\mathbb{E}_{j-1}\left[v_j^n\,(3)\right]=0$. Then, we note that term $k_n\left(\frac{v_j^n(3)}{k_n}\right)^2$ can be decomposed as

$$\begin{aligned}
k_n\left(\frac{v_j^n\,(3)}{k_n}\right)^2 &= \frac{4}{k_n^5}\sum_{i=0}^{2k_n-1}\left(\epsilon\,(2)_i\right)^2\left(\zeta\,(1)_{j+i}\right)^2\left(\zeta\,(2)_{j+i}\right)^2+\frac{8}{k_n^5}\sum_{j=0}^{2k_n-2}\sum_{i=0}^{2k_n-1}\epsilon(1)_j\epsilon(2)_j\zeta(1)_{i+j}\zeta(2)_{i+j}\epsilon(1)_l\epsilon(2)_l\zeta(1)_{i+l}\zeta(2)_{i+l}\\
&\doteq \mathcal{A}_{1,n}+\mathcal{A}_{2,n}
\end{aligned}$$

Now, by conditioning on (p_t), we readily obtain that $\mathbb{E}\left[\mathcal{A}_{2,n}\right]=0$. Concerning $\mathcal{A}_{1,n}$, we have

$$\mathbb{E}\left[|\mathcal{A}_{1,n}|\right] \le \mathbb{E}\left[\frac{4}{k_n^5}\sum_{i=0}^{2k_n-1}\left(\epsilon\,(2)_i\right)^2\left(\zeta\,(1)_{j+i}^2\left(\zeta\,(2)_{j+i}\right)^2\right)\right] \le \frac{C}{k_n^5}\Delta_n\sum_{i=0}^{2k_n-1}\left(\epsilon\,(2)_i\right)^2$$

By boundedness of Bernoulli random variables and (p_t) we have that $\left(\zeta\left(1\right)_{j+i}\right)^2 \leq C$ for some positive constant C. Hence

$$\mathbb{E}\left[\left|\mathcal{A}_{1,n}\right|\right] \leq \frac{C}{k_n^5}\Delta_n \sum_{j=0}^{2k_n-1}\left(\epsilon\left(2\right)_i\right)^2 = \frac{C}{k_n^5}\Delta_n \frac{2k_n^3+k_n}{3} \sim \frac{\Delta_n}{k_n^2}$$

Hence

$$\sum_{j=1}^{g_n}\mathbb{E}\left[\left|\mathcal{A}_{1,n}\right|\right] \leq \frac{C}{k_n} \to 0.$$

Consequently, by Lemma 2, $\frac{1}{k_n}v_j^n\left(3\right)$ is asymptotically negligible.

Part 1: Proof of the convergence in probability of $v_i^n\left(4\right)$

First, by conditioning on (p_t) we readily obtain $\mathbb{E}_{j-1}\left[v_j^n\left(4\right)\right] = 0$. Next, consider the decomposition:

$$\left(\frac{v_j^n\left(4\right)}{k_n}\right)^2 = \mathcal{A}_{1,n} + \mathcal{A}_{2,n},$$

where

$$\mathcal{A}_{1,n} = \frac{C}{k_n^6}\sum_{j=0}^{2k_n-2}\left(\sum_{l=i+1}^{2k_n-1}\epsilon(1)_i\epsilon(1)_l\zeta(1)_{j+i}\zeta(1)_{j+l}\right)^2,$$

and

$$\begin{aligned}\mathcal{A}_{2,n} &= \frac{C}{k_n^6}\sum_{i=0}^{2k_n-3}\sum_{m=i+1}^{2k_n-2}\left(\sum_{l=i+1}^{2k_n-1}\epsilon(1)_i\epsilon(1)_l\zeta(1)_{j+i}\zeta(1)_{j+l}\right)\left(\sum_{u=m+1}^{2k_n-1}\epsilon(1)_m\epsilon(1)_u\zeta(1)_{j+m}\zeta(1)_{j+u}\right),\\ &= \frac{C}{k_n^6}\sum_{i=0}^{2k_n-3}\sum_{m=i+1}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\sum_{u=i+2}^{2k_n-1}\epsilon(1)_i\epsilon(1)_l\epsilon(1)_m\epsilon(1)_u\zeta(1)_{j+i}\zeta(1)_{j+l}\zeta(1)_{j+u}\zeta(1)_{j+m}.\end{aligned}$$

By conditioning on (p_t) again, we have $\mathbb{E}\left[\zeta(1)_{j+i}\zeta(1)_{j+l}\zeta(1)_{j+u}\zeta(1)_{j+m}\right] = 0$ if at least two of the indexes i, l, u, m are different. Since in the sums that appear in $\mathcal{A}_{2,n}$ one among m, l or u is different from i, we have $\mathbb{E}\left[\mathcal{A}_{2,n}\right] = 0$. Analogously, the expected value of the cross-product terms in $\mathcal{A}_{1,n}$ is zero. Next, since $|\zeta(1)_{j+l}| \leq C$, for some constant $C > 0$,

$$\mathbb{E}\left[\mathcal{A}_{1,n}\right] = \frac{C}{k_n^6}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\mathbb{E}\left[\left(\zeta(1)_{j+i}\zeta(1)_{j+l}\right)^2\right] \leq \frac{C(2k_n-2)(2k_n-1)}{k_n^6} \sim \frac{1}{k_n^4}.$$

Hence,

$$k_n\sum_{i=1}^{g_n}\mathbb{E}\left[\left(\frac{v_j^n\left(4\right)}{k_n}\right)^2\right] \leq \frac{C}{k_n^3\Delta_n} \to 0.$$

Consequently, by Lemma 2, $\frac{1}{k_n}v_j^n\left(4\right)$ is asymptotically negligible. *Part 1: Proof of the convergence in probability of* $v_i^n\left(5\right)$

By successive conditioning and using Lemma 10, we obtain

$$\left|\mathbb{E}_{j-1}\left[v_j^n\left(5\right)\right]\right| \leq \frac{C}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l\Delta_n^2 = C\frac{\Delta_n^2}{k_n^2}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l \sim C\,\Delta_n^2k_n^2,$$

where we use the fact that $\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l \sim k_n^4$. Hence, we have

$$\sum_{j=1}^{g_n}\frac{1}{k_n}\left|\mathbb{E}_{j-1}\left[v_j^n\left(5\right)\right]\right| \sim \Delta_nk_n \to 0.$$

Next, we have

$$\left(\frac{v_j^n\,(5)}{k_n}\right)^2 = \mathcal{A}_{1,n} + \mathcal{A}_{2,n},$$

where

$$\mathcal{A}_{1,n} = \frac{C}{k_n^6}\sum_{i=0}^{2k_n-2}\left(\sum_{l=i+1}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l\zeta(2)_{j+i}\zeta(2)_{j+l}\right)^2,$$

$$\mathcal{A}_{2,n} = \frac{C}{k_n^6}\sum_{j=0}^{2k_n-3}\sum_{m=j+1}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\sum_{u=j+2}^{2k_n-1}\epsilon(2)_i\epsilon(2)_l\epsilon(2)_m\epsilon(2)_u\zeta(2)_{j+i}\zeta(2)_{j+l}\zeta(2)_{i+u}\zeta(2)_{i+m}.$$

Furthermore, we have

$$\mathcal{A}_{1,n} = \mathcal{A}_{1,1,n} + \mathcal{A}_{1,2,n},$$

where

$$\mathcal{A}_{1,1,n} = \frac{C}{k_n^6}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\left(\epsilon(2)_i\epsilon(2)_l\zeta(2)_{j+i}\zeta(2)_{j+l}\right)^2,$$

$$\mathcal{A}_{1,2,n} = \frac{C}{k_n^6}\sum_{i=0}^{2k_n-3}\sum_{l=i+1}^{2k_n-2}\sum_{m=i+2}^{2k_n-1}\left(\epsilon(2)_i\right)^2\epsilon(2)_l\epsilon(2)_m\left(\zeta(2)_{j+i}\right)^2\zeta(2)_{j+l}\zeta(2)_{j+m}.$$

Using the estimate (32) of Lemma 10, and the fact that $\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\left(\epsilon(2)_i\epsilon(2)_l\right)^2 \sim k_n^6$, we obtain:

$$\mathbb{E}\left[\mathcal{A}_{1,1,n}\right] \le C\,\frac{\Delta_n^2}{k_n^6}\sum_{i=0}^{2k_n-2}\sum_{l=i+1}^{2k_n-1}\left(\epsilon(2)_i\epsilon(2)_l\right)^2 \sim \Delta_n^2,$$

which implies that $k_n\sum_{i=1}^{g_n}\mathbb{E}\left[\mathcal{A}_{1,1,n}\right] \le k_n\,\Delta_n \to 0$. Next, using the estimates (32) and (33) of Lemma 10, we have

$$\mathbb{E}\left[\left(\zeta(2)_{j+i}\right)^2\zeta(2)_{j+l}\zeta(2)_{j+m}\right] \le C\begin{cases}\Delta_n^2, & l=m,\\ \Delta_n^3, & i\ne l\ne m.\end{cases}$$

Hence, we have

$$\mathbb{E}\left[\mathcal{A}_{1,2,n}\right] \le C\frac{\Delta_n^2}{k_n^6}S_1 + C\frac{\Delta_n^3}{k_n^6}S_2 \sim \Delta_n^2 \vee \Delta_n^3 k_n,$$

where

$$S_1 = \sum_{j=0}^{2k_n-3}\sum_{l=i+1}^{2k_n-2}\sum_{m=i+2}^{2k_n-1}\left(\epsilon(2)_i\right)^2\epsilon(2)_l\epsilon(2)_m\mathbb{I}\left(\{l=m\}\right) = \sum_{i=0}^{2k_n-3}\sum_{l=i+2}^{2k_n-2}\left(\epsilon(2)_i\right)^2\left(\epsilon(2)_l\right)^2 \sim k_n^6,$$

$$S_2 = \sum_{i=0}^{2k_n-3}\sum_{l=i+1}^{2k_n-2}\sum_{m=i+2}^{2k_n-1}\left(\epsilon(2)_i\right)^2\epsilon(2)_l\epsilon(2)_m\mathbb{I}\left(\{l\ne m\}\right) = \sum_{i=0}^{2k_n-3}\sum_{l=i+1}^{2k_n-2}\sum_{m=i+2}^{2k_n-1}\left(\epsilon(2)_i\right)^2\epsilon(2)_l\epsilon(2)_m - S_1 \sim k_n^7.$$

Consequently,

$$k_n\sum_{j=1}^{g_n}\mathbb{E}\left[\mathcal{A}_{1,2,n}\right] \le C\,\Delta_n k_n \to 0.$$

So, summing up $k_n \sum_{j=1}^{g_n} \mathbb{E}\left[\mathcal{A}_{1,n}\right] \to 0$. With a procedure similar to that used for $\mathcal{A}_{1,2,n}$, we obtain

$$k_n \sum_{j=1}^{g_n} \mathbb{E}\left[\mathcal{A}_{2,n}\right] \leq C\,\Delta_n k_n \to 0.$$

Thus, $\frac{1}{k_n} v_j^n\,(5)$ is asymptotically negligible by Lemma 2.

Proof of the convergence in probability of $v_i^n\,(6)$ and $v_i^n\,(7)$ First, by conditioning on (p_t) we readily obtain $\mathbb{E}_{j-1}\left[v_j^n\,(6)\right] = 0$. Next, consider the decomposition

$$\left(\frac{v_j^n\,(6)}{k_n}\right)^2 = \mathcal{A}_{1,n} + \mathcal{A}_{2,n},$$

where

$$\mathcal{A}_{1,n} = \frac{C}{k_n^6} \sum_{i=0}^{2k_n-2} \left(\sum_{l=i+1}^{2k_n-1} \epsilon(1)_i \epsilon(2)_l \zeta(1)_{j+i} \zeta(2)_{j+l} \right)^2,$$

and

$$\begin{aligned}
\mathcal{A}_{2,n} &= \frac{C}{k_n^6} \sum_{i=0}^{2k_n-3} \sum_{m=i+1}^{2k_n-2} \left(\sum_{l=i+1}^{2k_n-1} \epsilon(1)_i \epsilon(2)_l \zeta(1)_{j+i} \zeta(2)_{j+l} \right) \left(\sum_{u=m+1}^{2k_n-1} \epsilon(1)_m \epsilon(2)_u \zeta(1)_{j+m} \zeta(2)_{j+u} \right), \\
&= \frac{C}{k_n^6} \sum_{i=0}^{2k_n-3} \sum_{m=i+1}^{2k_n-2} \sum_{l=i+1}^{2k_n-1} \sum_{u=i+2}^{2k_n-1} \epsilon(1)_i \epsilon(2)_l \epsilon(1)_m \epsilon(2)_u \zeta(1)_{j+i} \zeta(2)_{j+l} \zeta(1)_{j+u} \zeta(2)_{j+m}.
\end{aligned}$$

By conditioning on (p_t), $\mathbb{E}\left[\mathcal{A}_{2,n}\right] = 0$, because $\mathbb{E}\left[\zeta(1)_{j+i}\zeta(1)_{j+u}\right] = 0$ for $u > i$. Analogously, the expected value of the cross-product terms in $\mathcal{A}_{1,n}$ is zero. Hence, we have

$$\mathbb{E}\left[\mathcal{A}_{1,n}\right] = \frac{C}{k_n^6} \sum_{i=0}^{2k_n-2} \sum_{l=i+1}^{2k_n-1} \mathbb{E}\left[\left(\epsilon(1)_i \epsilon(2)_l \zeta(1)_{j+i} \zeta(2)_{j+l}\right)^2 \right] \leq C\,\frac{\Delta_n}{k_n^6} \sum_{i=0}^{2k_n-2} \sum_{l=i+1}^{2k_n-1} \left(\epsilon(2)_l\right)^2 \sim \frac{\Delta_n}{k_n^2}.$$

Thus,

$$k_n \sum_{j=1}^{g_n} \mathbb{E}\left[\mathcal{A}_{1,n}\right] \leq \frac{C}{k_n} \to 0.$$

Consequently, $\frac{1}{k_n} v_j^n\,(6)$ is asymptotically negligible by Lemma 2. Analogously, $\frac{1}{k_n} v_j^n\,(7)$ is asymptotically negligible as well.